ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University

Department of Electrical and Computer Engineering

Lecture 25

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Second-Order Systems

1. Damping factor and natural frequency
2. Quality Q
3. -3 dB bandwidth
4. All-pass systems
5. Minimum phase systems

Frequency Response of General Second-Order Systems

A general second-order system has a transfer function of the form

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}.$$

It can be stable, unstable, causal or not, depending on the signs of the coefficients and the specified ROC.

Let's restrict our attention to causal, stable LTI second-order systems of this type.

Necessary and sufficient condition for stability: the coefficients a_i are all positive, or all negative. The poles are given by:

$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

The damping ratio and natural frequency

Assume that $b_1=b_2=0$, then the transfer function is

$$H(s) = \frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

where ζ is the damping ratio and ω_n is the undamped natural frequency. The poles are:

$$p_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^{\,2} - 4\omega_n^{\,2}}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Many physical systems such as the mass-spring-damper system or a RLC filter can be modeled using this transfer function, which corresponds to the differential equation:

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{d\ y(t)}{dt} + \omega_n^{\,2} y(t) = A\omega_n^{\,2} x(t)$$

Case ζ>1

In this case, the system is said to be *overdamped*.

-The step response doesn't exhibit any ringing.

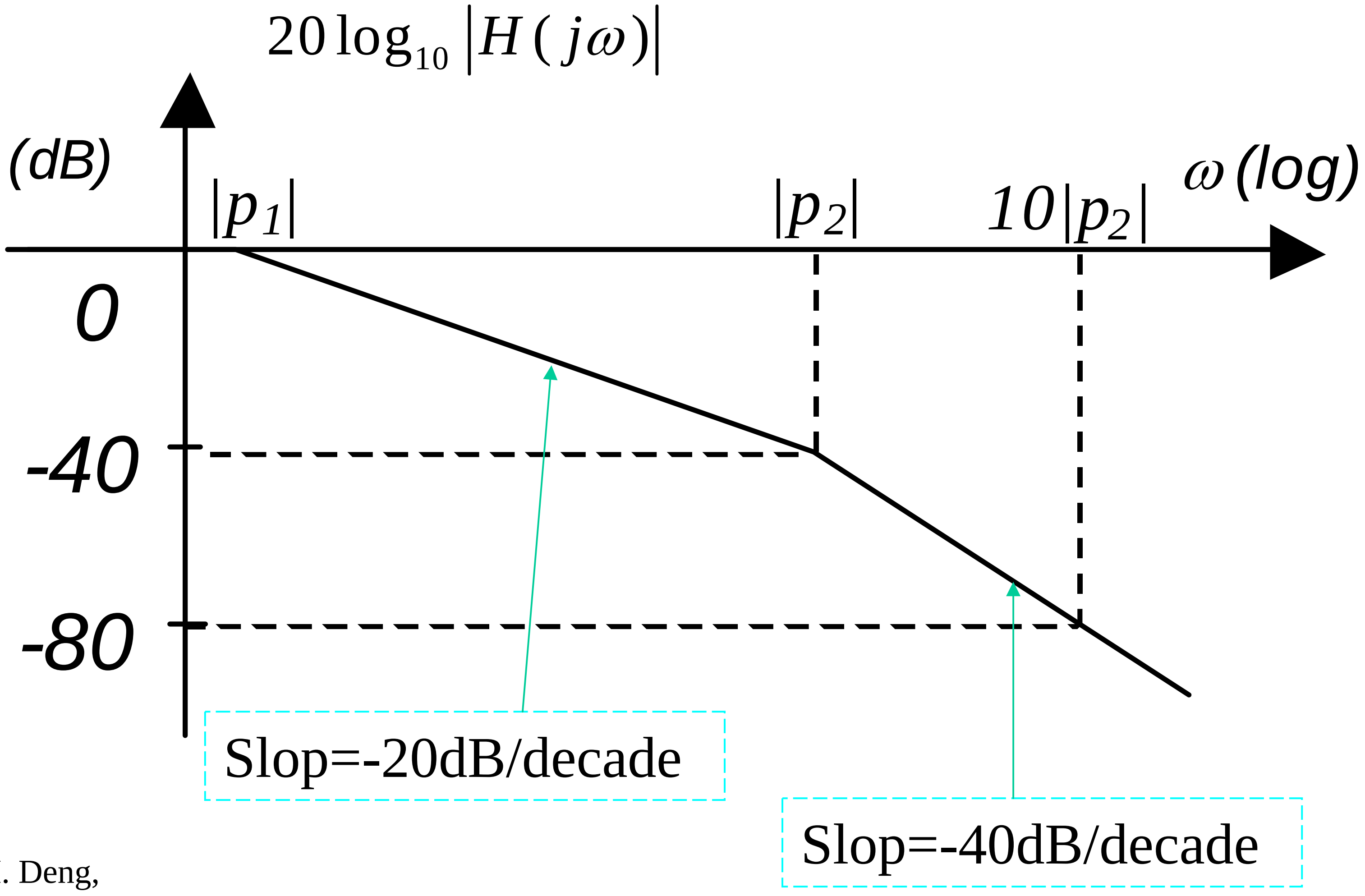
-The two poles are real, negative and distinct:

