ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University

Department of Electrical and Computer Engineering

Lecture 24

October 31, 2008

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Bode Plots of Multi-Order (multi-pole-zero) Systems

Frequency Responses of First-Order Lag, Lead Systems

Steps of drawing Bode plots

Step 1: Dividing multi-order system into cascade of multiple 1^{st}-order systems, each containing a single pole or zero;

Step 2: Determining the asymptotes and break frequencies of these 1^{st}-order systems;

Step 3: Adding up the Bode plots of these 1^{st}–order systems.

Example of a first-order system

Consider again the first-order system with frequency response

$$H(j\omega) = \frac{1}{j\omega + 2}.$$

It is convenient to write it as the product of a gain and a first-order transfer function with unity gain at DC:

$$H(j\omega) = \frac{1}{2}\frac{1}{j\omega/2 + 1}.$$

The Bode magnitude plot of the first-order system

The Bode plot of the magnitude is the graph of

$$20\log_{10}\left|H(j\omega)\right| = 20\log_{10}\left|\frac{1}{2}\right| + 20\log_{10}\left|\frac{1}{\frac{j\omega}{2}+1}\right| dB$$

$$= -20\log_{10} 2 - 20\log_{10}\left|\frac{j\omega}{2}+1\right| dB$$

$$= -6\,dB - 20\log_{10}\left|\frac{j\omega}{2}+1\right| dB$$

The Bode magnitude plot of a 1st-order system has 2 asymptotes: one straight line for very low frequencies, and one straight line for very high frequencies. The frequency at which the two asymptotes meet is called the break frequency.

The low- and high-frequency asymptotes of the first-order system

For low frequencies (ω<<2),

$$20\log_{10}|H(j\omega)| \approx -6\,dB - 20\log_{10}|1|\,dB = -6\,dB\,.$$

i.e., for very low frequencies the Bode magnitude plot approximates a straight line.

For high frequencies (ω>>2),

$$\begin{aligned} 20\log_{10}|H(j\omega)| &\approx -6\,dB - 20\log_{10}\left|\frac{\omega}{2}\right| dB \\ &= -6\,dB - 20\log_{10}|\omega|\,dB + 20\log_{10}2\,dB \\ &= -20\log_{10}|\omega|\,dB \end{aligned}$$

i.e., for very high frequencies, the Bode magnitude plot approximates a straight line with a slop –20 dB/decade, or –6 dB/octave.

