MATH 363
Discrete Mathematics
Winter 2009

Assignment 3
Solutions

1. Clearly, if $A = B$, $A \times B = A \times A = B \times A$. Now suppose $A \times B = B \times A$. Then for any $a \in A$ and $b \in B$, it must be the case that $(a, b) = (b, a)$. This implies equality in each component, namely, $a = b$ and $b = a$. Thus for any $a \in A$, $b \in B$, we have $a = b$, so $A = B$.

2. Let $A \oplus B = (A \cup B) \setminus (A \cap B)$.

   a) Take any $x \in (A \oplus B) \oplus B$. Then $x \in ((A \oplus B) \cup B) \setminus ((A \oplus B) \cap B)$, so:

   - $x \in A \oplus B \cup B$. So $x \in A \oplus B$ or $x \in B$
   - $x \notin A \oplus B \cap B$.

   Suppose $x \in B$, $x \notin A \oplus B$. But then $x \notin A \cup B$, so in particular, $x \notin B$, and we get a contradiction. So $x \notin B$. Then it must be the case that $x \in A \oplus B$. So we have $x \in A \cup B$ and $x \notin B$, so $x \in A$. This shows that $(A \oplus B) \oplus B \subseteq A$. It remains to show the reverse inclusion.

   Take $x \in A$. If $x \notin A \cap B$, then $x \in A \oplus B$ and hence $x \in (A \oplus B) \oplus B$. If $x \in A \cap B$, $x \notin A \oplus B$, so $x \notin (A \oplus B) \cap B$. But $x \in A \cap B$, so in particular, $x \in B$, so $x \in (A \oplus B) \cup B$. Thus $x \in (A \oplus B) \oplus B$. This shows that $A \subseteq (A \oplus B) \oplus B$ and we're done.

   b) It is true in general that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. The intuition is that symmetric difference of two sets "deletes" intersections, so it will delete all intersections that are in an even number of sets (e.g., $A \cap B \cap C$ will not be deleted). Draw a picture to convince yourselves. Here is the actual proof:

   Take $x \in (A \oplus B) \oplus C$. Then:

   - $x \in A \oplus B$ or $x \in C$
   - $x \notin (A \oplus B) \cap C$

If $x \in C$, then $x \notin (A \oplus B)$, i.e., $x \notin A \cup B$ or $x \in A \cap B$.

- If $x \in A \cap B$, then $x \in A \cap B \cap C$. So in particular, $x \in A$ and also, $x \in B \cap C$. But if $x \in B \cap C$, then $x \notin B \oplus C$. Thus $x \in A \cup (B \oplus C)$ (because $x \in A$), and $x \notin A \cap (B \oplus C)$ (because $x \notin B \oplus C$). So $x \in A \oplus (B \oplus C)$.
- If $x \notin A \cup B$ (and $x \notin A \cap B$), then $x \notin A$ and $x \notin B$. But $x \in B \oplus C$, since $x \in C$ (and so $x \in B \cup C$) and also, $x \notin B$ (so $x \notin B \cap C$). Now, since $x \in B \oplus C$, $x \in A \cup (B \oplus C)$. But $x \notin A$, so $x \notin A \cap (B \oplus C)$. Thus $x \in A \oplus (B \oplus C)$.

This shows that $(A \oplus B) \oplus C \subseteq A \oplus (B \oplus C)$. Similarly, $A \oplus (B \oplus C) \subseteq (A \oplus B) \oplus C$.

Note that with this identity, part a) becomes trivial:

$$(A \oplus B) \oplus B = A \oplus (B \oplus B) = A \oplus \emptyset = A.$$

3. Let $f : A \to B$ be a function, $S, T \subseteq A$.

   a) Suppose $f(x) \in f(S \cup T)$, so that $x \in S \cup T$. If $x \in S$, then $f(x) \in f(S)$. If $x \in T$, then $f(x) \in f(T)$. So $f(x) \in f(S) \cup f(T)$ and hence $f(S \cup T) \subseteq f(S) \cup f(T)$. Conversely, if $f(x) \in f(S) \cup f(T)$, $f(x) \in f(S)$ (and so $x \in S$), or $f(x) \in f(T)$ (and so $x \in T$). Thus $x \in S$ or $x \in T$, i.e., $x \in S \cup T$ (and so $f(x) \in f(S \cup T)$). This shows that $f(S) \cup f(T) \subseteq f(S \cup T)$ and we're done.

