Assignment 1

14.1 #17

Evaluate by inspection:

$$\iint_{x^2+y^2\le 1} (4x^2y^3 - x + 5)\,dA$$

i) $$\iint_{x^2+y^2\le 1} 4x^2y^3\,dA = 0$$

this is "odd" for the y-variable

and the disk $x^2+y^2\le 1$ is symmetric under the map that sends y to $-y$ (ie the reflection across the x-axis).

ii) $$\iint_{x^2+y^2\le 1} x\,dA = 0$$

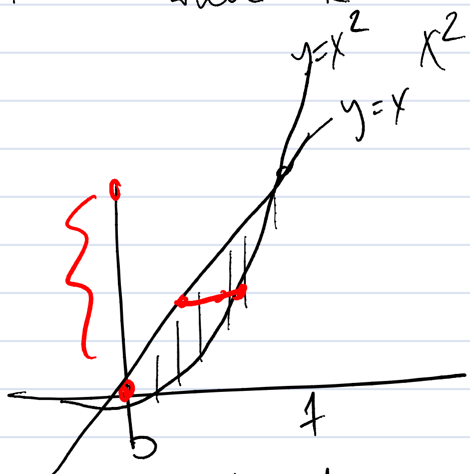
again odd & reflection across y-axis

iii) $$\iint_{x^2+y^2\le 1} 5\,dA = 5 \times \text{area of disk} = 5\pi$$

Ans = 5π.

14.2 # 13 evaluate by iteration:

$$\iint_R \frac{x}{y} e^y \, dA$$ where R: $0 \le x \le 1$, $x^2 \le y \le x$



looks like it's easier to integrate wrt x first

so R: $0 \le y \le 1$
$y \le x \le \sqrt{y}$

$$\int_0^1 dy \int_y^{\sqrt{y}} \frac{x}{y} e^y \, dx$$

$$= \int_0^1 dy \; \frac{1}{2}\frac{x^2}{y} e^y \Big|_y^{\sqrt{y}} = \int_0^1 \frac{1}{2}\frac{y}{y} e^y - \frac{1}{2}\frac{y^2}{y} e^y \, dy$$

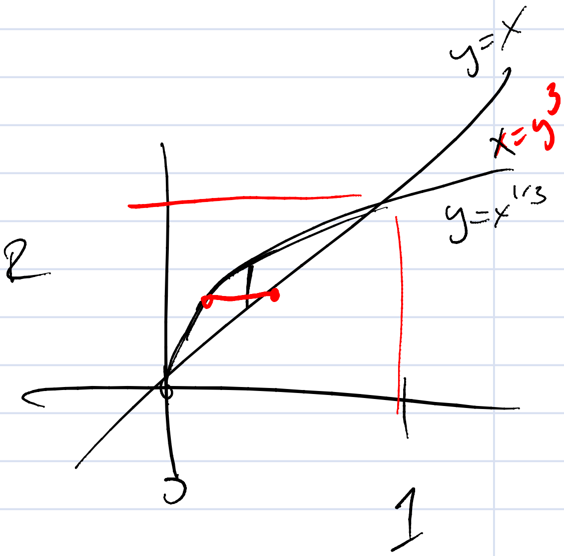
$$= \frac{1}{2}\int_0^1 e^y - y e^y \, dy = \frac{1}{2}(2-y)e^y \Big|_0^1$$

$$= \frac{1}{2}(2-1)e^1 - \frac{1}{2}(2-0)e^0$$

$$= \frac{e}{2} - 1$$

#18 sketch domain & evaluate

$$\int_0^1 dx \int_x^{x^{1/3}} \sqrt{1-y^4}\, dy$$



swap order:

$$\int_0^1 dy \int_{y^3}^{y} \sqrt{1-y^4}\, dx$$

$$= \int_0^1 x\left(\sqrt{1-y^4}\right)\Big|_{y^3}^{y} dy = \int_0^1 y\sqrt{1-y^4}\, dy + \int_0^1 -y^3\sqrt{1-y^4}\, dy$$