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}.$$

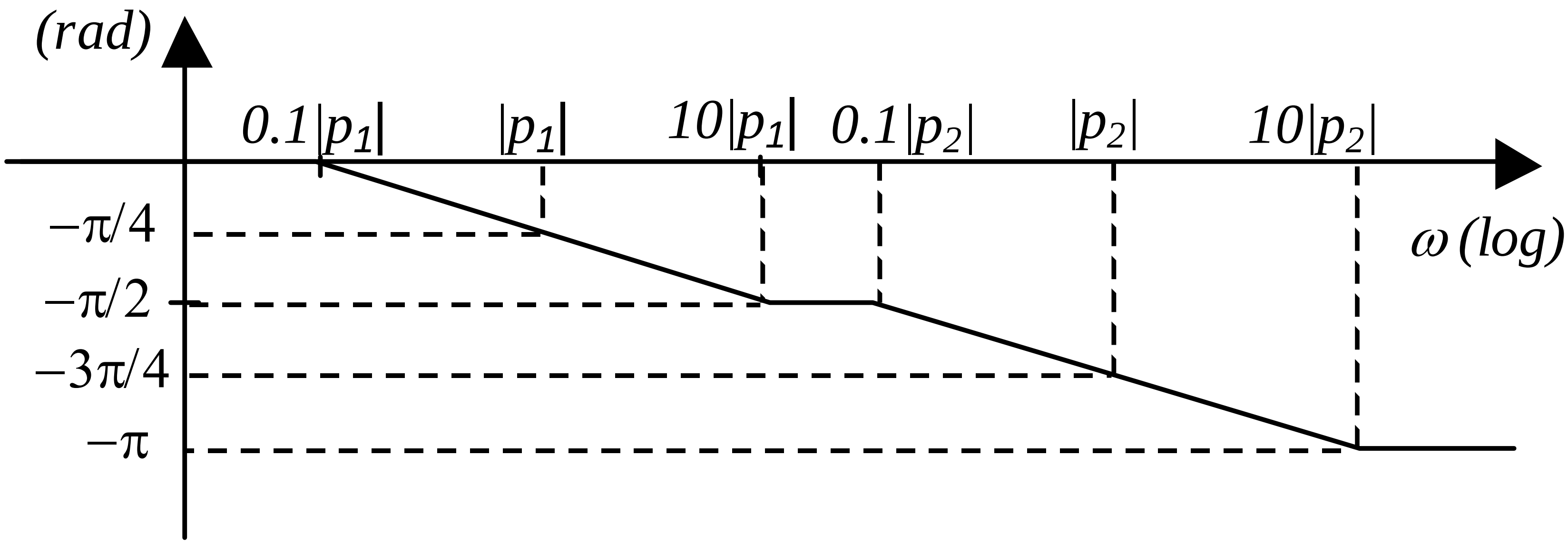
The second-order system can be seen as a cascade of two standard first-order systems (lags).

$$H(s) = \frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = A\left(\frac{1}{\frac{s}{-p_1} + 1}\right)\left(\frac{1}{\frac{s}{-p_2} + 1}\right)$$

The Bode plot of $H(j\omega) = A\frac{1}{\frac{j\omega}{-p_1}+1}\frac{1}{\frac{j\omega}{-p_2}+1}$ is easy to sketch



$\angle H(j\omega)$



Case ζ=1

In this case, the system is said to be *critically damped*.

-The two poles are negative and real, but they are the same.

$$p_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n = p_2 \, .$$

The second-order system can also be seen as a cascade of two first-order transfer functions having the same pole.

$$H(s) = A\frac{1}{(\frac{s}{-p_1}+1)^2}$$

Case ζ<1

In this case, the system is said to be *underdamped*.

The step response exhibits some ringing, although it really becomes visible only for $\zeta < 1/\sqrt{2} = 0.707$.