   This holds for $\cup$ replaced by $\cap$. The proof is the same, with "or" replaced by "and".

   b) Let $x \in f^{-1}(S \cup T)$, i.e., $f(x) \in S \cup T$. So $f(x) \in S$ or $f(x) \in T$, i.e., $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$. So $x \in f^{-1}(S) \cup f^{-1}(T)$. Hence $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$. Conversely, suppose $x \in f^{-1}(S) \cup f^{-1}(T)$. Then $x \in f^{-1}(S)$ (i.e., $f(x) \in S$), or $x \in f^{-1}(T)$ (i.e., $f(x) \in T$). So $f(x) \in S \cup T$, i.e., $x \in f^{-1}(S \cup T)$. Hence $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$.

   Showing that $f^{-1}(S) \cup f^{-1}(T) = f^{-1}(S \cup T)$ is similar (replace $\cup$ by $\cap$ and "or" by "and").

4. a) Let $f(x) = ax + b$ and $g(x) = cx + d$. Then:

   $$(f \circ g)(x) = f(g(x)) = ag(x) + b = a(cx + d) + b = acx + (ad + b)$$

   and

   $$(g \circ f)(x) = g(f(x)) = cf(x) + d = c(ax + b) + d = acx + (bc + d)$$

   So in order to have $f \circ g = g \circ f$, we need $ad + b = bc + d$. We can rewrite this as $b(1 - c) = d(1 - a)$. So given any three of $a, b, c, d \in \mathbb{R}$, we can find the fourth one so that $f \circ g = g \circ f$.

b) Let $f(x) = 2x^3 + 1$. First find the inverse of f:

$$f(x) = 2x^3 + 1 \to f(x) - 1 = 2x^3 \to x^3 = \frac{1}{2}(f(x) - 1).$$

Hence

$$f^{-1}(x) = \left(\frac{1}{2}(x - 1)\right)^{\frac{1}{3}}.$$

It is easy to check that $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$. So f is invertible and f^{-1} is its inverse.

5. a) Proceed by induction on n. First check the base case. For $n = 1$, $\sum_{k=1}^{1} k^3 = 1$ and $\frac{1^2 \times 2^2}{4} = 1$, so the statement is true for $n = 1$. Now suppose that the statement is true for $k = n$, i.e., suppose that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$. We show that it is also true for $k = n + 1$.

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left(\frac{1}{4}n^2 + n + 1\right) \\ &= \frac{1}{4}(n+1)^2\left(n^2 + 4n + 4\right) = \frac{1}{4}(n+1)^2(n+2)^2.\end{aligned}$$

b)

$$\sum_{k=99}^{200} k^3 = \sum_{k=1}^{200} k^3 - \sum_{k=1}^{98} k^3 = \frac{200^2(201)^2}{4} - \frac{98^2(99)^2}{4} = 380477799.$$

6. a) Let $f : [2,4] \to [1,2]$ be given by $f(x) = \frac{1}{2}x$. Clearly, $\bar{f} : [1,2] \to [2,4]$ given by $\bar{f}(x) = 2x$, is the inverse of f. So f is invertible and hence a bijection.

b) Let $f : \mathbb{R} \to (0,1)$ be given by $f(x) = \arctan x + \frac{\pi}{2}$. Clearly, $\bar{f} : (0,1) \to \mathbb{R}$, given by $\bar{f}(x) = \tan(x - \frac{\pi}{2})$ is the inverse of f, so f is a bijection.

7. Let A_n be countable sets, i.e., for each n, A_n can be enumerated as $A_n = \{a_{n_i} \mid i \in \mathbb{N}\}$. So we can write:

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{\infty} a_{n_i} = \bigcup_{k=2}^{\infty}\{a_{n_i} \mid k = i + n\}.$$

This is a countable union of finite sets, which can be enumerated as $\{a_{1_1}, a_{1_2}, a_{2_1}, a_{1_3}, a_{3_1}, a_{2_2} \ldots\}$.

8. (Bonus Problem) Let $S = \{x \mid x \notin x\}$.

- Suppose $S \in S$. Then by definition of S, $S \notin S$, so we get a contradiction.
- Suppose $S \notin S$. But then S satisfies the condition for being an element of S, so $S \in S$ and we get a contradiction again.