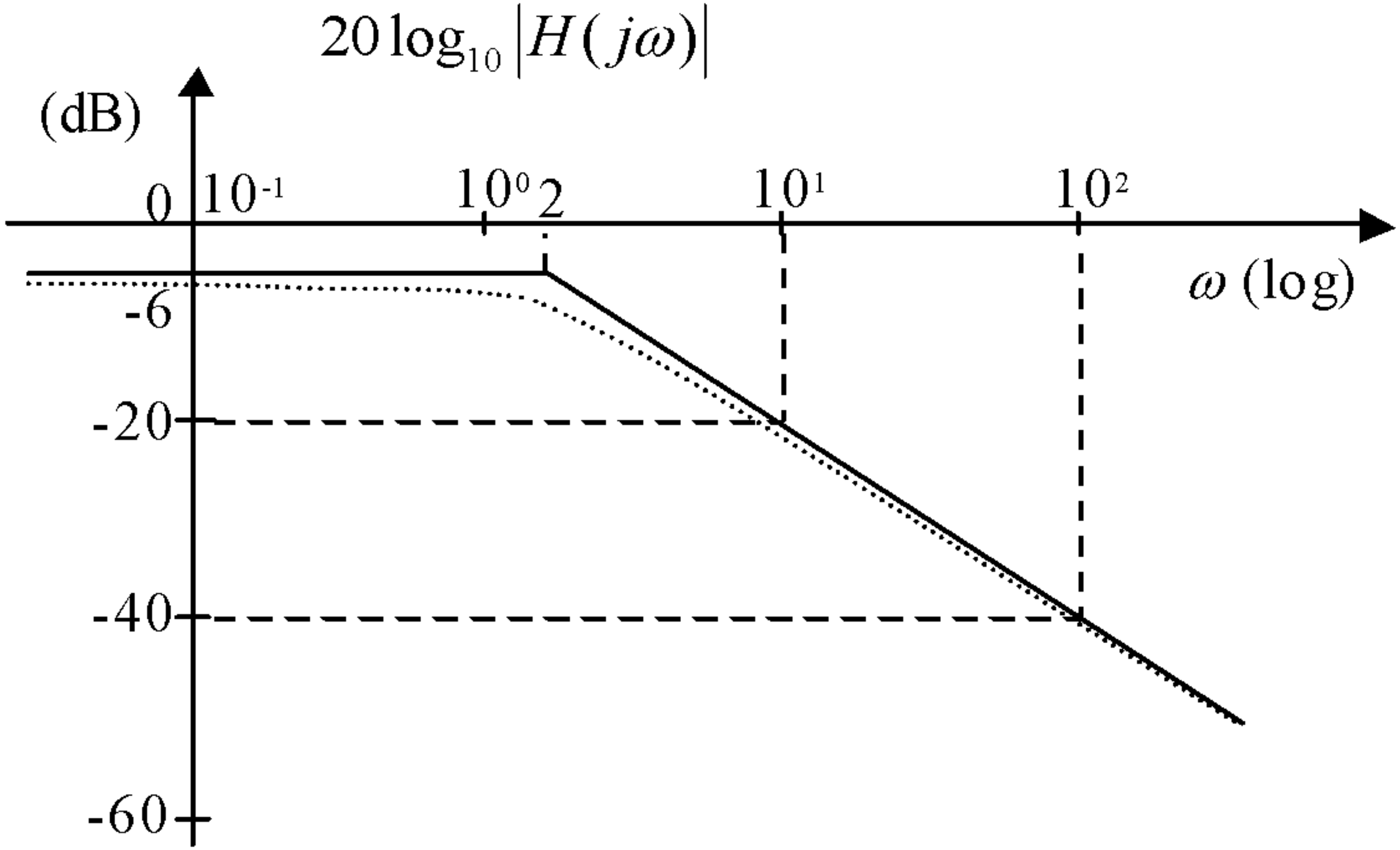
The slop of the high-frequency asymptotes

For ω>> 2, say $\omega = 10$, we get -20 dB; for $\omega = 100$, we get -40 dB, etc.

Therefore, the slope of the asymptote is -20dB/decade.

The 2 asymptotes meet at the break frequency 2 radians/s, at which the magnitude drops from the DC gain by 3 dB.

Given the 2 asymptotes and the drop at the break frequency, we can sketch the magnitude Bode plot (the dashed line as follows).



$20\log_{10}|H(j\omega)|$
(dB)
0
10^{-1}
10^{0}
2
10^{1}
10^{2}
ω (log)
-6
-20
-40
-60

The asymptotes of the Bode phase plot of the 1st-order system

The phase response is given by:

$$\angle H(j\omega) = \angle \frac{1}{2}\frac{1}{j\omega/2+1} = \angle \frac{1}{j\omega/2+1} = \arctan(\frac{-\omega}{2})$$

For ω<<2, the phase approximates 0; for ω>>2 the phase is approximates $-\frac{\pi}{2}$. i.e., at very low and very high frequencies the phase response approximates to 2 parallel asymptotes, respectively

