Sets and intervals

$6x^2 - 5x \leq 1$ What is the solution set of this inequality

$\Rightarrow \underbrace{6x^2 - 5x - 1}_{} \leq 0$

We want to factor this.

$\Delta = b^2 - 4ac = 25 + 4 \cdot 6 = 49 = 7^2$

$\Rightarrow x_{\frac{1}{2}} = \frac{-b + \sqrt{\Delta}}{2a} = \frac{5+7}{2 \cdot 6} = 1$

$\frac{-b - \sqrt{\Delta}}{2a} = \frac{5-7}{12} = -\frac{1}{6}$

So we have $(x-1)(x+\frac{1}{6}) \leq 0$

2 possibilities:

A • $(x-1)(x+\frac{1}{6}) = 0$

$\Rightarrow (x-1) = 0$ or $(x+\frac{1}{6}) = 0$

$\Rightarrow x = 1$ $\boxed{A_1}$ $\Rightarrow x = -\frac{1}{6}$ $\boxed{A_2}$

B • $(x-1)(x+\frac{1}{6}) < 0$

$\Rightarrow (x-1)$ and $(x+\frac{1}{6})$ have opposite signs and are both different from 0.

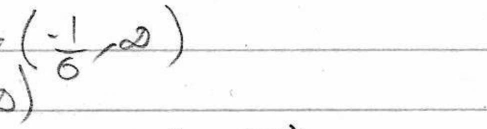
2 sub-possibilities:

▪ $(x-1) < 0$ & $(x+\frac{1}{6}) > 0$

$\Rightarrow x < 1$ & $x > -\frac{1}{6}$

$\Rightarrow x \in (-\infty, 1)$ & $x \in (-\frac{1}{6}, \infty)$

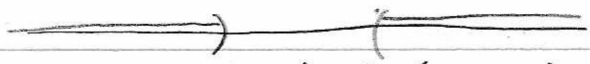
$\Rightarrow x \in (-\infty, 1) \cap (-\frac{1}{6}, \infty)$



$\Rightarrow x \in (-\frac{1}{6}, 1)$ $\boxed{B_1}$

■ $(x-1)>0$ & $\left(x+\frac{1}{6}\right)<0$

$\Rightarrow x>1$ & $x<-\frac{1}{6}$



$\Rightarrow x\in\left(-\infty,-\frac{1}{6}\right)\cap(1,\infty)$

$=\emptyset$ $\boxed{B_2}$

(So no new solution)

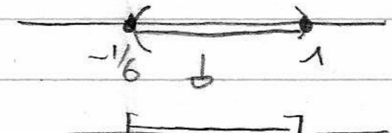
Now we regroup all the solutions:

$x\in A_1\cup A_2\cup B_1\cup B_2$

because we said it can be solution A_1 or A_2 or B_1 or B_2

$=\{1\}\cup\left\{-\frac{1}{6}\right\}\cup\left(1,-\frac{1}{6}\right)\cup\emptyset$

$=\left[-\frac{1}{6},1\right]$



So $\boxed{S=\left[-\frac{1}{6},1\right]}$

Similarly: What is S for $\left|2-\frac{x}{2}\right|<\frac{1}{2}$?

$\Rightarrow -\frac{1}{2}<2-\frac{x}{2}<\frac{1}{2} \Rightarrow 2-\frac{x}{2}<\frac{1}{2}$ and $2-\frac{x}{2}>-\frac{1}{2}$

$$\frac{x}{2} > \frac{3}{2} \quad \text{and} \quad \frac{x}{2} < \frac{5}{2}$$

$$\Rightarrow x > 3 \quad \text{and} \quad x < 5$$

$$\Rightarrow S = (3, \infty) \cap (5, \infty) =$$

$$= (3, 5)$$

Functions

Give an intuitive notion: for instance, a rule that assign to any number another number.

