Exercises from the quizes

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1. Compute the limits

   (a)
   $$\lim_{x\to-\infty} \frac{x^3-4x^2+1}{(\sqrt{2}x^4-17x^3+1)^{3/4}}.$$

   (b)
   $$\lim_{x\to\infty} \frac{x^3-4x^2+1}{(\pi x^4-17x^3+1)^{3/4}}.$$

2. The function $f(x):\mathbb{R}\to\mathbb{R}$ is defined by the following formula

$$f(x)=\begin{cases} \frac{\cos(\pi x)}{x}, & x>0 \\ 3, & x=0 \\ x\sin\left(\pi\sqrt{x^2+3}\right), & -1\le x<0 \\ \sqrt{-(x+1)}, & x<-1 \end{cases}$$

   (a) Compute the limit $\lim\limits_{x\to a} f(x)$ for every real $a\in\mathbb{R}$ (including infinite limits). For every point $a\in\mathbb{R}$ that the limit $\lim\limits_{x\to a} f(x)$ doesn't exist, compute the one-sided limits $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to a-} f(x)$. Explain.

   (b) Compute the limits $\lim\limits_{x\to\pm\infty} f(x)$. Explain.

3. For every value of the parameter $p\in\mathbb{R}$, the function $f_p(x):\mathbb{R}\to\mathbb{R}$ is defined by

$$f_p(x)=\begin{cases} (x-p)^2, & x\ge 2p \\ x^3, & x<2p \end{cases}$$

(a) Find all the values of the parameter p, such that the function f_p is <u>continuous</u> on $\mathbb{R}$. Explain.

(b) For every value of p you found in (a), determine whether the function f is differentiable on $\mathbb{R}$. Explain.

4. What are the (global and local) extrema of

$$f(x) = \cosh x?$$

Use the definition of the hyperbolic cosine. Deduce that $\cosh x \geq 1$ on $\mathbb{R}$.

5. (a) Write the linear approximation $L(x)$ of $f(x) = e^x$ about $x = 0$ and the error term $E(x) = f(x) - L(x)$ in Lagrange form.

   (b) Using the Lagrange form, bound $|E(x)|$ from above for any $x > 0$. Prove the inequality

$$e^x - (1 + x) > \frac{x^2}{2}$$

   for $x > 0$. (Note that this is stronger than the inequlity in the midterm).

   (c) Write the second order Taylor polynomial $P_2(x)$ of $f(x)$ about $x = 0$ and the error term $E_2(x) = f(x) - P_2(x)$ in the Lagrange form. Approximate the Euler constant e by $P_2(1)$. Use the Lagrange form to determine if this approximation above or below (in other words, is $P_2(1) \leq e$ or $P_2(1) \geq e$). Bound $|E_2(x)|$ to estimate how many correct decimal digits it gives you.

6. Prove the following inequality:

$$\sqrt{1 - x^2} + x \arcsin x \geq 1$$

on $[-1, 1]$.

7. Let $f : I \to \mathbb{R}$ be a real valued function defined on the interval $I = (-1, 1)$ by

$$f(x) = \frac{1}{1 - x}.$$

(a) Write the linear approximation $L(x)$ of $f(x)$ about $x = 0$ and the error term $E(x) = f(x) - L(x)$ in Lagrange form. Using the Lagrange form, bound $|E(x)|$ from above for any $|x| < 1/2$.

(b) Using the Lagrange form of the remainder, check if the approximation $L(x)$ is above or below. Validate your answer by computing explicitly $E(x) = f(x) - L(x)$ in its simplest form.

(c) Using the explicit computation in (b), for any $x \in I$ find a number s between 0 and x, such that

$$\frac{f''(s)}{2!}x^2 = E(x),$$

guaranteed by the theorem on the remainder in the Lagrange form.

8. (a) What are the (global and local) extrema of the function

$$f(x) = \sin x + \cos x?$$

Note that f is periodic.

(b) Deduce from (a) that for every $x \in \mathbb{R}$ we have

$$|\sin x| + |\cos x| \leq \sqrt{2},$$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function defined on the whole real line by

$$f(x) = \sin(x) - \cos x.$$

(a) Write the linear approximation $L(x)$ to f about $x = \pi/4$. Write the expression of the error term $E(x) = f(x) - L(x)$ in the Lagrange form. Using the inequality in (8b), give an upper bound to $|E(x)|$ in the interval $(\pi/4 - 0.01, \pi/4 + 0.01)$.