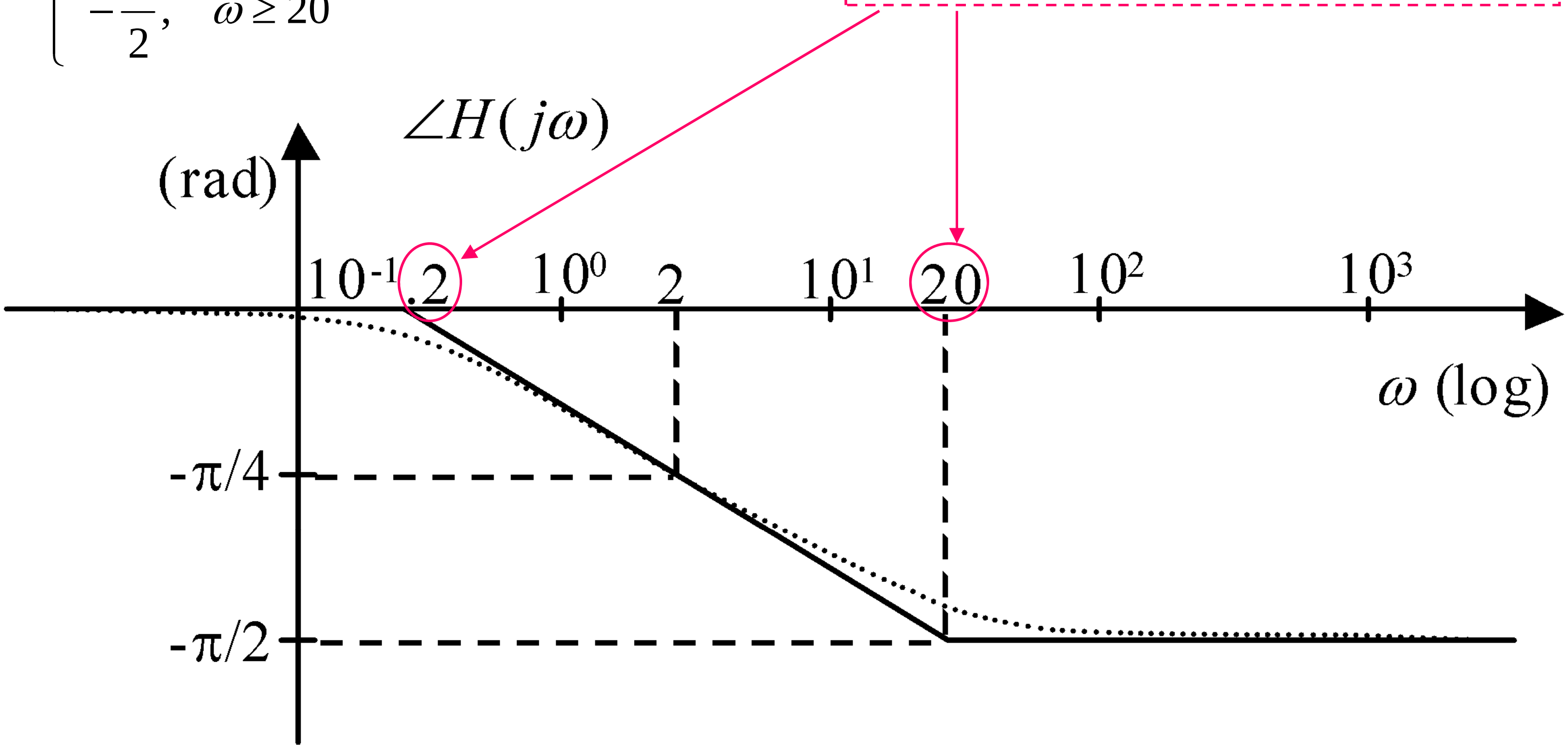
Connecting the 2 parallel asymptotes is a straight line, the third asymptotes. The three asymptotes are given by the following piece wise linear function:

$$\angle H(j\omega) = \begin{cases} 0, & \omega \le \frac{2}{10} \\ -\frac{\pi}{4}[\log_{10}(\frac{\omega}{2})+1], & \frac{2}{10} \le \omega \le 20 \\ -\frac{\pi}{2}, & \omega \ge 20 \end{cases}$$