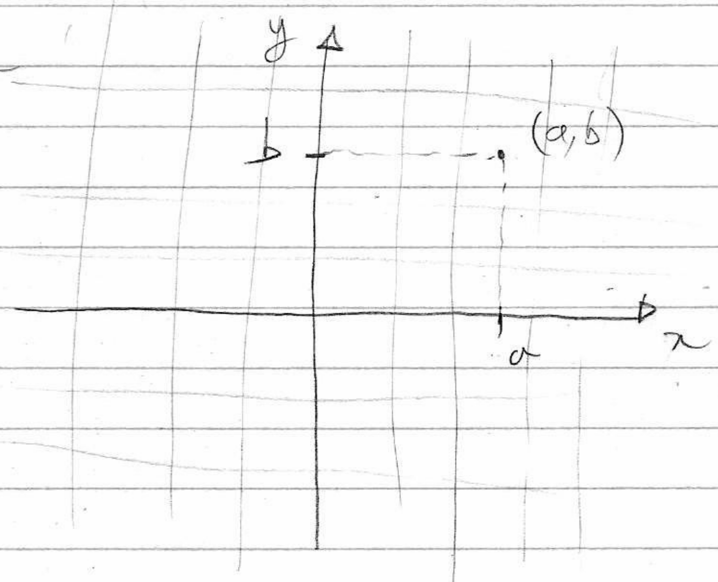
Can present a function with a formula and a domain of definition. e.g. $f(x) = x^2 + \cos(x)$

Then the rule assign to a number r the value obtained when you replace x by r in the formula

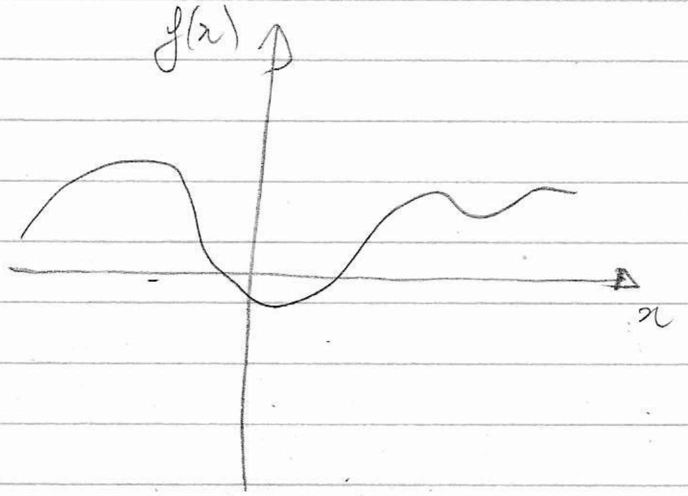
e.g. $f(2) = 2^2 + \cos(2)$.

Can see a function as a graph.

