

McGILL UNIVERSITY

FACULTY OF ENGINEERING

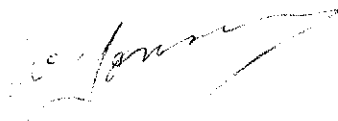
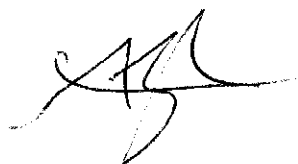
FINAL EXAMINATION

MATH 265

Advanced Calculus

Examiner: Professor A. Humphries  
Associate Examiner: Professor W. Jonsson

Date: Tuesday, April 15, 2003  
Time: 9:00 A.M. - 12:00 P.M.



INSTRUCTIONS

**Attempt all questions.  
Calculators are not permitted.  
Questions are not necessarily of equal value.**

This exam comprises the cover and 1 page of 6 questions.

1. Is it possible to solve the system

$$\begin{aligned}xy^2 + xu + yv^2 &= 3 \\ u^3y + 2xv - u^2v^2 &= 2,\end{aligned}$$

for  $(u, v)$  as functions of  $(x, y)$  near  $(x, y, u, v) = (1, 1, 1, 1)$ ? If so compute the determinant of the Jacobian  $\partial(u, v)/\partial(x, y)$  at this point.

2. Assume that  $a$  and  $b$  are fixed positive numbers. Find the extreme values of  $x + y$  subject to the constraint  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .
3. (a) Determine whether the vector field

$$\mathbf{F} = \sin yz\mathbf{i} + (xz \cos yz + \exp z)\mathbf{j} + (xy \cos yz + y \exp z)\mathbf{k}$$

is conservative. If  $\mathbf{F}$  is conservative find a scalar field  $\varphi$  such that  $\mathbf{F} = \nabla\varphi$ . Evaluate  $\int_{\gamma} \mathbf{F} \cdot d\mathbf{r}$  where  $\gamma$  is the curve starting from  $(0, 0, 0)$  and ending at  $(1, 1, 1)$  satisfying  $x^4 = y^2 = z$ .

- (b) Do the same in the case where

$$\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k}.$$

4. (a) The coordinates  $(\bar{x}, \bar{y})$  of the centroid of a plane region  $D$  are given by

$$\bar{x} = \frac{1}{A} \iint_D x \, dx dy, \quad \bar{y} = \frac{1}{A} \iint_D y \, dx dy,$$

where  $A$  is the area of  $D$ . Use Green's Theorem to obtain the equivalent formulas

$$\bar{x} = \frac{1}{2A} \int_{\partial D} x^2 \, dy, \quad \bar{y} = -\frac{1}{2A} \int_{\partial D} y^2 \, dx.$$

- (b) Use the above to find the centroid of the semicircular region  $D$  whose boundary comprises the semi-circular arc given by

$$x = a \cos t, \quad y = a \sin t, \quad t \in [-\pi/2, \pi/2],$$

and the piece of the  $y$ -axis between  $-a$  and  $a$ .

5. Show that

$$\nabla \cdot (\varphi \mathbf{F}) = \varphi \nabla \cdot \mathbf{F} + \nabla \varphi \cdot \mathbf{F}.$$

Show that if  $\varphi = f(|\mathbf{r}|)$  then  $\nabla \varphi = f'(|\mathbf{r}|)\hat{\mathbf{r}}$  (where  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , and  $\hat{\mathbf{r}}$  is the unit vector in the direction of  $\mathbf{r}$ ).

Find  $\nabla \cdot (|\mathbf{r}|^k \mathbf{r})$  where  $k$  is a constant.

6. (a) Evaluate  $\iint_{\Sigma} \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS$  where  $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + xyz\mathbf{k}$ , and  $\Sigma$  is that part of the sphere  $x^2 + y^2 + z^2 = 1$  such that  $z \leq 1/\sqrt{2}$  and  $\mathbf{n}$  is the outward pointing normal.
- (b) Find the flux of the vector field  $\mathbf{F} = xyz\mathbf{i} + y^2\mathbf{j} + x^2z\mathbf{k}$  out of the cube  $0 \leq x, y, z \leq 1$ .
- (c) Find the flux of the vector field  $\mathbf{F} = \nabla \left( \frac{1}{\sqrt{x^2 + (y-1)^2 + z^2}} \right)$  out of the sphere centered at  $(0, 0, 0)$  with radius 2.

[Note that you may assume Gauss' Law provided you verify the hypotheses which allow it to be derived from the Divergence Theorem.]