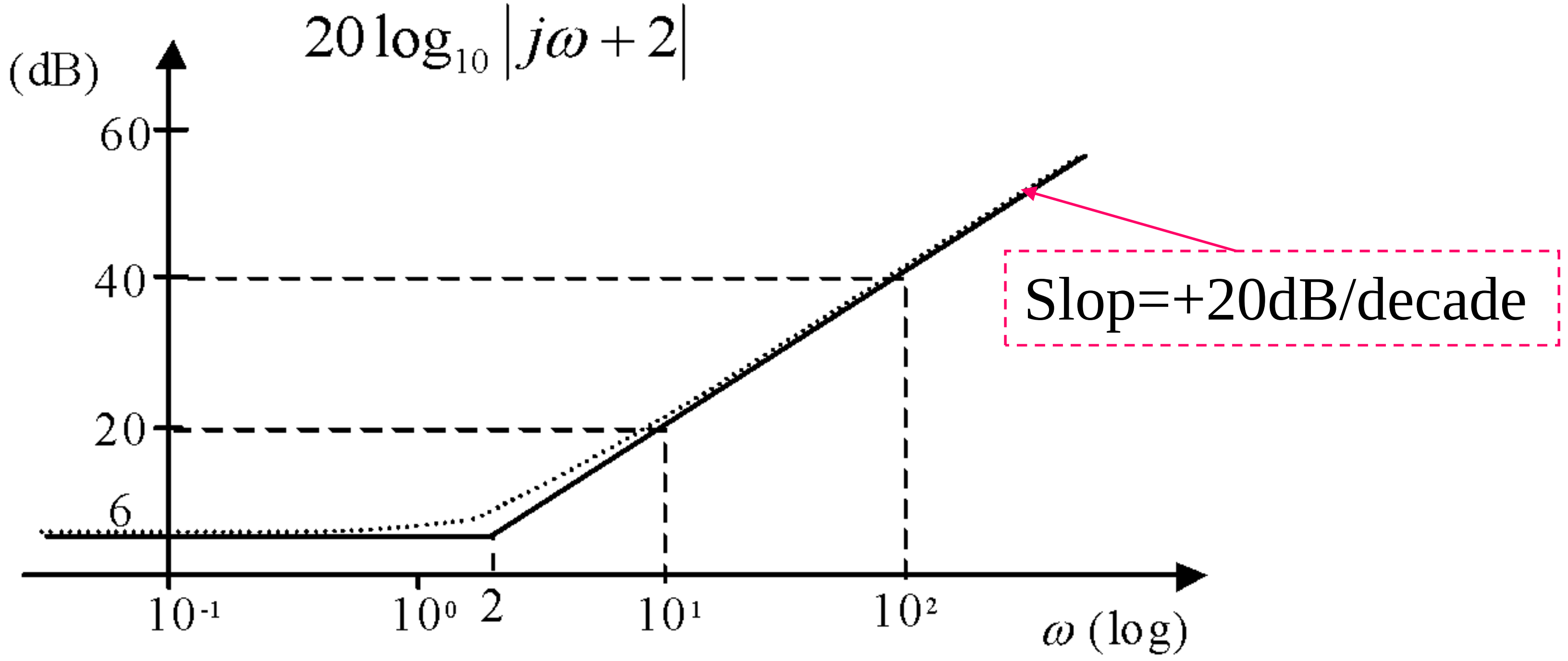
The break frequencies of the Bode phase plot of the 1st-order system

$$\angle H(j\omega) = \begin{cases} 0, & \omega \le \frac{2}{10} \\ -\frac{\pi}{4}[\log_{10}(\frac{\omega}{2})+1], & \frac{2}{10} \le \omega \le 20 \\ -\frac{\pi}{2}, & \omega \ge 20 \end{cases}$$

