McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 265

Advanced Calculus

Examiner: Professor A. Humphries
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Date: Tuesday, April 15, 2003
Time: 9:00 A.M. - 12:00 P.M.

INSTRUCTIONS

Attempt all questions.
Calculators are not permitted.
Questions are not necessarily of equal value.

This exam comprises the cover and 1 page of 6 questions.

1. Is it possible to solve the system

$$xy^2 + xu + yv^2 = 3$$

$$u^3y + 2xv - u^2v^2 = 2,$$

for (u, v) as functions of (x, y) near $(x, y, u, v) = (1, 1, 1, 1)$? If so compute the determinant of the Jacobian $\partial(u, v)/\partial(x, y)$ at this point.

2. Assume that a and b are fixed positive numbers. Find the extreme values of $x + y$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. (a) Determine whether the vector field

$$\mathbf{F} = \sin yz\mathbf{i} + (xz\cos yz + \exp z)\mathbf{j} + (xy\cos yz + y\exp z)\mathbf{k}$$

is conservative. If $\mathbf{F}$ is conservative find a scalar field φ such that $\mathbf{F} = \nabla\varphi$. Evaluate $\int_\gamma \mathbf{F}\cdot d\mathbf{r}$ where γ is the curve starting from $(0, 0, 0)$ and ending at $(1, 1, 1)$ satisfying $x^4 = y^2 = z$.

(b) Do the same in the case where

$$\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k}.$$

4. (a) The coordinates $(\bar{x}, \bar{y})$ of the centroid of a plane region D are given by

$$\bar{x} = \frac{1}{A}\iint_D x\,dxdy, \qquad \bar{y} = \frac{1}{A}\iint_D y\,dxdy,$$

where A is the area of D. Use Green's Theorem to obtain the equivalent formulae

$$\bar{x} = \frac{1}{2A}\int_{\partial D} x^2\,dy, \qquad \bar{y} = -\frac{1}{2A}\int_{\partial D} y^2\,dx.$$

(b) Use the above to find the centroid of the semicircular region D whose boundary comprises the semi-circular arc given by

$$x = a\cos t, \quad y = a\sin t, \qquad t \in [-\pi/2, \pi/2],$$

and the piece of the y-axis between $-a$ and a.

5. Show that

$$\nabla\cdot(\varphi\mathbf{F}) = \varphi\nabla\cdot\mathbf{F} + \nabla\varphi\cdot\mathbf{F}.$$

Show that if $\varphi = f(|\mathbf{r}|)$ then $\nabla\varphi = f'(|\mathbf{r}|)\hat{\mathbf{r}}$ (where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$).

Find $\nabla\cdot(|\mathbf{r}|^k\mathbf{r})$ where k is a constant.

6. (a) Evaluate $\iint_\Sigma \nabla\times\mathbf{F}\cdot\mathbf{n}dS$ where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$, and Σ is that part of the sphere $x^2 + y^2 + z^2 = 1$ such that $z \leqslant 1/\sqrt{2}$ and $\mathbf{n}$ is the outward pointing normal.

(b) Find the flux of the vector field $\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k}$ out of the cube $0 \leqslant x, y, z \leqslant 1$.

(c) Find the flux of the vector field $\mathbf{F} = \nabla\left(\frac{1}{\sqrt{x^2+(y-1)^2+z^2}}\right)$ out of the sphere centered at $(0, 0, 0)$ with radius 2.

[Note that you may assume Gauss' Law provided you verify the hypotheses which allow it to be derived from the Divergence Theorem.]