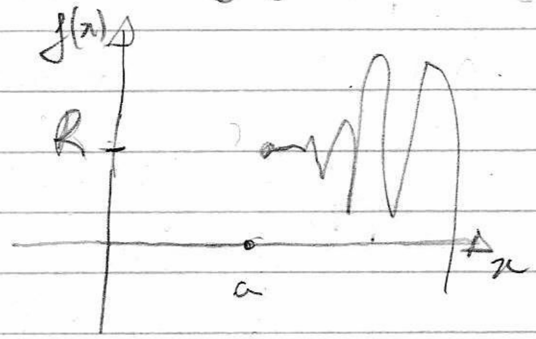
Definitions of limits

Ask if everybody understands what a limit is.

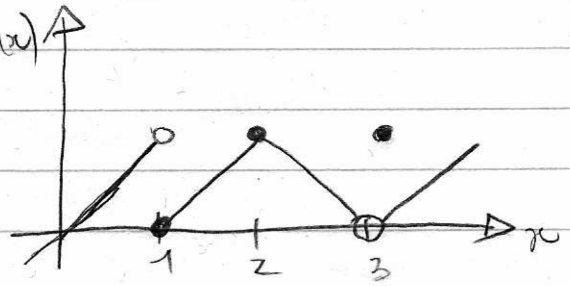
Define: a real number R is the limit from the right of a function f at a point a (and we denote that by $R = \lim_{x \to a^+} f(x)$) if "when we follow the graph of the function from the right ~~of a~~ towards a and we get really close to a, the values of f becomes all really close to R.



Same thing for limit from the left $L = \lim_{x \to a^-} f(x)$

We say $\lim_{x \to a} f(x) = X$ if both one-sided limits exist and are equal.

Interactive exercise:



What are:

$\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$,

$\lim_{x \to 3^+} f(x)$, $\lim_{x \to 3^-} f(x)$

Rules for calculating limits

Recall: what can we do?

We can sum limits, take their differences, take their product, take a multiple of them, take their quotient ONLY IF the limit of the function at the denominator is not 0,

and we can take their powers, i.e. $\lim_{x \to a} (f(x))^{m/n}$

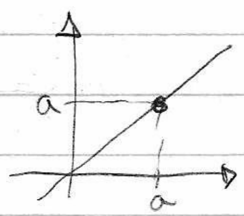
ONLY IF $L > 0$ if n is even b/c $(f(x))^{m/n} = \left(\sqrt[n]{f(x)}\right)^m$

and $\sqrt[n]{f(x)}$ is not a real number if n is even and $f(x) < 0$.

or $L \neq 0$ if $m > 0$, b/c $(f(x))^{-m/n} = \dfrac{1}{(f(x))^{m/n}}$

Compute $\lim_{x \to a} \dfrac{x^2 + x - 4 + 2\sqrt{x}}{x^3 - 2x^2 + 7}$

$\lim_{x \to a} x = a$



So $\lim_{x \to a} x^2 = \left(\lim_{x \to a} x\right)^2 = a^2$, (*)

$\lim_{x \to a} x^3 = a^3$

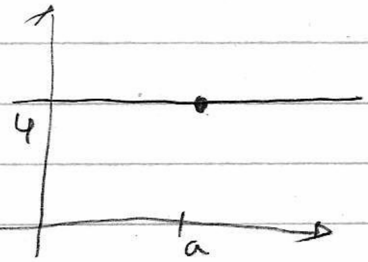
$\lim_{x \to a} \sqrt{x} = \lim_{x \to a} x^{1/2}$

$= \left(\lim_{x \to a} x\right)^{1/2} = \sqrt{a}$

(*) or alternatively

$\lim_{x \to a} x^2 = \lim_{x \to a} x \cdot x = \left(\lim_{x \to a} x\right)\left(\lim_{x \to a} x\right) = a^2$

$$\lim_{x \to a} 4 = 4$$



So we have successively

$$\lim_{x \to a} x^2 + x = \lim_{x \to a} x^2 + \lim_{x \to a} x$$
$$= a^2 + a,$$

$$\lim_{x \to a} x^2 + x - 4 = a^2 + a - 4,$$

$$\lim_{x \to a} x^2 + x - 4 + 2\sqrt{x} = a^2 + a - 4 - \sqrt{a}$$

Similarly, $\lim_{x \to a} x^3 - 2x^2 + 7 = a^3 - 2a^2 + 7$.