Break frequencies of the Bode phase plot



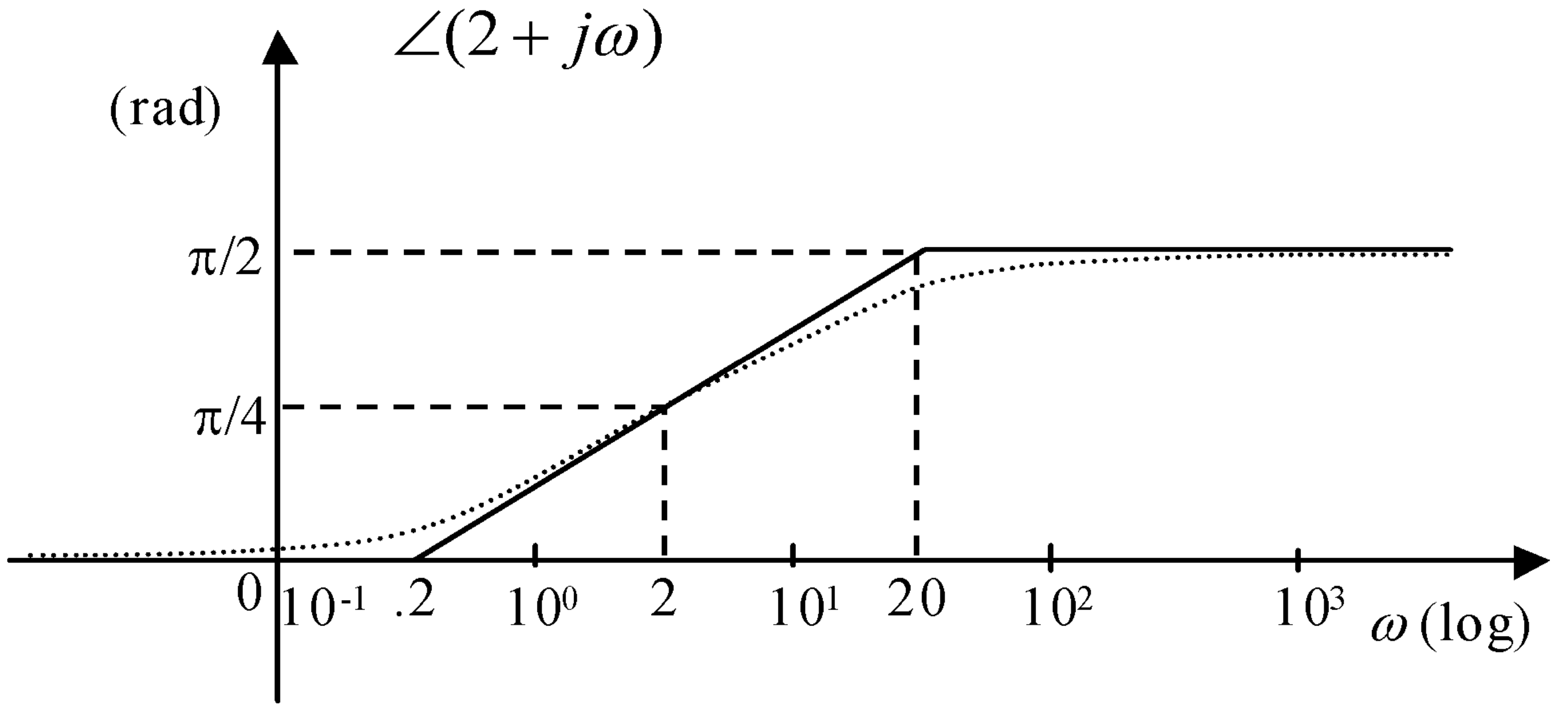
The Bode magnitude plot of a single-zero system



