McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 265

ADVANCED CALCULUS

Examiner: Professor A.R. Humphries — Date: December 8, 2003
Associate Examiner: Professor W. Jonsson — Time: 9:00 A.M. - 12:00 P.M. hours

INSTRUCTIONS

Calculators are not permitted.
Answer in exam booklets.
This is a closed book exam.
Dictionaries are permitted.

This exam comprises the cover and 2 pages of 7 questions.

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1. The following equations define y and v implicitly as functions of x and u near $(x, y, u, v) = (1, -1, -1, 1)$

$$\begin{aligned} xy + yu^2 + u^2v + x^2v &= 0 \\ x + y^2 + u + v^2 &= 2 \end{aligned}$$

At this choice of values of the variables, compute $\frac{\partial y}{\partial x}$ and $\frac{\partial v}{\partial u}$.

2. With the aid of differentiation under the integral sign show that the if $x(t)$ satisfies the following integral equation,

$$x(t) + \int_0^t \sin(t-\tau)x(\tau)d\tau = \cos 2t$$

Then it also satisfies the ordinary differential equation

$$x'' + x = -3\cos 2t \text{ with initial conditions } x(0) = 1 \quad x'(0) = 0$$

3. With the aid of Lagrange Multipliers, find the shortest distance from the origin to a point on the curve of intersection of the surfaces

$$xy + z^2 = 0 \text{ and } x^2 + y^2 = 1$$

4. Let C be the circle in the xy-plane of radius $a = 2$ centered at $(x, y) = (-1, 3)$

   (a) Compute the line integral

$$\oint_C \frac{-(y-3)dx}{(x+1)^2 + (y-3)^2} + \frac{(x+1)dy}{(x+1)^2 + (y-3)^2}$$

   where C is traversed in the positive direction.

   (b) Evaluate the following line integrals (Green's Theorem may be useful).

   i.

$$\oint_\Gamma \frac{-(y-3)dx}{(x+1)^2 + (y-3)^2} + \frac{(x+1)dy}{(x+1)^2 + (y-3)^2}$$

   Where Γ is the boundary of the square $|x| \le 2, \quad |y| \le 4$

   ii.

$$\oint_\Gamma \frac{-(y-3)dx}{(x+1)^2 + (y-3)^2} + \frac{(x+1)dy}{(x+1)^2 + (y-3)^2}$$

   Where Γ is the circumference of the circle of radius $a = 1$ centered at the origin.

5. Consider the region $\mathcal{D}$ in the $u-v$ plane bounded by the circle $u^2+v^2=1$ where the positive direction around this circle defines the positive direction around the boundary of the surface S in $\mathcal{R}^3$ defined parametrically by

$$\mathbf{r}(u,v) = (u+v)\mathbf{i} + (u^2+v^2)\mathbf{j} + uv\mathbf{k} \text{ for } (u,v) \in \mathcal{D}$$

For the vector field $\mathbf{F} = (yz+x)\mathbf{i} + (xz+y)\mathbf{j} + (xy+z)\mathbf{k}$, evaluate $\int_{\partial S} \mathbf{F}\cdot\mathbf{dr}$ where ∂S, the boundary of S, is traversed in the positive direction.

6. Let $\rho^2 = x^2+y^2+z^2$

   (a) Calculate $\nabla\frac{1}{\rho}$ for $\rho \neq 0$.

   (b) Show that, if $\rho \neq 0$,then $\nabla\cdot\nabla\frac{1}{\rho} = 0$.

   (c) For the vector field $\mathbf{F} = -\nabla\frac{1}{\rho}$ and the surface S of the sphere of radius $a>0$, $x^2+y^2+z^2=a^2$, show that

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = 4\pi$$

.

7. Let $\mathbf{F} = (y^2+2xz, z^2+2xy, x^2+2yz)$.

   (a) Compute $\nabla\times\mathbf{F}$.

   (b) Is $\mathbf{F}$ conservative? If yes, then find a corresponding potential.

   (c) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve $\mathbf{r} = t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ for $0\leq t\leq 1$.