① let $u = y^2$ $du = 2y\,dy$ $\left(\begin{matrix} y=0 & u=0 \\ y=1 & u=1 \end{matrix}\right.$

$$\int_0^1 y\sqrt{1-y^4}\,dy = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du = \frac{1}{2}\left(\frac{u}{2}\sqrt{1-u^2} + \sin^{-1} u\right)\Big|_0^1$$

$$= \frac{1}{2}\sin^{-1}(1) - \frac{1}{2}\sin^{-1}(0) = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \pi/4$$

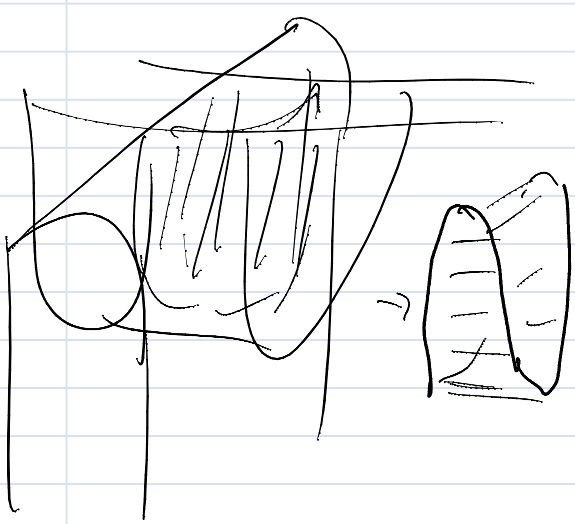
(2) let $u = 1-y^4$ $\quad du = -4y^3$ $\quad \begin{cases} y=0 & u=0 \\ y=1 & \end{cases}$

$$\int_0^1 -y^3\sqrt{1-y^4}\,dy = \frac{1}{4}\int_1^0 \sqrt{u}\,du = \frac{1}{6}u^{3/2}\Big|_1^0 = \frac{1}{6}(0-1)$$
$$= -1/6$$

ANS = (1) + (2) = $\pi/4 - 1/6$.

Note on (1): this can also be done by making the change of variables $y = u^2$ and then $u = \sin\theta$, if you don't want to look in the integral tables

22) Find the volume of the indicated solid under $z = 1-y^2$ and above $z = x^2$



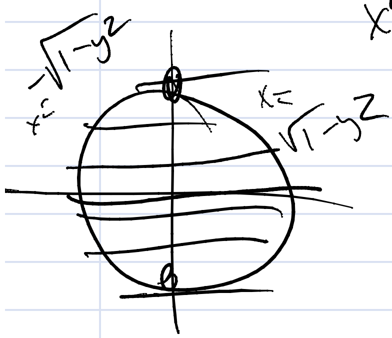
where is our domain?

the two graphs intersect when $z = z$ or

$1-y^2 = x^2$ or

$x^2+y^2=1$

volume $\displaystyle\iint_{x^2+y^2\le 1} (1-y^2) - x^2 \, dA$



$$\int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(1-y^2-x^2\right) dx$$

$$= \int_0^1 \left(x - xy^2 - \frac{x^3}{3}\right)\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx$$

$$= \int_0^1 x\left(1-y^2-\frac{x^2}{3}\right)\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx$$