Given a numerator (*jω+2*), i.e., the inverse of *1/(jω+2)*, the Bode magnitude plot is simply the Bode magnitude plot of *1/(jω+2)* flipped around the frequency axis, because:

$$20\log_{10}|j\omega+2| = -20\log_{10}\frac{1}{|j\omega+2|}$$

The Bode phase plot of a single-zero system



Given a single-zero system (*s+2*), i.e., the inverse of *1/(s+2)*, the Bode phase plot is simply the Bode phase plot of *1(s+2)* flipped around the frequency axis, because:

$$\angle(2+j\omega) = -\angle\frac{1}{2+j\omega}$$

The Bode plots of a second-order system

Second-Order Example:

$$H(s) = \frac{s+100}{\left(s^2+11s+10\right)} = \frac{s+100}{\left(s+1\right)\left(s+10\right)}$$

$$= 10\frac{\frac{s}{100}+1}{\left(s+1\right)\left(\frac{s}{10}+1\right)}, \ \mathrm{Re}\{s\} > -1$$

which has the frequency response

$$H(j\omega) = 10\frac{\frac{j\omega}{100}+1}{(j\omega+1)(\frac{j\omega}{10}+1)}$$

The break frequencies are 1, 10 and 100 radians/s.

The Bode magnitude plot of the second-order system

The Bode magnitude plot is the graph of

$$20\log_{10}\left|H(j\omega)\right| = 20\log_{10}\left|10\right| + 20\log_{10}\left|\tfrac{j\omega}{100}+1\right|$$

$$+20\log_{10}\left|\frac{1}{\tfrac{j\omega}{10}+1}\right| + 20\log_{10}\left|\frac{1}{j\omega+1}\right| \; dB$$

$$= 20 + 20\log_{10}\left|\tfrac{j\omega}{100}+1\right| - 20\log_{10}\left|\tfrac{j\omega}{10}+1\right|$$

$$-20\log_{10}\left|j\omega+1\right| \; dB$$

At low frequencies (ω<<1),

$$20\log_{10}\left|H(j\omega)\right| \approx 20\,dB + 20\log_{10}\left|1\right| - 20\log_{10}\left|1\right|$$

$$-20\log_{10}\left|1\right| dB = 20\,dB \quad .$$

For high frequencies (ω>>100),

$$
\begin{aligned}
20\log_{10}\left|H(j\omega)\right| &\approx 20+20\log_{10}\left|\frac{\omega}{100}\right|-20\log_{10}\left|\frac{\omega}{10}\right|-20\log_{10}\left|\omega\right| dB \\
&= 20+20\log_{10}\left|\omega\right|-40-20\log_{10}\left|\omega\right|+20-20\log_{10}\left|\omega\right| dB \\
&= -20\log_{10}\left|\omega\right| dB
\end{aligned}
$$