The two poles are distinct, complex conjugate:

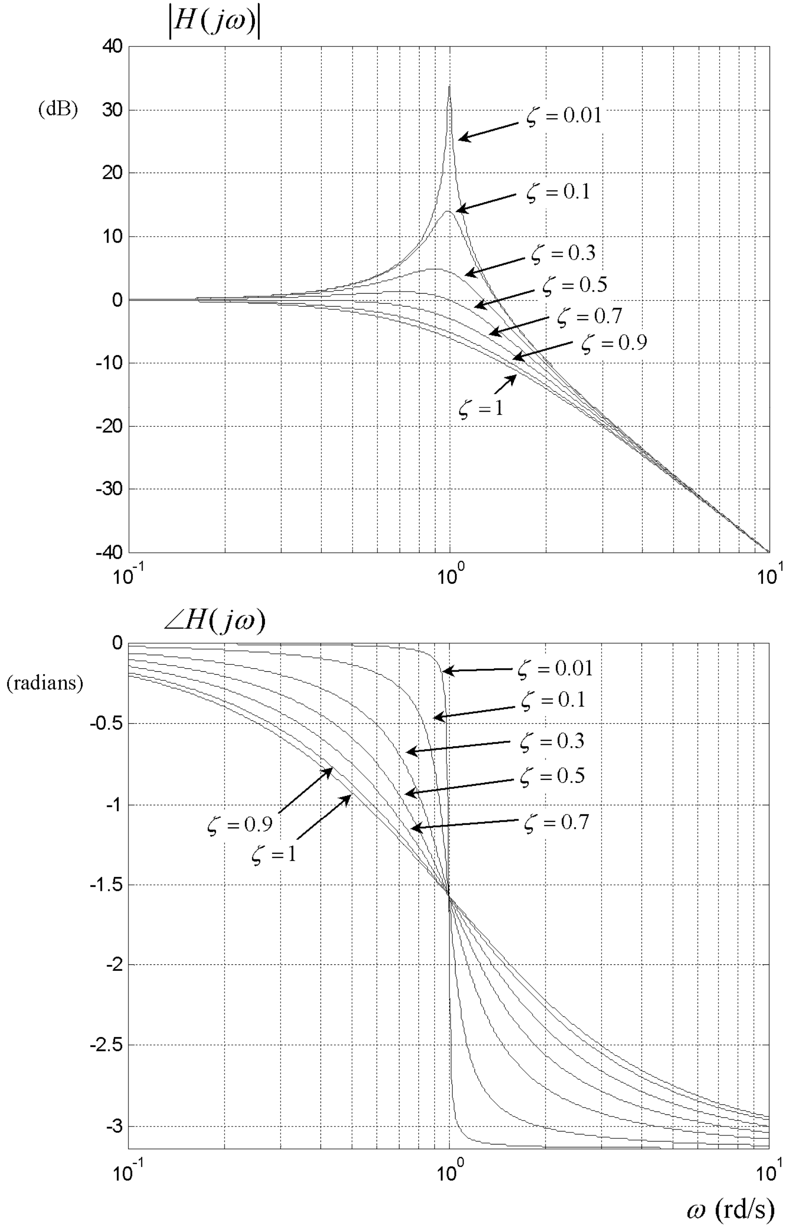
$$p_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}, \quad p_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}.$$

The Bode plots of general second-order systems with different damping factors

Note:

1. For ζ<1, the ***approximation error*** of the asymptotes increases greatly around the break frequency.

2. For ζ=0.707, the magnitude response has maximal flatness, and corresponds to a second-order lowpass Batterworth filter with cutoff frequency ω_n.



Example

Consider the second-order transfer function

$$H(s) = \frac{1}{-2s^2 - 6s - 9} = -\frac{1}{2}\frac{1}{s^2 + 3s + 9/2}$$

Where $\omega_n = 3\sqrt{2}/2$, and the damping ratio is

$$\zeta = \frac{3}{2\omega_n} = \frac{3}{2\frac{3\sqrt{2}}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0.707$$

Since the damping ratio is less than one, the two poles are complex.

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\frac{3}{2} \pm j\frac{3}{2}$$

Quality Q

In the field of communications, the underdamped second-order filter has played an important role as a simple frequency-selective bandpass filter.

