MATH 363
Discrete Mathematics
Winter 2009

Assignment 6
Solutions

1. The graphs shown in Figure 1 are isomorphic. To construct an isomorphism, recall that any isomorphism preserves the degree. This gives a good indication as to what the map should be, as there is only one vertex of degree 2, and 2 vertices of degree 4. Here is an explicit isomorphism:

$$f : \begin{cases} u_1 & \mapsto & v_1 \\ u_2 & \mapsto & v_3 \\ u_3 & \mapsto & v_2 \\ u_4 & \mapsto & v_5 \\ u_5 & \mapsto & v_4 \end{cases}$$

Now check that this is indeed an isomorphism, i.e., check that for every edge (u_i, u_j), $(f(u_i), f(u_j))$ is also an edge:

$$f : \begin{cases} (u_1, u_2) & \mapsto & (v_1, v_3) \\ (u_1, u_4) & \mapsto & (v_1, v_5) \\ (u_1, u_5) & \mapsto & (v_1, v_4) \\ (u_2, u_3) & \mapsto & (v_3, v_2) \\ (u_2, u_4) & \mapsto & (v_3, v_5) \\ (u_2, u_5) & \mapsto & (v_3, v_4) \\ (u_3, u_4) & \mapsto & (v_2, v_5) \\ (u_4, u_5) & \mapsto & (v_5, v_4) \end{cases}$$

These are all edges in the second graph, and both graphs have eight edges, so these are precisely all the edges in the second graph. Thus f is an isomorphism.

2. Suppose c is a cut vertex. Denote by $\Gamma(c)$ the neighbours of c. Let C_1 and C_2 be two components that are obtained after the deletion of c. Take some $u \in \Gamma(c) \cap C_1$ and $v \in \Gamma(c) \cap C_2$. Note that $\Gamma(c) \cap C_i \neq \emptyset$ if the

original graph was connected. Suppose there is a path, p, which connects u and v and doesn't go through c. Then p contains an edge (u', v') with $u' \in C_1$ and $v' \in C_2$. But then after removing c, this edge will still be present in the graph and it will connect C_1 to C_2. But these ought to be disconnected, so we get a contradiction.

Conversely, suppose there are two vertices $u, v \in G$ such that every path between u and v passes through c. Let G' be the graph obtained by deleting all edges touching c, i.e., $V_{G'} = V_G$ and

$$E_{G'} = E_G \setminus \{(w, c) \mid w \in V \setminus \{c\}\}.$$

Suppose that c is not a cut-vertex. Then G' is connected. In particular, there is a path, p, in G' between u and v. But $G' \subseteq G$, so p is also a path in G between u and v. Since p was chosen in G', it contains no edge of the form (w, c) or (c, w), i.e., p does not pass through c. This is a contradiction.

3. Build the graph corresponding to his situation. The vertices are disjoint pieces of land, namely the two islands and both banks of the river. Edges are bridges. The question then asks whether there exists an Euler tour on this graph. We know that an euler tour exists if and only if Every vertex has even degree. It is the case in this situation, so we're done. See Figure 2, for an example of a path that does the job.

4. The graph in Figure 3 has no Hamiltonian path. There are three vertices of degree one (e, f and g). Once we've visited any one of them, we need to go out, in order to visit one of the others. But in order to "go out", we can only use the one edge that we "came in" through. So if we visit any of these vertices, while trying to build a Hamiltonian path, we will get stuck in the first one we visit.

5. Note: the problem asks for a circuit, i.e., we need to get back to the starting point. Also, it is implicit that every city must be visited exactly once.

   This is a travelling salesman problem, and as we know well, there is only one way to do it deterministically, namely compute the weight of all possible routes and then pick the one with minimal weight. So we consider all permutations of length five on the set of cities. Note that since we are looking at circuits, the starting point doesn't matter, so a cyclic permutation of a chosen route will yield the same total airfare. So we can fix a starting point (say Seattle, without loss of generality) and it is enough to consider all cycles starting at Seattle. There are $4! = 24$ possible ones, but since it doesn't matter in which direction we travel on this circuit, going clockwise and counterclockwise will yield the same airfare, so we only need to consider 12 possible routes. Here they are:

Route						Weight
Seattle	Phoenix	New Orleans	New York	Boston	Seattle	\$1175
Seattle	Phoenix	New Orleans	Boston	New York	Seattle	\$1165
Seattle	Phoenix	New York	New Orleans	Boston	Seattle	\$1315
Seattle	Phoenix	New York	Boston	New Orleans	Seattle	\$1215
Seattle	Phoenix	Boston	New Orleans	New York	Seattle	\$1355
Seattle	Phoenix	Boston	New York	New Orleans	Seattle	\$1265
Seattle	New Orleans	Phoenix	New York	Boston	Seattle	\$1575
Seattle	New Orleans	Phoenix	Boston	New York	Seattle	\$1615
Seattle	New Orleans	New York	Phoenix	Boston	Seattle	\$1765
Seattle	New Orleans	Boston	Phoenix	New York	Seattle	\$1755
Seattle	New York	Phoenix	New Orleans	Boston	Seattle	\$1665
Seattle	New York	New Orleans	Phoenix	Boston	Seattle	\$1715

6. Neither of these graphs is planar. The first one is called Petersen's graph and is a well-known in graph theory. It is not planar because it contains a homeomorphic copy of $K_{3,3}$, with vertices as shown in Figure 4.