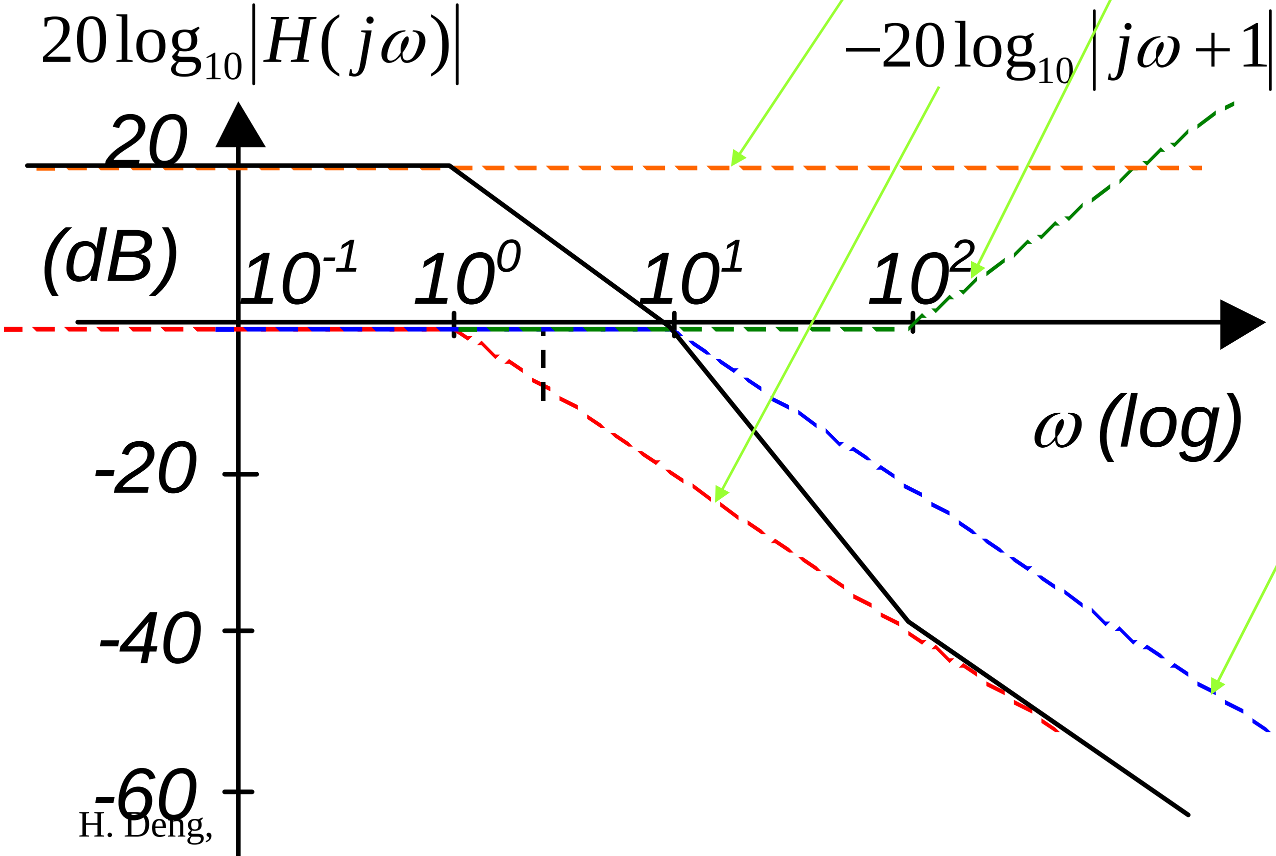
We can plot the asymptotes of each first-order term on the same magnitude graph (dashed lines) and then add them together to obtain the Bode magnitude plot (solid line).

$$20\log_{10}\left|H(j\omega)\right| = 20\log_{10}\left|10\right| + 20\log_{10}\left|\frac{j\omega}{100}+1\right|$$