When the damping ratio is very low, the filter becomes highly selective due to its high peak resonance at

$$\omega_{\max} = \omega_n \sqrt{1 - 2\xi^2} \, .$$

The *quality Q* of the filter is defined as

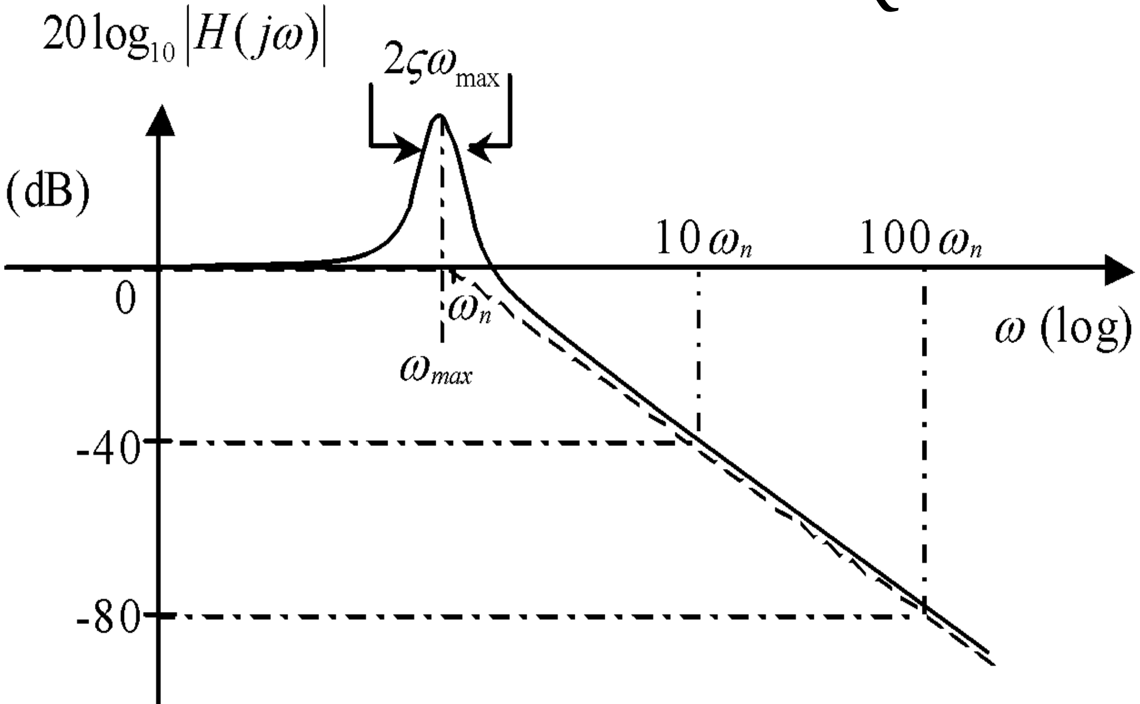
$$Q = \frac{1}{2\varsigma} \, .$$

The –3 dB bandwidth

The –3dB bandwidth = the frequency difference between the two frequencies where the magnitude is 3 dB lower than the peak magnitude.

For the second-order filter, the –3 dB bandwidth is:

$$\Delta\omega \approx \frac{\omega_{\max}}{Q} = 2\varsigma\omega_{\max}$$



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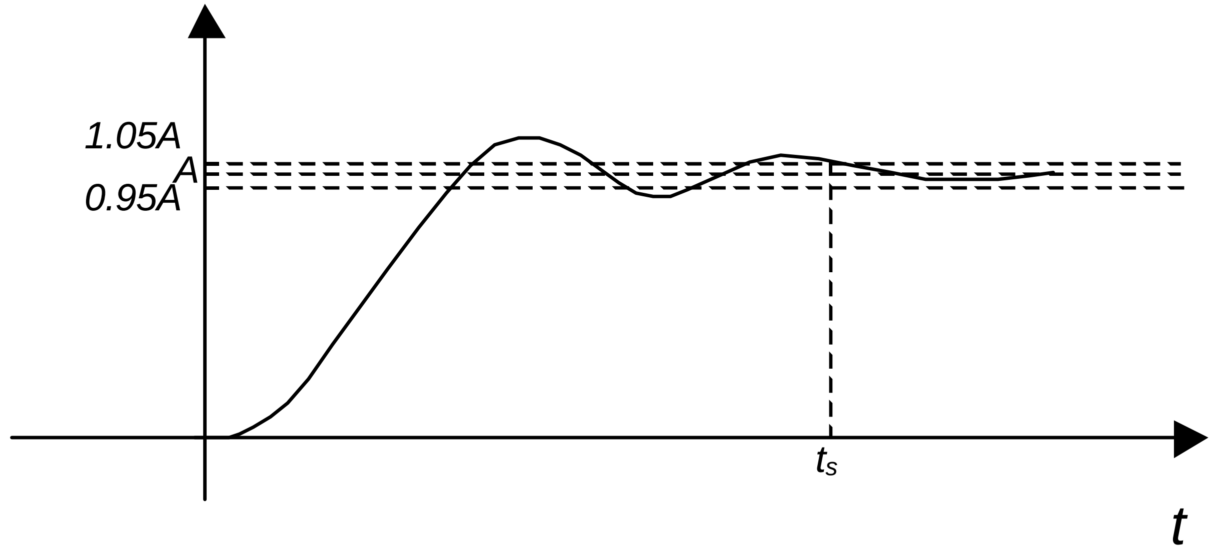
The step response of an under-damped second-order system