So $$\lim_{x \to a} \frac{x^2 + x + 4}{x^3 - 2x^2 + 7} = \frac{a^2 + a - 4 + 2\sqrt{a}}{a^3 - 2a^2 + 7}$$

PROVIDED $a^3 - 2a^2 + 7 \neq 0$.

Note: Can do all this directly for rational functions $P(x)/Q(x)$

More interesting exercise:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{when } f(x) = x^4$$

$$= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{\overbrace{x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4}^{\text{cf exercise with binomial.}} - x^4}{h}$$

$$= \lim_{h \to 0} 4x^3 + \underbrace{6hx^2}_{\xrightarrow{h \to 0} 0} + \underbrace{4h^2x}_{\xrightarrow{h \to 0} 0} + \underbrace{h^3}_{\xrightarrow{h \to 0} 0}$$

$$= 4x^3$$

Why important ? Because that's what we will do we want to find derivatives.

$\boxed{\text{There seems to be a problem with the denominator}}$

$$\lim_{x\to -1} \frac{x^2-1}{\underbrace{x+1}}$$

this is 0, but

$$x^2-1 = (x-1)(x+1)$$

$$\Rightarrow \lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} x-1 = -2$$

More subtle

$$\lim_{x\to 8} \frac{x^{2/3}-4}{x^{1/3}-2} = \lim_{x\to 8} \frac{(x^{1/3}-2)(x^{1/3}+2)}{x^{1/3}-2}$$

$$= 8^{1/3}+2 = 4$$

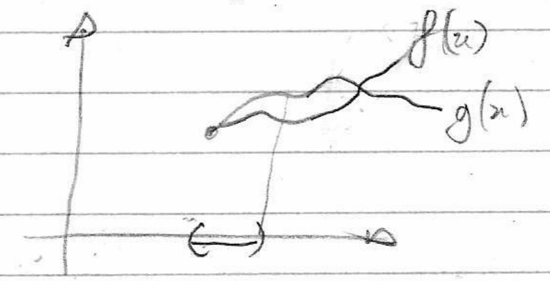
Doesn't always work:

$$\lim_{t\to 1} \frac{t^2-1}{t^2-2t+1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{(t-1)(t-1)} \qquad \left(\begin{array}{l} \Delta^2-4ac=0 \\ \Rightarrow x_{\frac{1}{2}} = 1 \end{array}\right)$$

$\to$ undefined

Squeeze theorem

Order is preserved: if $f(x) \leq g(x)$ on one open interval starting at a point a, then, if they exist, $\lim_{x \to a^+} f(x) \leq \lim_{x \to a^+} g(x)$



$\Rightarrow$ squeeze theorem

If $f(x) \leq g(x) \leq h(x)$ on an open interval containing a, then, if they exist, $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x)$

So if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x)$ equals them

What is $\lim_{x \to 0} x \sin\left(x^3 + x + \sqrt{x^2+2}\right)$?

We have $x^3 + x + \sqrt{x^2+2}$ exists for all $x \in \mathbb{R}$

(We have to check that the function is defined near 0. $\sqrt{\ }$ could be a problem).