$$= 2\int_0^1 \sqrt{1-y^2}\left(1-y^2-\frac{(1-y^2)}{3}\right) dx$$

$$= 2\int_0^1 \sqrt{1-y^2}\,\frac{2}{3}\left(1-y^2\right) dy$$

at $y=0$ $\sin=0$

$y=1$ $\sin=\pi/2$

$$= \frac{4}{3}\int_0^1 \left(1-y^2\right)^{3/2} dy$$

let $y=\sin\theta$

then $dy=\cos\theta\, d\theta$

$$= \frac{4}{3}\int_0^{\pi/2} (\cos^2\theta)^{3/2} \cos\theta \, d\theta$$

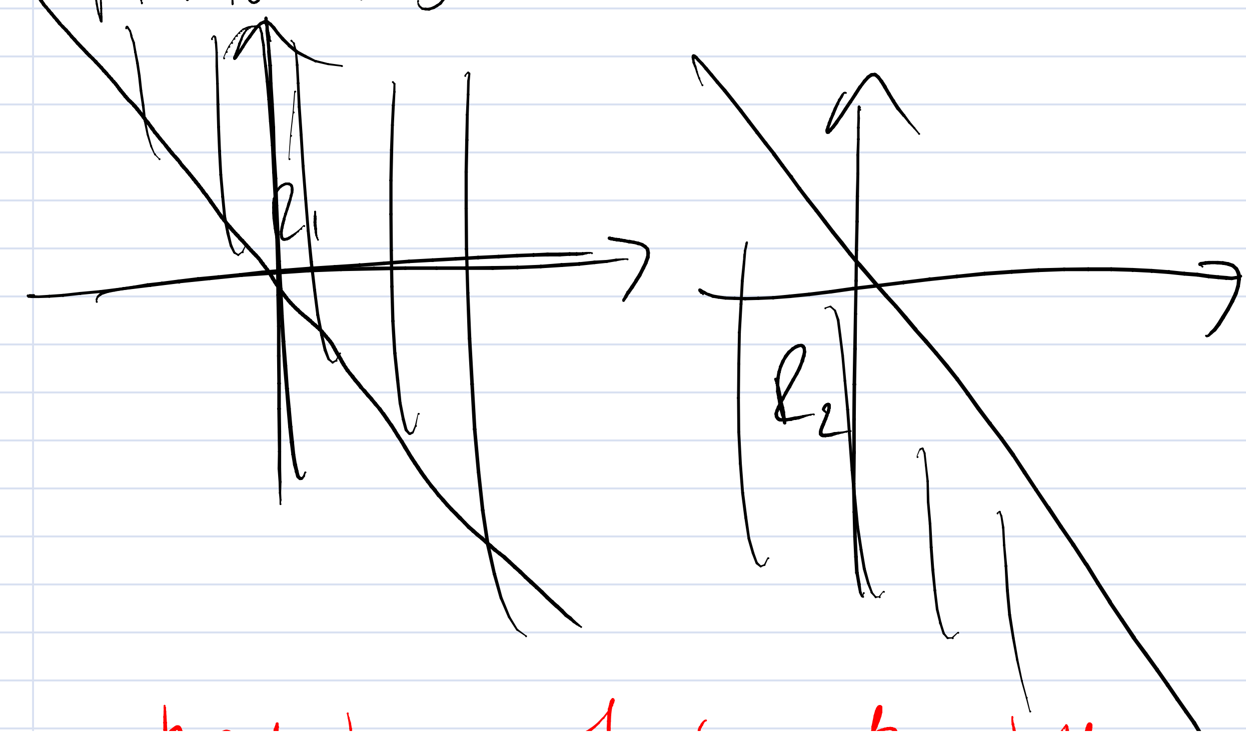
$$= \frac{4}{3}\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{\cancel{4}}{\cancel{3}}\,\frac{1}{24}\cdot\frac{\pi}{2} = \pi/4$$

here we cheat & check integral tables.

14.3 # 8 determine whether the following converges or diverges:

$$\iint_{\mathbb{R}^2} e^{-|x+y|}\,dA = \iint_{R_1} e^{-|x+y|}\,dA + \iint_{R_2} e^{-|x+y|}\,dA$$

Split into $x+y \geq 0$ and $x+y \leq 0$



notice: because of symmetry both integrals are the same.

$$\iint_{R_1} e^{-|x+y|}\, dA = \int_{-\infty}^{\infty} d \int_{x}^{\infty} e^{-(x+y)}\, dA$$

$= e^{-x} \cdot e^{-y}$

$$= \int_{-\infty}^{\infty} dx \; -e^{-(x+y)} \Big|_{x}^{\infty}$$

$$= \int_{-\infty}^{\infty} -e^{-x} \left(\lim_{y \to \infty} \left(e^{-y} - e^{-x} \right) \right) dx$$

$$= \int_{-\infty}^{\infty} -e^{-x} \left(0 - e^{-x} \right) dx$$

$$= -2 \int_{-\infty}^{\infty} e^{-x}\, d \;=\; -2 \left(\lim_{x \to \infty} e^{-x} \right) \to 0$$