Settling time is the time response first reaches its final value within a certain percentage, as shown by t_s.

For a second-order system, t_s depends primarily on ω_n but also on ζ . For a given ω_n, the settling time is a nonlinear function of ζ.

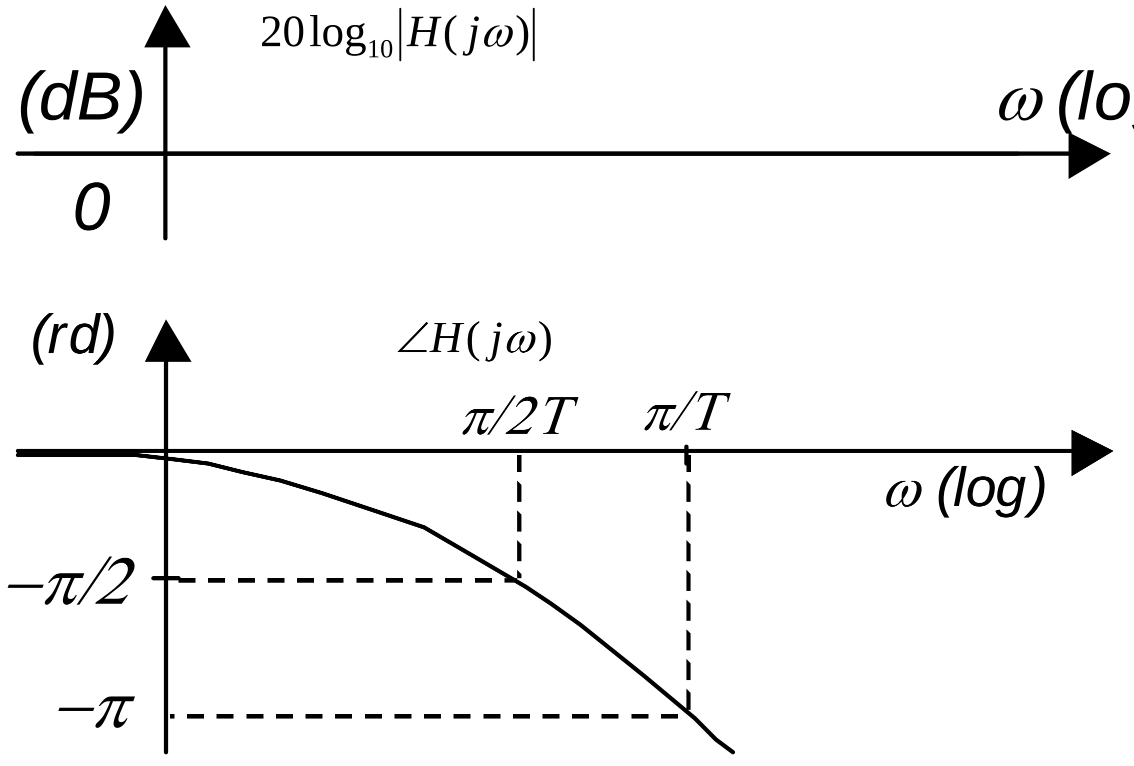
Ideal delay system

The transfer function of an ideal delay of *T* time units is

$$H(s) = e^{-sT}.$$

- Its frequency response is $H(j\omega) = e^{-j\omega T}$.
- Its magnitude its is 1 for all frequencies.
- Its phase is $\angle H(j\omega) = -\omega T$, which is ***linear*** and negative for positive frequencies.