Also $-1 \leq \sin\left(x^3 + x + \sqrt{x^2+2}\right) \leq 1$

So $\underbrace{-x}_{\substack{\downarrow\, x\to 0 \\ 0}} \leq x\sin\left(x^3+x+\sqrt{x^2+2}\right) \leq \underbrace{x}_{\substack{\downarrow\, x\to 0 \\ 0}}$

So by the squeeze theorem,

$$\lim_{x\to 0} x\sin\left(x^3+x+\sqrt{x^2+2}\right) = 0$$

Limits at infinity of rational functions

Let $f(x) = \dfrac{P(x)}{Q(x)}$ with $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$

Then
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

$$= \lim_{x\to\infty} \frac{x^n \left(a_n + a_{n-1} \overset{\to 0}{\frac{1}{x}} + \cdots + a_1 \overset{\to 0}{\frac{1}{x^{n-1}}} + a_0 \overset{\to 0}{\frac{1}{x^n}}\right)}{x^m \underbrace{\left(b_m + b_{m-1} \underset{\to 0}{\frac{1}{x}} + \cdots + b_1 \underset{\to 0}{\frac{1}{x^{m-1}}} + a_0 \underset{\to 0}{\frac{1}{x^m}}\right)}_{\to b_m}}$$

$$= \lim_{x\to\infty} x^{n-m} \cdot \frac{a_n}{b_m}$$

So if $n - m = 0$, $\lim_{x\to\infty} f(x) = \dfrac{a_n}{b_m}$

< 0, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{a_n}{b_m} \dfrac{1}{x^{m-n}}$
$= 0$

> 0, if $\dfrac{a_n}{b_m} > 0$, $\lim_{x\to\infty} f(x) = +\infty$

if $\dfrac{a_n}{b_m} < 0$, $\lim_{x\to\infty} f(x) = -\infty$

For $\lim\limits_{x \to \infty}$, it's the same except for the >0 case:

- if $n-m$ is odd: if $\frac{a_n}{b_m} > 0$, $\lim\limits_{x \to -\infty} f(x) = -\infty$

  if $\frac{a_n}{b_m} < 0$, $\lim\limits_{x \to -\infty} f(x) = +\infty$

- if $n-m$ is even: if $\frac{a_n}{b_m} > 0$, $\lim\limits_{x \to -\infty} f(x) = +\infty$

  if $\frac{a_n}{b_m} < 0$, $\lim\limits_{x \to -\infty} f(x) = -\infty$

Limits at infinity

If we can get as close as we want of L by taking x large enough, then we say $\lim_{x\to\infty} f(x) = L$.

- Quick answer questions: ask the class

- $\lim_{x\to\infty} \frac{x^2-2}{x-2x^2} \longrightarrow = -1/2$
- $\lim_{x\to\infty} \frac{x}{x^2-4} \longrightarrow = 0$ b/c $\deg P(x) = 1$, $\deg Q(x) = 2$

Detail ~~this~~ one:

$$\lim_{x\to\infty} \frac{x^3+3}{x^2+2} = \lim_{x\to\infty} \frac{x^2(x + \overbrace{3/x^2}^{\to 0})}{x^2(1 + \underbrace{2/x^2}_{\to 0})}$$

$$= \frac{\lim_{x\to\infty} x + 3/x^2}{1}$$

$$= \infty.$$

Harder

$$\lim_{x\to\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-2x}\right)$$

Ask if anyone has an idea

$$= \lim_{x\to\infty} \left[\frac{\left(\sqrt{x^2+2x} - \sqrt{x^2-2x}\right)\left(\sqrt{x^2+2x} + \sqrt{x^2-2x}\right)}{\sqrt{x^2+2x} + \sqrt{x^2-2x}}\right]$$

$$= \lim_{x\to\infty} \frac{(x^2+2x)-(x^2-2x)}{\sqrt{\quad} + \sqrt{\quad}}$$

$$= \lim_{x\to\infty} \frac{4x}{x\left(\sqrt{1+\underbrace{2/x}_{\to 0}} + \sqrt{1-\underbrace{2/x}_{\to 0}}\right)}$$

$$= 2.$$