MATHEMATICS 264 Fall 2006

SOLUTIONS

(1) With the aid of a change of variables, compute the value of the integral

$$\iint_{(D)} \left(\frac{1}{y^2}\right) \mathrm{d}A$$

region (D) in the first quadrant of the x, y plane bounded by the curves

$$x^2 + y^2 = 1, x^2 + y^2 = 4, x = 2y, x = 4y$$

SOLUTIONS: Let $x = r\cos\theta, y = r\sin\theta$. Then,

$$(D): r = 1 \to 2; \theta = \theta_1 \to \theta_2,$$

where $\theta_2 = \arctan(1/2), \theta_1 = \arctan(1/4)$.

$$\iint_{(D)} \left(\frac{1}{y^2}\right) \mathrm{d}A = \iint_{(D)} \left(\frac{1}{r^2\sin^2\theta}\right) r\mathrm{d}r\mathrm{d}\theta = \ln r\Big|_1^2 (-\cot\theta)\Big|_{\theta_1}^{\theta_2}$$
$$= -\ln 2(\tan\theta)^{-1}\Big|_{\theta_1}^{\theta_2} = \ln 4.$$

(2) Find the area of that portion of the surface $z^2 = x^2 + y^2$ above the first quadrant of the (x, y)-plane for which $x \le 1$ and $y \le 2$.

SOLUTIONS: The projection of (S) on xy plane is $(A): (0 \le x \le 1; 0 \le y \le 2)$.

$$Area = \iint_{(S)} \mathrm{d}S = \iint_{(A)} \sqrt{1 + z_x^2 + z_y^2}\mathrm{d}x\mathrm{d}y = \iint_{(A)} \sqrt{2}\, \mathrm{d}x\mathrm{d}y = 2\sqrt{2}.$$

(3) Compute the line integral

$$\oint_C 2y\mathrm{d}y - y\mathrm{d}x$$

where C is the boundary of the half disc $x^2 + y^2 \le 1$ with $y \ge 0$ traversed in the positive sense

(a) by parameterizing the boundary curve (there are two pieces, a straight line segment and a semi-circle) and then evaluating the integral directly.

(b) by applying Green's Theorem, then evaluating the resulting double integral.

SOLUTIONS:

- Let $(C) = (C)_1 \bigcup (C)_2$, where $(C)_1$: $-1 \le x \le 1; y = 0$; $(C)_2$:

$$x = \cos\theta; y = \sin\theta; \quad (0 \le \theta \le \pi).$$

Thus,

$$\int_{(C)_1} = 0.$$

On the other hand, along $(C)_2$, we have

$$\mathrm{d}\vec{r} = (-\sin\theta\vec{i} + \cos\theta\vec{j})\mathrm{d}\theta,$$

and

$$\vec{F} = F_1\vec{i} + F_2\vec{j}, \qquad F_1 = -y, F_2 = 2y.$$

Hence

$$\int_{(C_2} \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{\pi} (2\sin\theta\cos\theta + \sin^2\theta)\mathrm{d}\theta = \frac{\pi}{2}.$$

- On the other hand, by Green's theorem, we have

$$\iint_{(A)} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}x\mathrm{d}y = \iint_{(A)} (1)\mathrm{d}x\mathrm{d}y = \frac{\pi}{2}.$$

(4) (a) State Gauss' theorem.

(b) Let (V) be the region inside the paraboloid $z = 4 - x^2 - y^2$, in the first octant (i.e. x, y, and z are all non-negative) and (S) its boundary. Furthermore, let $\mathbf{F}$ be the vector valued function $\mathbf{F} = yz\mathbf{i} + xy\mathbf{j} + \mathbf{k}$.

Verify Gauss' theorem for this function and the region (V) by

(i) evaluating the appropriate surface integral.

(ii) by evaluating the appropriate volume integral,

and showing that both have the same value.

SOLUTIONS:

- Gauss's Theorem:

$$\iiint_{(V)} \mathrm{div}\vec{F}\mathrm{d}V = \iint_{(S)} \vec{F} \cdot \vec{n}\mathrm{d}S$$

where (S) is closed simple connected surface; (V) is the enclosed volume; $\vec{F}$ is differentiable vector field; $\vec{n}$ is outward unit normal vector of the surface.