The Bode plots of an ideal delay system



Note: the linear phase response is an exponential function of the log-scale frequency:

$$\angle H(j\omega) = -\omega T$$

$$\log_{10}[-\angle H(j\omega)] = \log_{10}(\omega T)$$

$$\angle H(j\omega) = -10^{\log_{10} T} 10^{\log_{10} \omega}$$

Group delay

The group delay is defined as follows:

$$\tau(\omega) := -\frac{d}{d\omega}\angle H(j\omega) \qquad \text{(second)}$$

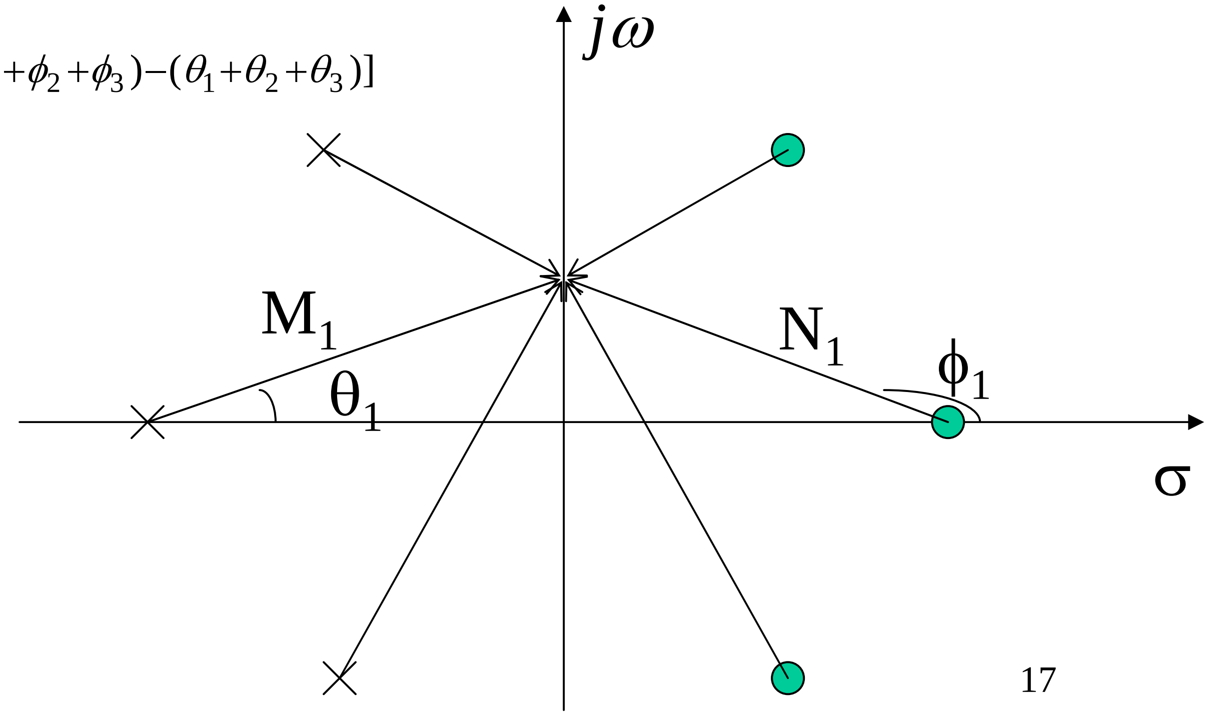
- A pure delay system has a constant group delay of *T* seconds.
- Group delay gives an idea of how much the bulk of a signal is delayed in a given frequency band.
- Non-constant group delay leads to output waveform distortion caused by the phase.

All-Pass Systems

A system is said to be an all-pass system, if its transfer functions has poles on the left half s-plane and zeros on the right half s-plane, and if each pole (zero) is a "mirror" of a zero (pole).

An example of an all-pass system is shown below.

$$H(j\omega) = \frac{N_1 N_2 N_3}{M_1 M_2 M_3} e^{i[(\phi_1+\phi_2+\phi_3)-(\theta_1+\theta_2+\theta_3)]}$$



As $N_i = M_i$, then

$$|H(j\omega)| = \frac{N_1 N_2 N_3}{M_1 M_2 M_3} = 1$$

Applications of all-pass system

- Phase correction

Minimum phase system

1. A system is called as minimum phase system if its all zeros are in the left half s-plane or $j\omega$ axis. Otherwise, the system is called non-minimum phase system.

2. A non-minimum phase system can be viewed as a cascade of a minimum phase system and an all-pass system.

3. A minimum phase system has less absolute phase shift to input signals than a non-minimum phase system with the same magnitude response.

Examples

Example 1. Consider the minimum-phase system $H(s)=1$. The magnitude of its frequency response if 1, and its phase is zero for all frequencies.