(b) Write the second order Taylor polynomial $P_2(x)$ of $f(x)$ about $x = \pi/4$ and the error term $E_2(x) = f(x) - P_2(x)$ in the Lagrange form. Using the inequality in (1b), bound $|E_2(x)|$ from above in the interval $(\pi/4 - 0.01, \pi/4 + 0.01)$.

(c) In light of (b), can you improve your bound in (a)?

10. Compute the following limits or show they don't exist:

   (a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4 \sin(x^2-y^2)}{|x|^3+|y|^3}$.

   (b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4 \sin(x^2+y^2)}{x^3+y^3}$.
   Note: there is no absolute value in the denominator.

   (c) $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^3+2xy+y^3)}{x^3+2xy+y^3}$.

   (d) $\lim\limits_{(x,\,y)\to(0,0)} \frac{x^3+y^2x}{|x|^3+|y|^3}$.

   (e) $\lim\limits_{(x,\,y)\to(1,0)} \frac{\sin^2(xy)}{\cos(xy)-1}$

   (f) $\lim\limits_{(x,\,y)\to(0,\,0)} f(x,y)$, where

$$f(x,\,y) = \begin{cases} \sqrt{x^2+y^2-1}, & x^2+y^2 \geq 1 \\ 0 & (x,\,y) = (0,\,0) \end{cases}.$$

11. Let $f(x,y) : D \to \mathbb{R}$ be the 2-variable function defined by

$$f(x,y) = 2\cos(3x+4y) - \sin(3x+4y).$$

   (a) For a point $P = (x_0, y_0)$, compute the partial derivatives $\frac{\partial z}{\partial x}|_P$ and $\frac{\partial z}{\partial y}|_P$. Write the equation of the tangent plane π of the surface defined by f at $Q = (\pi/4, \pi/4, -\frac{3\sqrt{2}}{2})$. Compute the parametric form of the normal line to π at Q.

   (b) Compute the partial derivatives of the second order $\frac{\partial^2 z}{\partial x^2}|_P$, $\frac{\partial^2 z}{\partial y^2}|_P$, $\frac{\partial^2 z}{\partial x \partial y}|_P$ and $\frac{\partial^2 z}{\partial y \partial x}|_P$ (Hint: you can compute just 3 derivatives and get the remaining one "for free" using the corresponding theorem).

   (c) Using the results you got in (b), check that the function f satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 25f = 0.$$

12. Let $f(x,y) : D \to \mathbb{R}$ be the 2-variable function defined by

$$f(x,y) = \cosh(x^2+y^2) + \sinh(x^2+y^2).$$

(a) For a point $P = (x_0, y_0)$, compute the partial derivatives $\frac{\partial z}{\partial x}|_P$ and $\frac{\partial z}{\partial y}|_P$. Write the equation of the tangent plane π of the surface defined by f at $Q = (0, 1, e)$. Compute the parametric form of the normal line to π at Q.

(b) Compute the partial derivatives of the second order $\frac{\partial^2 z}{\partial x^2}|_P$, $\frac{\partial^2 z}{\partial y^2}|_P$, $\frac{\partial^2 z}{\partial x \partial y}|_P$ and $\frac{\partial^2 z}{\partial y \partial x}|_P$ (Hint: you can compute just 3 derivatives and get the remaining one "for free" using the corresponding theorem).

(c) Using the results you got in (b), check that the function f satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial x \partial y} - 4xyf = 0.$$

13. Let $f(x, y) : D \to \mathbb{R}$ be the 2-variable function defined by

$$f(x, y) = e^{x^2 - y^2}.$$

(a) For a point $P = (x_0, y_0)$, compute the partial derivatives $\frac{\partial z}{\partial x}|_P$ and $\frac{\partial z}{\partial y}|_P$. Write the equation of the tangent plane π of the surface defined by f at $Q = (1, 0, e)$. Compute the parametric form of the normal line to π at Q.

(b) Compute the partial derivatives of the second order $\frac{\partial^2 z}{\partial x^2}|_P$, $\frac{\partial^2 z}{\partial y^2}|_P$, $\frac{\partial^2 z}{\partial x \partial y}|_P$ and $\frac{\partial^2 z}{\partial y \partial x}|_P$.

(c) Using the results you got in (b), check that the function f satisfies the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 2\frac{\partial^2 f}{\partial x \partial y} - 4(x - y)^2 f = 0.$$