$$-\left(-2 \lim_{x \to -\infty} e^{-x} \right) \nearrow \infty$$

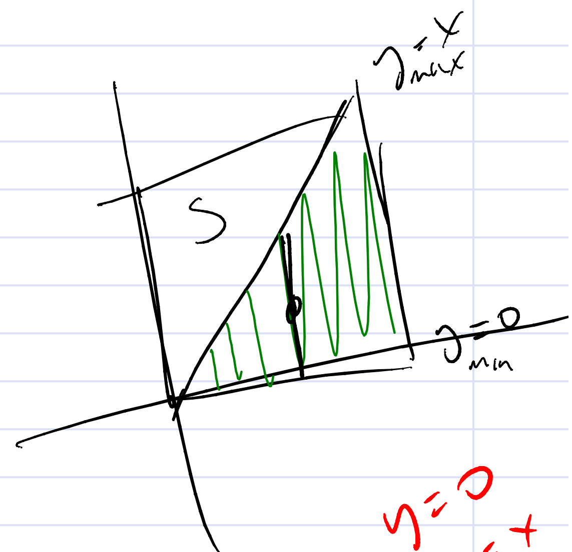
$:=$ diverges

#21 Evaluate both iterations of the improper integral: $\displaystyle\iint_S \frac{x-y}{(x+y)^3}\,dA$

where S is the square $0<x<1$, $0<y<1$. Show that the above improper integral does not exist by considering

$$\iint_T \frac{x-y}{(x+y)^3}\,dA$$

where T is in $\rightarrow$



$$\int_0^1 dx \int_0^x \frac{x-y}{(x+y)^3}\,dy$$

let $u = y+x$

$du = dy$

$$= \int_0^1 dx \int_x^{2x} \frac{x-(u-x)}{u^3}\,du$$

$y=0$
$u=x$
$y=x$
$u=2x$

$$= \int_0^1 dx \int_x^{2x} -\frac{1}{u^2}\, du = \int_0^1 \left. \frac{1}{u} \right|_x^{2x} dx$$

$$= \int_0^1 \frac{1}{2x} - \frac{1}{x}\, dx = \int_0^1 -\frac{1}{2x}\, dx = \ln\left|-2x\right| \Big|_0^1$$

$$= \ln(2) - \lim_{x \to 0} \ln(x) \nearrow \infty$$

diverges.

thus, Integral over S diverges.

$\boxed{14.4}$ #18

For what values of k does the following converge (& converges to what)?

$$\iint_{\mathbb{R}^2} \frac{dA}{(1+x^2+y^2)^k}$$

Convert to polar

$$= \int_0^{2\pi}\int_0^{\infty} \frac{1}{(1+r)^k}\, r\, dr\, d\theta$$

as $r \to \infty$ $\quad \lim \frac{r}{(1+r)^k} = \begin{cases} \infty & k<1 \\ 1 & k=1 \\ 0 & k>1 \end{cases}$

so the only time the integral can be finite is if $k>1$. In that case

$$\int_0^{2\pi}\int_0^{\infty} \frac{1}{(1+r)^k}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\infty} \frac{1}{(1+r)^{k-1}} - \frac{1}{(1+r)^k}\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2-k}(1+r)^{2-k} - \frac{1}{1-k}(1+r)^{1-k} \Big|_0^{\infty} \qquad (k-1>0)$$

$$= \int_0^{2\pi} \lim_{r\to\infty} \left(\frac{1}{2-k}(1+r)^{2-k} - \frac{1}{1-k}(1+r)^{1-k} \right) - \left(\frac{1}{2-k} - \frac{1}{1-k} \right) .$$

$\to 0$

Since $k>1$, but if $1<k<2$

this part goes

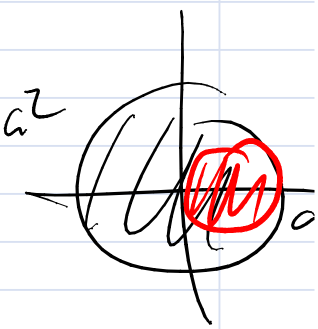
to ∞

thus we must have $k \geq 2$.

$$\begin{cases} \text{if } k = 2 & \text{answer} = 2\pi \cdot \left(\frac{1}{1-k}\right). \\ \text{if } k > 2 & \text{answer} = 2\pi \cdot \left(\frac{1}{1-k} - \frac{1}{2-k}\right) \end{cases}$$

$\boxed{\#22}$ find the volume lying in both the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y=ax$.