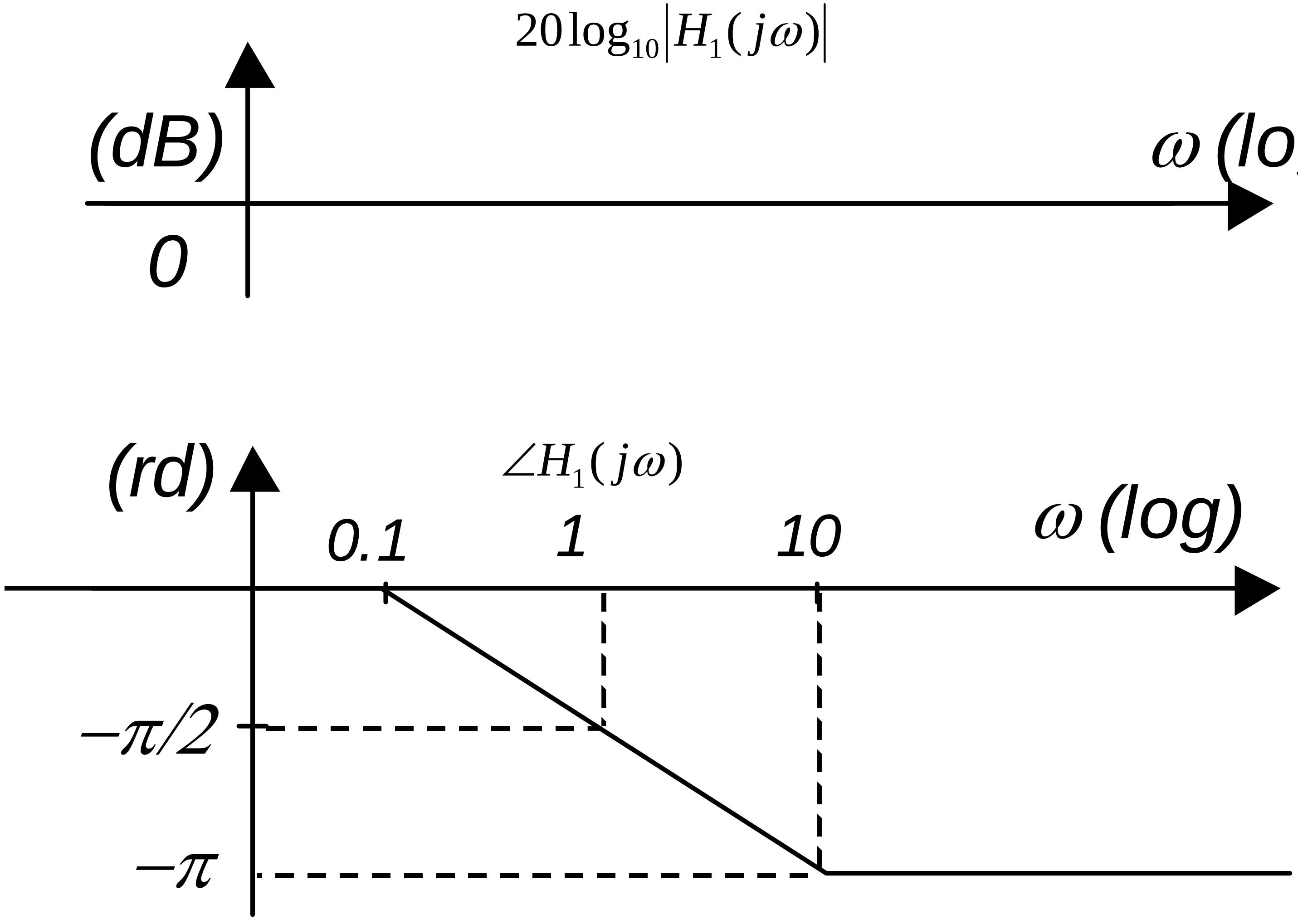
Example 2. Now consider the system

$$H_1(s)=\frac{-s+1}{s+1}.$$

This is a non-minimum phase and all-pass system. Its magnitude of the frequency response is 1 for all frequencies. Its phase is given by

$$\angle H(j\omega)=\arctan\left(\frac{-\omega}{1}\right)+\arctan\left(\frac{-\omega}{1}\right)=-2\arctan(\omega)$$

which tends to $-\pi$ as $\omega\to\infty$.



Such a system is called an *allpass system* because it passes all frequencies with unity gain.

