1. Consider the function $f(x, y) = 1 + e^x \sin y + e^{-y} \cos x$

   (a) Using differentials (i.e. the Taylor expansion of $(f(0+h, 0+k)$ up to degree one) estimate $f(0.01, -0.02)$.

   (b) Use the Lagrange form of the remainder (second degree terms) to estimate the error in using differentials to approximate $f(0.01, -0.02)$ as in the first part of this question.

2. (a) With the aid of differentiation under the integral sign (or otherwise) evaluate

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} dx \text{ for } a > 0$$

   You may assume that $\int_0^\infty e^{-ax} \sin x dx = 1/(a^2 + 1)$.

   (b) Now show that $\int_0^\infty \frac{\sin x}{x} dx = \pi/2$.

3. The following equations define x and y implicitly as functions of u and v near $x = 1, y = -1, u = 0, v = 1$

$$\begin{aligned} x^2u + y^2v^2 + uv &= 1 \\ v^2y - xy + v^2 &= 1 \end{aligned}$$

   At this choice of values of the variables compute

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

4. Using Lagrange Multipliers, find the extrema of

$$G(x, y) = x^2 + 4y^2 + 16z^2 \text{ subject to } xy = 1$$

   Now, if possible, classify them.

5. With the aid of a change of variables, find the area of the region in the x, y plane bounded by the parabolas

$$y = 2x^2, y = 4x^2, x = 3y^2, x = 5y^2$$

6. Let C be the circle of radius $\epsilon > 0$ centered at the origin (i.e. $x^2 + y^2 = \epsilon^2$) and Γ any simple closed curve which is piecewise smooth and encloses a region D which is connected and simply connected.

   (a) compute the line integral $\oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ where C is traversed in the positive direction.

   (b) Hence or otherwise show that

      i. if the origin is in the interior of D, then $\oint_\Gamma \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy = \oint_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$.

      ii. if the origin lies outside the region D (and the origin is not on Γ either), then $\oint_\Gamma \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy = 0$

7. (a) State Gauss' theorem.

   (b) Let V be the solid tetrahedron in the first octant bounded by the co-ordinate planes and by the plane $3x + 4y + z = 12$ and let S be its boundary. Let $\mathbf{F}$ be the vector valued function $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3y\mathbf{j} + 6z\mathbf{k}$. Verify Gauss' theorem for this function $\mathbf{F}$ and region V by

      i. evaluating the appropriate surface integral,

      ii. evaluating the appropriate volume integral.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-314A

ADVANCED CALCULUS

Examiner: Professor W. Jonsson
Associate Examiner: Professor N.G.F. Sancho

Date: Friday, December 17, 1999
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators are not permitted.
Questions are not necessarily of equal weight.

This exam comprises the cover and one page of questions.