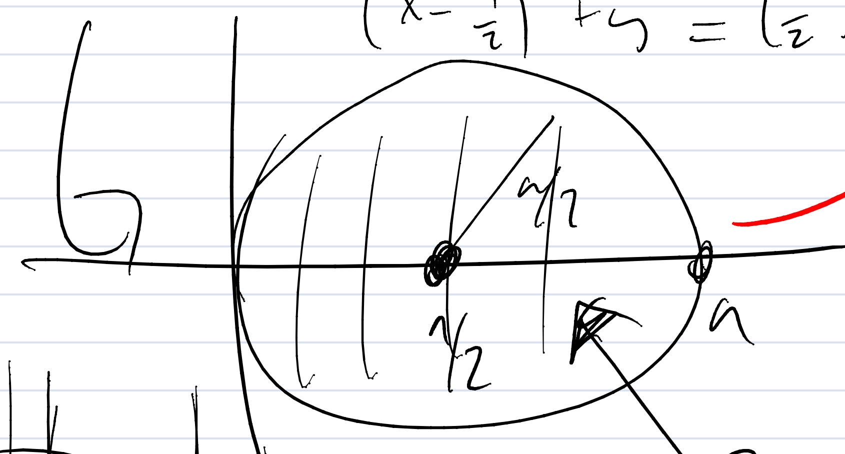
the sphere lies above $x^2+y^2 \leq a^2$



the cylinder: $x^2-ax+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+y=0$

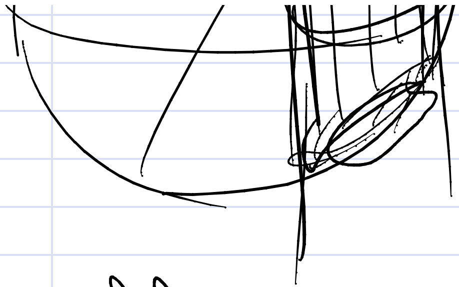
$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

lies above this



$z=+\sqrt{a^2-x^2-y^2}$

this is our region



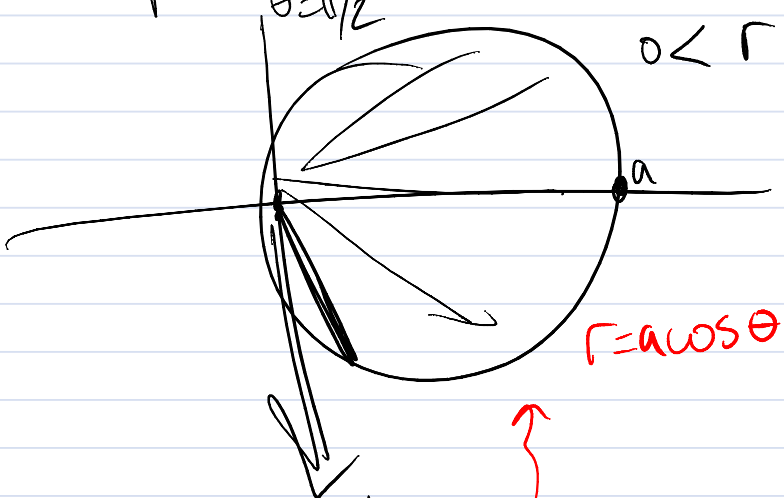
assume symmetry & look at

$$2\iint_R \sqrt{a^2-x^2-y^2}\;dA$$

want to switch to polar coordinates: how do we graph R in polar?