Example 3

The non-minimum phase system

$$H_1(s) = \frac{-s+100}{(s+1)(s+10)}$$

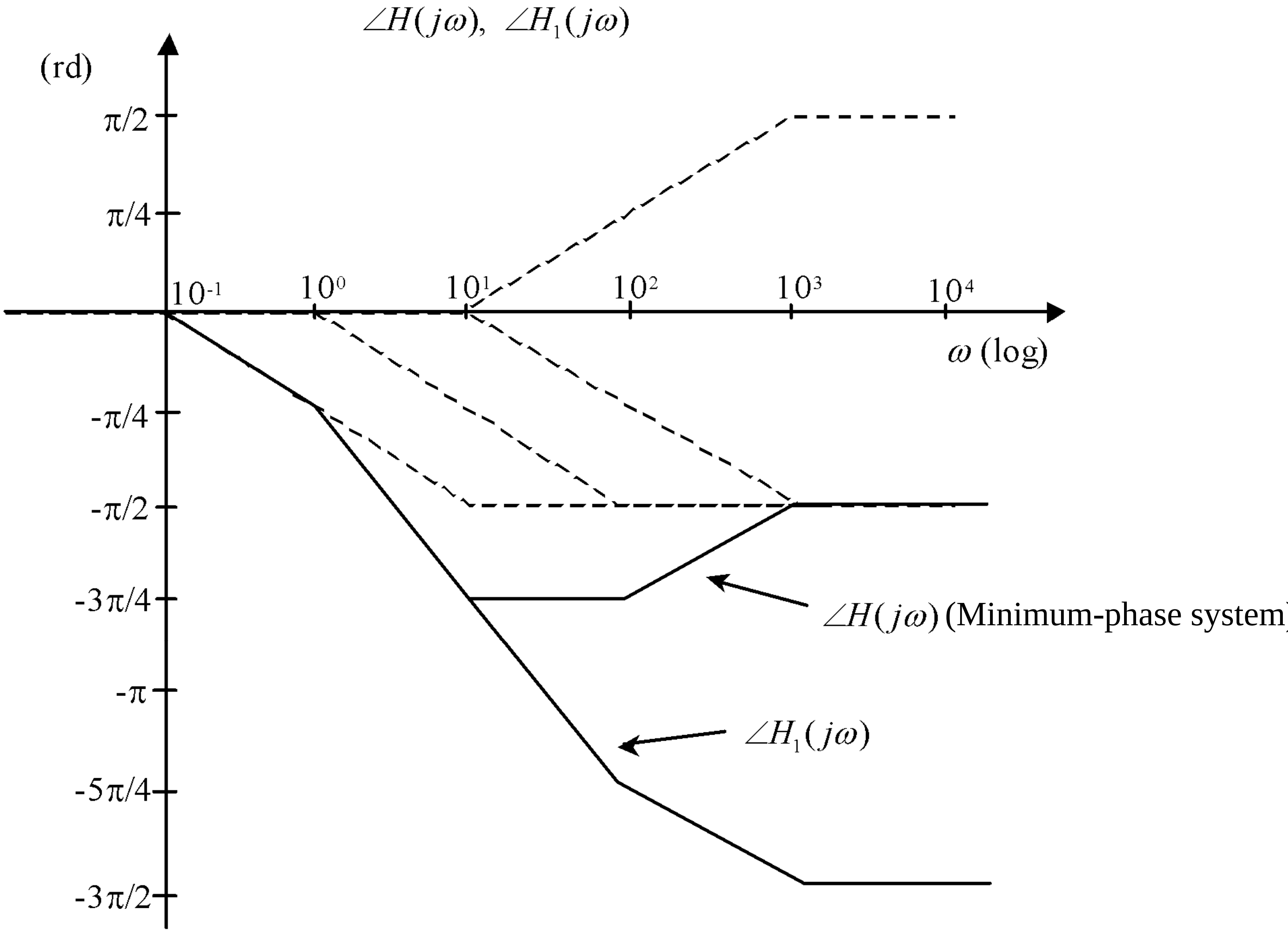
has the same magnitude as the minimum-phase system

$$H(s) = \frac{s+100}{(s+1)(s+10)}$$

Note that any non-minimum phase transfer function can be expressed as the product of a minimum-phase transfer function and an allpass transfer function.

For the example above, we can write

$$H_1(s) = \underbrace{\frac{s+100}{(s+1)(s+10)}}_{\text{minimum-phase}} \underbrace{\frac{-s+100}{s+100}}_{\text{allpass}}$$



$\angle H(j\omega),\ \angle H_1(j\omega)$
(rd)
π/2
π/4
-π/4
-π/2
-3π/4
-π
-5π/4
-3π/2
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
ω (log)
$\angle H(j\omega)$ (Minimum-phase system)
$\angle H_1(j\omega)$