   The second graph is not planar either. It contains no three-cycles, so if it were, the identity $|E| \leq 2|V| - 4$ would hold. But it has $|E| = 14$ and $|V| = 8$, so we get a contradiction.

7. We are given a graph which is simple and planar and since it is bipartite, it has no three-cycles. Then it satisfies the identity $e \leq 2v - 4$.

8. Build a graph, where each station is a vertex and two stations are connected by an edge if they are less than 150m apart. Vertex-colour this graph. The number of colours needed is the number of different stations needed. See Figure 6 for a colouring that uses three colours. This is the best we can do because the graph contains a three-cycle. So its chromatic number is 3, so we need three different channels.

9. (Bonus Problem)

   Claim. $\chi(K_m) = \begin{cases} m & \text{if} \quad m = 2n+1 \\ m-1 & \text{if} \quad m = 2n. \end{cases}$

   Proof. It is known that $\Delta \leq \chi(G) \leq \Delta + 1$, for any graph G, where Δ is the highest degree a vertex can have in G. So $m - 1 \leq \chi(K_m) \leq m$, since K_m is complete and so its highest degree is $m - 1$.

   First, consider the odd case, $m = 2n + 1$. Every set of edges of the same colour, has at most $\frac{1}{2}(m-1)$ edges, otherwise, there would be more than m vertices in the graph. The total number of edges is $|E| = \frac{m(m-1)}{2}$. So if K_m is k-coloured, we have

$$k\frac{m-1}{2} \geq \frac{m(m-1)}{2}.$$

Thus we get $k \geq m$, and we already know $k \leq m$, so for $m = 2n+1$, $\chi(K_m) = m$.

Now consider K_{2n}. Let $\{v_0, \dots, v_{2n-2}, v_{2n-1}\}$ be its vertices. Pick a vertex, say v_{2n-2} and delete all its incoming edges. This leaves us with a complete graph on $2n-1$ vertices. We can colour such a graph with $2n-1$ colours, so that vertex v_i is not touching an edge of colour i (see below for an explicit colouring with this property). Now add back the deleted edges (while keeping the old colouring), and colour (v_i, v_{2n-1}) with colour i. By construction, no edge adjacent to this one will hae colour i, so this gives a colouring on K_{2n} with $2n-1$ colours.

Remark. We give a colouring by $2n+1$ colours of K_{2n+1} such that every vertex is missing a different colour. Let $\{v_0, \dots, v_{2n}\}$ be the vertices of K_{2n+1}. Let $f : E \to \{0, 1, \dots, 2n\}$ be defined as follows:

$$f : \begin{cases} (v_i, v_j) & \mapsto i+j \quad \mod 2n+1 \quad \text{if} \quad i,j \neq 2n+1 \\ (v_i, v_j) & \mapsto 2i \quad \mod 2n+1 \quad \text{if} \quad j = 2n+1 \\ (v_i, v_j) & \mapsto 2j \quad \mod 2n+1 \quad \text{if} \quad i = 2n+1 \end{cases}.$$

Note that f is symmetric, as expected, since (u, v) and (v, u) denote the same edge. Now, let $e_1 = (v_{i_1}, v_{j_1})$ and $e_2 = (v_{i_2}, v_{j_2})$ be two adjacent edges. They must have a common vertex, without loss of generality, $v_{i_1} = v_{i_2} = v_i$. It is easy to see that $f(e_1) \neq f(e_2)$:

- If $i = 2n$, then $f(e_1) = 2j_1 \neq 2j_2 = f(e_2)$.
- If $i, j_1, j_2 \neq 2n$, then $f(e_1) = i + j_1 \neq i + j_2 = f(e_2)$.
- If $i \neq 2n$, but $j_1 = 2n$, then $f(e_1) = 2i \neq i + j_2 = f(e_2)$.

So f gives a colouring on K_{2n+1} in $2n+1$ colours. Note that in the colouring we gave, for every $0 \leq i \leq 2n$, vertex v_i has no incoming edge which is coloured with i.

□

Claim. $\chi(K_{n,m}) = \max(n, m)$.

Proof. Without loss of generality $n \leq m$. Let the vertices on the two sides be $\{v_0, \dots, v_{n-1}\}$ and $\{u_0, \dots, u_{m-1}\}$.

Colour each edge (v_0, u_i) with i. In the same vein, colour edges (v_j, u_i) with colour $i + j \mod m$. In other words, we use a different cyclic permutation of colours for the neighbours of each v_j. Since the graph is bipartite, this is enough, as we are assigning colours stating on one side and we ensure that they don't touch on the other side. □