$\theta=\pi/2$

$0<r<a\cos\theta$



$r=a\cos\theta$

$\theta=-\pi/2$

Skim section 8.5 ...

$$=2\int_{-\pi/2}^{\pi/2} d\theta \int_{0}^{\cos\theta} \sqrt{a^2-r^2}\; r\,dr$$

let $u=a^2-r^2$

$du=-2r\,dr$

$$= 2\int_{-\pi/2}^{\pi/2} d\theta \int -\tfrac{1}{2} u^{1/2}\, du = 2\int_{-\pi/2}^{\pi/2} \left(-\tfrac{1}{3} u^{3/2}\right)\Big|\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \left(-\tfrac{1}{3}\left(a^2-r^2\right)^{3/2}\right)\Bigg|_0^{a\cos\theta} d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} -\tfrac{1}{3}\left(a^2-a^2\cos^2\theta\right)^{3/2} - \left(-\tfrac{1}{3}\left(a^2\right)^{3/2}\right) d\theta$$

$$= \frac{2}{3}\int_{-\pi/2}^{\pi/2} a^3 - \left(a^2-a^2\cos^2\theta\right)^{3/2} d\theta$$

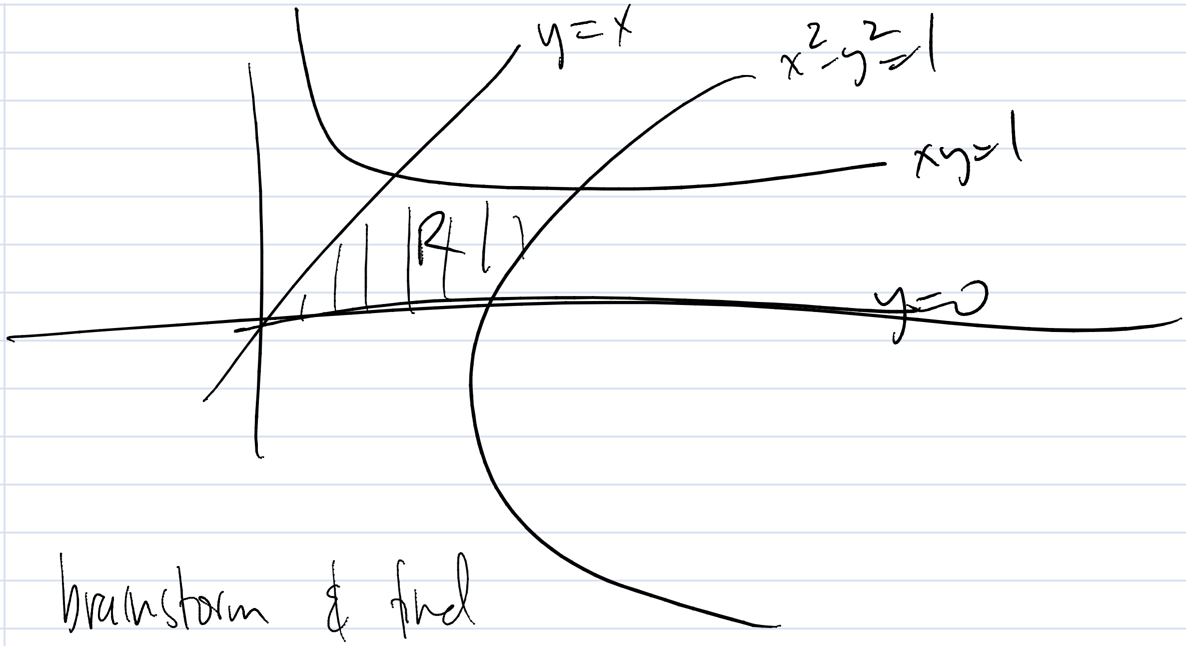
$$= \frac{2}{3}\int_{-\pi/2}^{\pi/2} a^3 - a^3\sin^3\theta\, d\theta = \frac{2a^3}{3}\cdot\pi - 0 = \boxed{\frac{2a^3}{3}\pi}$$

$\sin^3\theta$ is odd so $\int_0^{\pi/2} = -\int_{-\pi/2}^{0}$

#34 Evaluate $\iint_R (x^2+y^2)\,dA$ where

R is the region in the first quadrant bounded by

$y=0$, $y=x$, $xy=1$, and $x^2-y^2=1$.