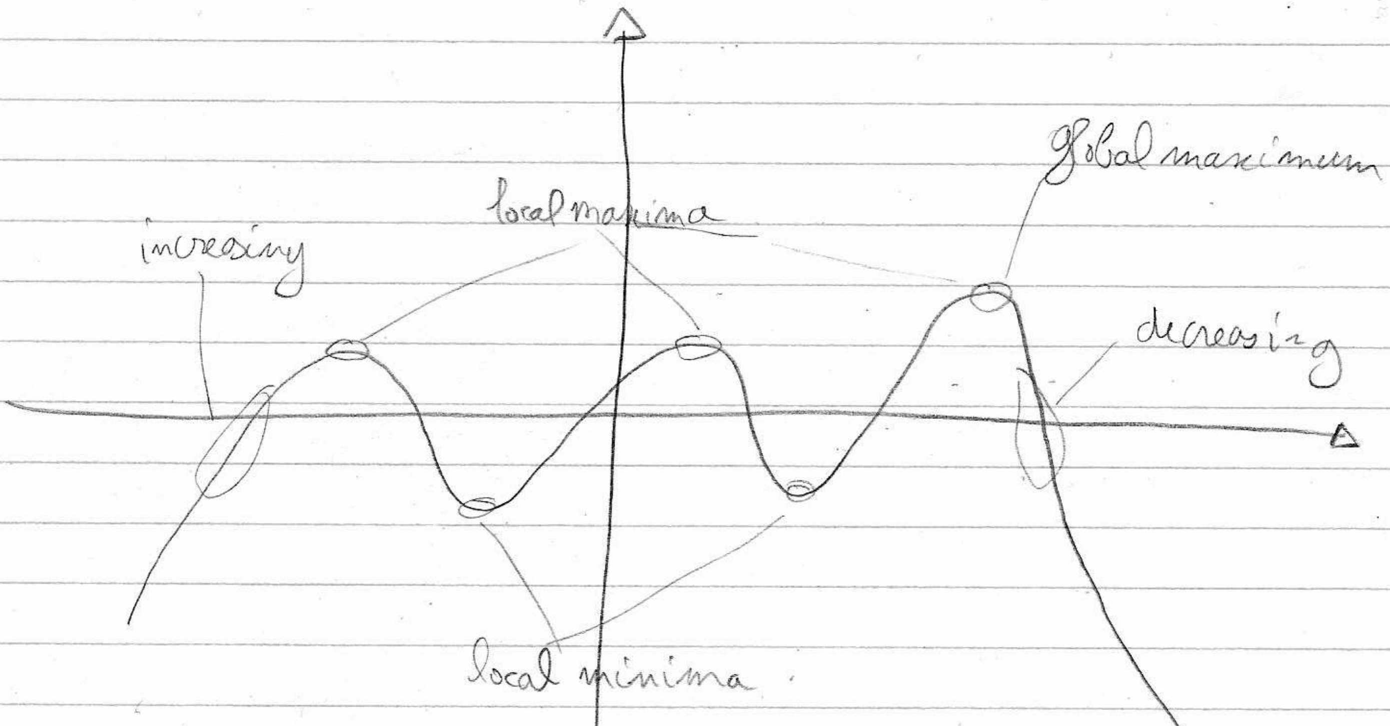
Cartesian plane



So a function can be seen as



Connection of properties of function with how it looks on the graph



▸ Quick questions:

What's the domain of $f(x) = \sqrt{x+2}$ $\quad [-2, \infty)$

What's the domain of $f(x) = \frac{1}{x-1}$? $\quad (-\infty, 1) \cup (1, \infty)$

Can make new functions by adding them, substracting, multiplying, dividing provided that for the domain of the quotient function, the denominator function is never 0, and composing functions.

E.g: if $f(x) = x^2$, $g(x) = 2x-1$, $h(x) = \frac{1}{x}$,

What's the domain of $k(x) = f(x) \cdot h \circ g(x)$?

$$k(x) = \frac{x^2}{2x-1}$$

$$\Rightarrow (-\infty, \tfrac{1}{2}) \cup (\tfrac{1}{2}, \infty)$$

Binomial Formula

Recall the formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

where $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

<u>compute $(x+2)^4$:</u>

$$(x+2)^4 = \binom{4}{0} x^0 2^4 + \binom{4}{1} x^1 2^3 + \binom{4}{2} x^2 2^2 + \binom{4}{3} x^3 2^1 + \binom{4}{4} x^4 2^0$$

$$\binom{4}{0} = \frac{4!}{0!(4-0)!} = \frac{4!}{4!} = 1$$

$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4!}{3!} = \frac{4\cdot\cancel{3\cdot2\cdot1}}{\cancel{3\cdot2\cdot1}} = 4$$

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\cdot3\cdot\cancel{2\cdot1}}{2\cdot1\cdot\cancel{2\cdot1}} = \frac{12}{2} = 6$$

Notice symmetry

$$\binom{4}{3} = \frac{4!}{3!1!} = 4$$

$$\binom{4}{4} = \frac{4!}{4!0!} = 1.$$

So $(x+2)^4 = 16 + 4\cdot x\cdot 8 + 6\cdot x^2\cdot 4 + 4\cdot x^3\cdot 2 + x^4$

$= x^4 + 8x^3 + 24x^2 + 32x + 16$

Induction

Recall what induction is:

if a property is true for a given integer m,

and if the fact that the property is true for an integer n implies it is true for the integer $n+1$,

then the property is true for all the integers $\geq m$.

Prove by induction that for all $n \geq 1$:

$$\frac{2}{1\cdot 3} + \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} + \dots + \frac{2}{(2n-1)(2n+1)} = 1 - \frac{1}{(2n+1)}$$

→ ▪ If $n = 1$, $\frac{2}{(2n-1)(2n+1)} = \frac{2}{1\cdot 3} = \frac{2}{3}$

$1 - \frac{1}{2n+1} = 1 - \frac{1}{3} = \frac{2}{3}$

So it holds for $n = 1$.

▪ If it holds for a given n, then:

$$\underbrace{\frac{2}{1\cdot 3} + \frac{2}{3\cdot 5} + \dots + \frac{2}{(2n-1)(2n+1)}} + \frac{2}{(2(n+1)-1)(2(n+1)+1)}$$

$$= 1 - \frac{1}{2n+1} + \frac{2}{(2n+1)(2(n+1)+1)}$$

(By the induction hypothesis)

$$= 1 + \frac{2 - (2(n+1)+1)}{(2n+1)(2(n+1)+1)}$$

$$= 1 + \frac{2 - 2n - 3}{(2n+1)(2(n+1)+1)}$$

$$= 1 + \frac{-(2n+1)}{(2n+1)(2(n+1)+1)}$$

$$= 1 - \frac{1}{2(n+1)+1}$$

so the property also holds for $n+1$

$\Rightarrow$ Conclusion : the property holds for all $n \geqslant 1$.