MATH 363 001
Discrete Mathematics
Winter 2009

Solutions to Assignment 5.

1) How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

   Solution: Let a_n be the number of messages that can be transmitted in n microseconds. Then $a_n = a_{n-1} + 2a_{n-2}$, where $a_1 = 1, a_2 = 3$. Solving the characteristic equation $x^2 - x - 2 = 0$ we get $x_1 = -1, x_2 = 2$. Hence $a_n = \alpha_1(-1)^n + \alpha_2 2^n$. Solving the initial conditions for α_1, α_2 we get $\alpha_1 = 1/3, \alpha_2 = 2/3$. Hence,

$$a_n = \frac{1}{3}(-1)^n + \frac{2}{3}2^n.$$

2) Solve the recurrence relation:

   a) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with the initial conditions $a_0 = 3, a_1 = 6, a_2 = 0$.
   Solution: Solve the characteristic equation $x^3 - 2x^2 - x + 2 = 0$. Rewrite as $x^2(x-2) - (x-2) = 0$, so $(x-2)(x^2-1) = 0$. It follows that the solutions are $x_1 = 2, x_2 = -1, x_3 = 1$. Hence $a_n = \alpha_1 2^n + \alpha_2(-1)^n + \alpha_3$. Solving the initial conditions for α_i-s we get $\alpha_1 = -1, \alpha_2 = -2, \alpha_3 = 6$. Therefore, $a_n = -2^n - 2(-1)^n + 6$.

   b) $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3, a_1 = 2, a_2 = 6, a_3 = 8$.
   Solution: Observe, that the degree of the recurrence relation is 4 (not 2!). Hence the characteristic equation is $x^4 - 5x^2 + 4 = 0$. Solving for $y = x^2$ we get $y_1 = 1, y_2 = 4$. Hence the solutions are $x_1 = -1, x_2 = 1, x_3 = -2, x_4 = 2$. This implies that $a_n = \alpha_1(-1)^n + \alpha_2 + \alpha_3(-2)^n + \alpha_4 2^n$. It remains to solve the initial conditions for α_i-s: $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 1$.

3) What is the generating function for the sequence $\{c_k\}$, where c_k is the number of ways to make change for k pesos using bills worth 10, 20, 50, and 100 pesos?

   Solutions: c_k is the coefficient of x^k in the following product:

$$G(x) = (1 + x^{10} + x^{20} \ldots)(1 + x^{20} + x^{40} \ldots)(1 + x^{50} + x^{100} \ldots)(1 + x^{100} + x^{200} \ldots).$$

   Using the formula for geometric series we get

$$G(x) = \frac{1}{(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}.$$

4) Use generating functions to solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6$ with $a_0 = 20, a_1 = 60$.

Solution: From the the recurrence relation (multiplying the sides by x^n) we get

$$a_n x^n = 2a_{n-1}x^n + 3a_{n-2}x^n + 4^n x^n + 6x^n.$$

Summing up from $n = 2$ to ∞ we get

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 2a_{n-1}x^n + \sum_{n=2}^{\infty} 3a_{n-2}x^n + \sum_{n=2}^{\infty} 4^n x^n + \sum_{n=2}^{\infty} 6x^n.$$

Hence

$$\sum_{n=2}^{\infty} a_n x^n = 2x \sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 3x^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2} + 16x^2 \sum_{n=2}^{\infty} (4x)^{n-2} + 6x^2 \sum_{n=2}^{\infty} x^{n-2}.$$

Hence

$$G(x) - a_0 - a_1 x = 2x(G(x) - a_0) + 3x^2 G(x) + \frac{16x^2}{1-4x} + \frac{6x^2}{1-x}.$$

Therefore,

$$G(x)(1 - 2x - 3x^2) = 20 + 20x + \frac{16x^2}{1-4x} + \frac{6x^2}{1-x}.$$

It follows that

$$G(x) = \frac{20+20x}{1-2x-3x^2} + \frac{16x^2}{(1-4x)(1-2x-3x^2)} + \frac{6x^2}{(1-x)(1-2x-3x^2)}.$$

Notice that $1 - 2x - 3x^2$ has roots $x_1 = -1, x_2 = 1/3$, so $1 - 2x - 3x^2 = (x+1)(1-3x)$. Hence,

$$G(x) = \frac{20+20x}{(x+1)(1-3x)} + \frac{16x^2}{(1-4x)(x+1)(1-3x)} + \frac{6x^2}{(1-x)(x+1)(1-3x)}.$$

Now we need to represent the left-hand side via elementary fractions. The first summand is easy:

$$\frac{20+20x}{(x+1)(1-3x)} = \frac{20}{1-3x}.$$

For the second one we are looking for constants A, B, C such that

$$\frac{1}{(1-4x)(x+1)(1-3x)} = \frac{A}{1-4x} + \frac{B}{x+1} + \frac{C}{1-3x}$$

Solving for A, B, C we get the required decomposition. Similar for the third term. Now decomposing the elementary fractions as power series (using the formula for the geometric series) we get $G(x)$ as a sum of power series, which allows one to find the coefficients a_n.

5) Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.

Solutions: Let A and B be the sets of positive integers not exceeding 1000 that are correspondingly squares and cubes. Then $|A \cup B| = |A| + |B| - |A \cap B|$. Since $\lfloor \sqrt{1000} \rfloor = 31$ we have $|A| = 31$, similarly, $|B| = 10$. Clearly, among these 10 cubes in B only $1^3, 4^3$ and 9^3 are squares, so $|A \cap B| = 3$. Hence $|A \cup B| = 31 + 10 - 3 = 38$.

6) In a survey of 270 college students, it is found that 64 like brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussels sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?

Solution: Let A, B, and C be the sets of students that like brussels sprouts, broccoli, and cauliflower. Then the required number is equal to $270 - |A \cup B \cup C|$. Now use the inclusion-exclusion principle to find $|A \cup B \cup C|$.

(7) (Bonus problem) Find the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 17$, where x_i are nonnegative integers such that $x_1 \leq 3, x_2 \leq 4, x_3 \leq 5, x_4 \leq 8$.

Each problem is 10 points.