- Let $(S) = (S)_1 \bigcup (S)_2 \bigcup (S)_2 \bigcup (S)_4$. We have that

On $(S)_1$ on yoz plane: $\vec{n} = (-1, 0, 0), x = 0$,

$$\int_{(S)_1} \vec{F} \cdot \vec{n}\mathrm{d}y\mathrm{d}z = -\int_0^2 \mathrm{d}y \int_0^{4-y^2} yz\mathrm{d}z = -\frac{64}{12}.$$

On $(S)_2$ on xoz plane: $\vec{n} = (0, -1, 0), y = 0$,

$$\int_{(S)_2} \vec{F} \cdot \vec{n}\mathrm{d}x\mathrm{d}z = \int_{(S)_2} xy\mathrm{d}x\mathrm{d}z = 0.$$

On $(S)_3$ on xoy plane: $\vec{n} = (0, 0, -1), z = 0$,

$$\int_{(S)_3} \vec{F} \cdot \vec{n}\mathrm{d}x\mathrm{d}z = \int_{(S)_3} (-1)\mathrm{d}x\mathrm{d}y = -\pi.$$

The side $(S)_4$ on the surface $z = 4 - x^2 - y^2$: $\vec{r} = x\vec{i} + y\vec{j} + (4 - x^2 - y^2)\vec{k}$,

$$\vec{r}_x \times \vec{r}_y = (2x, 2y, 1),$$

So that,

$$\int_{(S)_4} \vec{F} \cdot \vec{n}\mathrm{d}S = \int_{(S)_3} (yz, xy, 1) \cdot (2x, 2y, 1)\mathrm{d}x\mathrm{d}y$$
$$= \int_0^2 \mathrm{d}y \int_0^{\sqrt{4-y^2}} (8xy - 2x^3y - 2xy^3 + 2xy^2 + 1)\mathrm{d}x = \frac{64}{12} + \frac{64}{15} + \pi.$$

Therefore is is derived that

$$\iint_{(S)} \vec{F} \cdot \vec{n}\mathrm{d}S = \frac{64}{15}.$$

- Note that

$$\text{div}\vec{F} = x.$$

With cylindrical system, we have

$$\iiint_{(V)} \text{div}\vec{F}\text{d}V = \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r\cos\theta\text{d}z = \frac{64}{15}.$$

(5) Consider the domain in the (u, v) plane bounded by the circle $u^2 + v^2 = 1$ and the surface (S) in $\mathbb{R}^3$ defined perimetrically by

$$\mathbf{r}(u, v) = (u^2 + v^2)\mathbf{i} + uv\mathbf{j} + (u + v)\mathbf{k}$$

where the positive sense around the boundary is determined by the positive sense around the boundary of the region in the (u, v) plane.

Furthermore, set $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.

(a) Compute

$$\iint_{(S)} \nabla\times\mathbf{F}\cdot\,\mathbf{n}\text{d}S,$$

where (S) is the surface described above.

(b) To verify Stokes' Theorem, now compute

$$\oint_C \mathbf{F}\cdot\text{d}\mathbf{r}.$$

SOLUTIONS:

- One derives

$$\nabla\times\vec{F} = (0, 0, 0).$$

Hence,

$$\iint_{(S)} \nabla\times\vec{F}\cdot n\text{d}S = 0.$$

- On the other hand, The boundary (γ) of the surface (S) mapping to the (u, v) plane can be expressed in the form:

$$u = \cos\theta, \quad v = \sin\theta.$$

On the closed boundary

$$x = u^2 + v^2 = 1, \quad y = uv = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta, \quad z = u + v = \cos\theta + \sin\theta.$$

Hence,along the boundary,

$$\begin{aligned}\text{d}\vec{r} &= \text{d}x\vec{i} + \text{d}y\vec{j} + \text{d}z\vec{k}\\ &= \Big[\cos 2\theta\vec{j} + (-\sin\theta + \cos\theta)\vec{k}\Big]\text{d}\theta\end{aligned}$$

and

$$\begin{aligned}\vec{F} &= \frac{1}{2}\sin 2\theta(\cos\theta + \sin\theta)\vec{i} + \frac{1}{2}\sin 2\theta\vec{j} + \vec{k}\\ \vec{F}\cdot\text{d}\vec{r} &= (\frac{1}{4}\sin 4\theta + \cos\theta - \sin\theta)\text{d}\theta\end{aligned}$$

We get

$$\oint \vec{F}\cdot\text{d}\vec{r} = \int_0^{2\pi}(\frac{1}{4}\sin 4\theta + \cos\theta - \sin\theta)\text{d}\theta = 0.$$

The Stokes Theorem is verified.

(6) Solve the following problems by the method of separation of variables:

(a)

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u, \quad u(x,0) = 3\mathrm{e}^{-5x} + 2\mathrm{e}^{-3x}.$$

(b)

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} - 2u, \quad u(0,t) = 0, u(3,t) = 0, \; u(x,0) = 2\sin \pi x - \sin 4\pi x.$$

(7) Use Fourier series to solve the following heat conduction equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u_x(0,t) = 0, u_\pi(\pi,t) = 0, \; u(x,0) = 25x.$$