brainstorm & find

the right coordinate change to get

at $u=0$ have $y=x$ or $x^2-y^2=0$

$u=1$ have $x^2-y^2=1$

let $u=x^2-y^2$

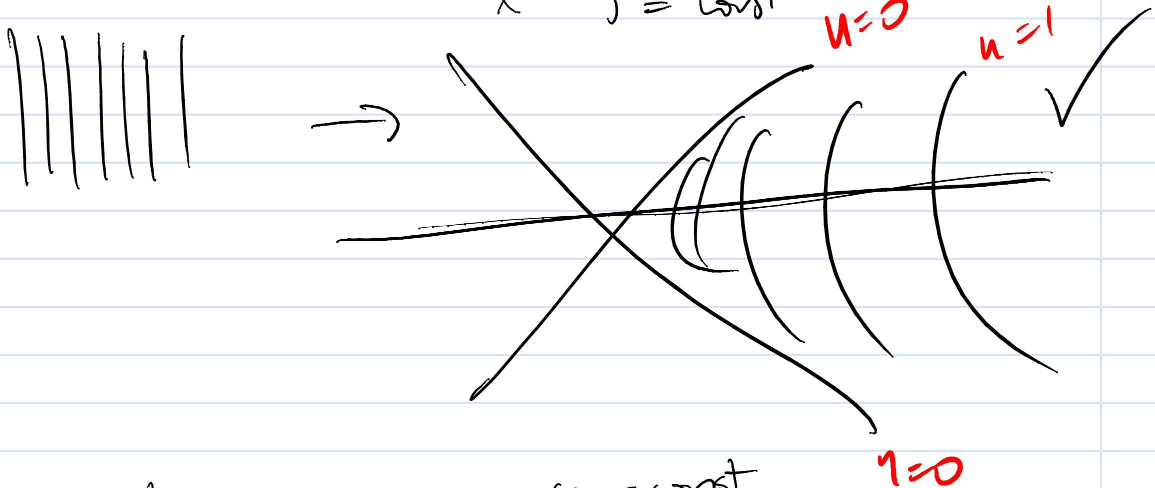
at $v=0$ $y=0$

$v=1$ $xy=1$

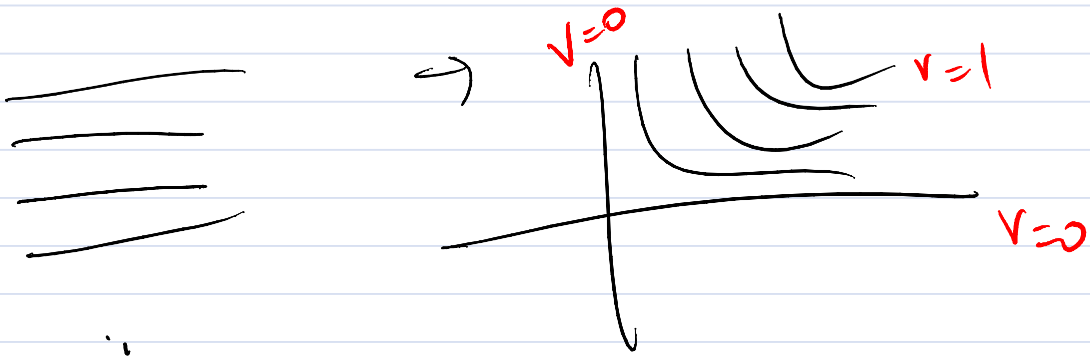
let $v=xy$

what is the map? (ie is it one to one.

$u = \text{const}$ $\qquad\qquad$ $x^2 - y^2 = \text{const}$



$v = \text{const}$ $\qquad\qquad$ $xy = \text{const}$



(note: this is where a graphical program like mathematica might come in handy)

so Jacobian is $\begin{bmatrix} 2x & y \\ -2y & x \end{bmatrix}$

with

determinant $= 2x^2 + 2y^2$

since $dx\,dy = \dfrac{1}{\left|\frac{\partial u}{\partial x}\right|}\,du\,dv$

$$\iint_R x^2+y^2 \, dx\,dy = \int_0^1\int_0^1 x^2+y^2 \frac{1}{2x^2+2y^2} \, du\,dv$$

$$= \frac{1}{2}\int_0^1\int_0^1 du\,dv = 1/2 \quad \blacksquare$$