MATH 363
Discrete Mathematics
Winter 2009

Midterm
Solutions

1. Recall that $x \to y$ is equivalent to $\neg x \vee y$.

   a)

$$\begin{aligned}
(\neg p \wedge (p \to q)) \to \neg p \quad &\Leftrightarrow \quad \neg\left(\neg p \wedge (\neg p \vee q)\right) \vee \neg p \Leftrightarrow (p \vee \neg(\neg p \vee q)) \vee \neg p \\
&\Leftrightarrow \quad (p \vee (p \wedge \neg q)) \vee \neg p \qquad \Leftrightarrow (p \wedge (p \vee \neg q)) \vee \neg p \\
&\Leftrightarrow \quad (p \vee \neg p) \wedge (p \vee \neg q \vee \neg p) \\
&\Leftrightarrow \quad \text{true.}
\end{aligned}$$

   So the original formula is a tautology.

   b) Denote

   - $\varphi(p, q, r) = (p \wedge q \to r)$
   - $\psi(p, q, r) = (p \to r) \wedge (q \to r)$.

   If the two formulas were equivalent, then they would be so for any assignment of truth values to (p, q, r). Check that for $(p, q, r) = (\text{false}, \text{true}, \text{false})$, $\varphi \neq \psi$:

$$\varphi(F, T, T) = (F \to F) = T$$

$$\psi(F, T, T) = (F \to F) \wedge (T \to F) = F.$$

   Note that 'False' implies anything. So $\varphi \neq \psi$, as desired.

2. If there were a duck with a lace collar in the village, it would have to be branded "B". If it is branded "B", it belongs to Mrs. Bond. But she has no grey ducks in the village. So if a duck had a lace collar it would not be grey. So there are no grey ducks in the village.

3. We proceed by induction. Let $f(n) = 1 \cdot 2 + \ldots + n(n + 1)$

- For $n = 1$, $\frac{1}{3}n(n+1)(n+2) = \frac{1}{3}1 \cdot 2 \cdot 3 = 2 = f(1)$.
- Suppose for n = k,

$$f(k) = 1 \cdot 2 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

- Show that the above identity also holds for $n = k+1$.

$$\begin{aligned} f(k+1) &= 1 \cdot 2 + \ldots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)\left(\frac{k(k+2)}{3} + k + 2\right) = (k+1)\left(\frac{k^2+2k+3k+3}{3}\right) \\ &= \frac{1}{3}(k+1)(k^2+5k+3) = \frac{1}{3}(k+1)(k+2)(k+3). \end{aligned}$$

4. Let $x \in (A \setminus B) \setminus C$. Then $x \in (A \setminus B)$ and $x \notin C$. So $x \in A$ and $x \notin B$ and $x \notin C$. Using this we get:

   - $x \in A$ and $x \notin C$, so $x \in A \setminus C$;
   - $x \notin B$, so $x \notin B \setminus C$.

   Thus, $(A \setminus B) \setminus C \subseteq (A \setminus C) \setminus (B \setminus C)$. Conversely, suppose $x \in (A \setminus C) \setminus (B \setminus C)$. Then $x \in A \setminus C$ and $x \notin B \setminus C$

   - $x \in A \setminus C$, so $x \in A$ and $x \notin C$
   - $x \notin B \setminus C$, so $x \notin B$ or $x \in C$. Since we already know that $x \notin C$, it must be the case that $x \notin B$.

   Thus, $x \in A$ and $x \notin B$ and $x \notin C$, which gives $x \in (A \setminus B) \setminus C$. Hence $(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C$.

   Note. $x \notin B \setminus C$ is equivalent to $\neg(x \in B \setminus C)$, which in turn is equivalent to $\neg\big((x \in B) \wedge (x \notin C)\big)$. This, by De Morgan's laws is equivalent to $\big(\neg(x \in B) \vee \neg(x \notin C)\big)$. Finally this gives $(x \notin B) \vee (x \in C)$. The key point is that the negation changes the 'and' into an 'or'.

5. - $f(x) = x^5+1$ is a bijection because it has an inverse, $\bar{f}(x) = (x-1)^{1/5}$.
   - $g(x) = \frac{x^2+1}{x^2+2}$ is not a bijection on $\mathbb{R}$ for many reasons. It is not injective (one-to-one), and it is not surjective (onto). Indeed,

   $$g(-x) = \frac{(-x)^2+1}{(-x)^2+2} = \frac{x^2+1}{x^2+2} = g(x).$$

   So $g(x) = g(-x)$, so g is not injective. Also notice that for all $x \in \mathbb{R}$, $x^2+2 > x^2+1$, so $0 < g(x) < 1$, so g is very much not surjective, either.

6. Let $X = \{x \mid ax^2 + bx + c = 0, a, b, c \in \mathbb{Z}\}$. We all know and love the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$. It gives solutions when they exist, but in particular, it tells us that for any triple of integers (a, b, c), there are at most two solutions to the quadratic equation $ax^2 + bx + c$. There are countably many such triples (There are $|\mathbb{Z}|$ of each component, so there are $3|\mathbb{Z}|$ triples, but $3|\mathbb{Z}| = \mathbb{Z}$). So $|X| \leq 2\mathbb{Z} = \mathbb{Z}$. Since $\mathbb{Z}$ is countable, X is a countable set.