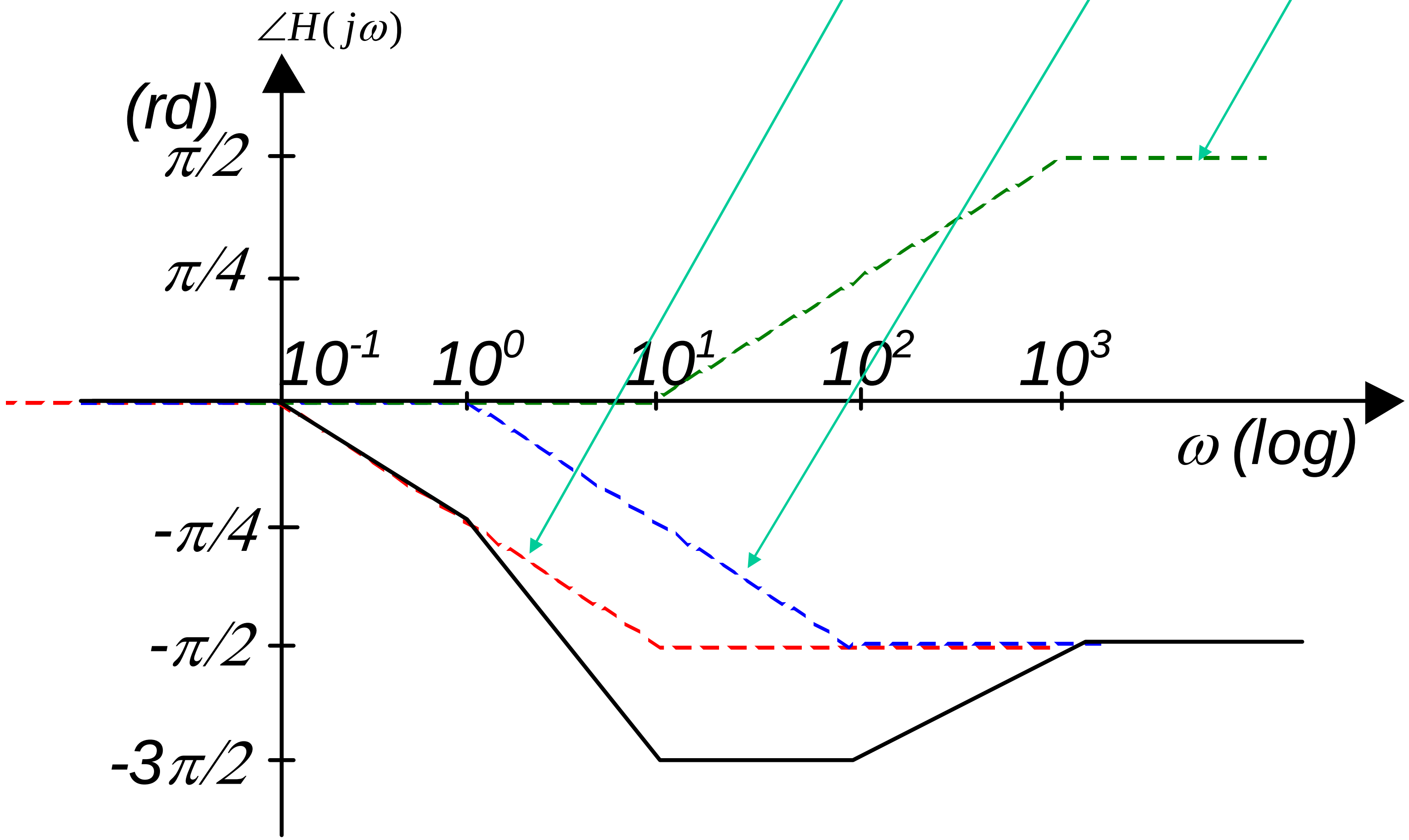
$$+20\log_{10}\left|\frac{1}{\frac{j\omega}{10}+1}\right| + 20\log_{10}\left|\frac{1}{j\omega+1}\right| \; dB$$

$$= 20 + 20\log_{10}\left|\frac{j\omega}{100}+1\right| - 20\log_{10}\left|\frac{j\omega}{10}+1\right|$$

$$-20\log_{10}\left|j\omega+1\right| \; dB$$



$$\angle H(j\omega) = \angle\frac{1}{j\omega+1} + \angle\frac{1}{\frac{j\omega}{10}+1} + \angle(\frac{j\omega}{100}+1)$$



First-order lag system

A first-order lag has a transfer function of the form

$$H(s) = \frac{\alpha\tau s + 1}{\tau s + 1},$$

where $0 \le \alpha < 1$, $\tau > 0$ is the time constant.

This system is called a *lag* because it has an effect similar to a pure delay $e^{-\tau s}$ at low frequencies for $\alpha = 0$. To the first order, the two systems are the same, as can be seen from the Taylor series around *s*=0:

$$\frac{1}{\tau s + 1} = 1 - \tau s + (\tau s)^2 - (\tau s)^3 + \ldots \approx 1 - \tau s$$

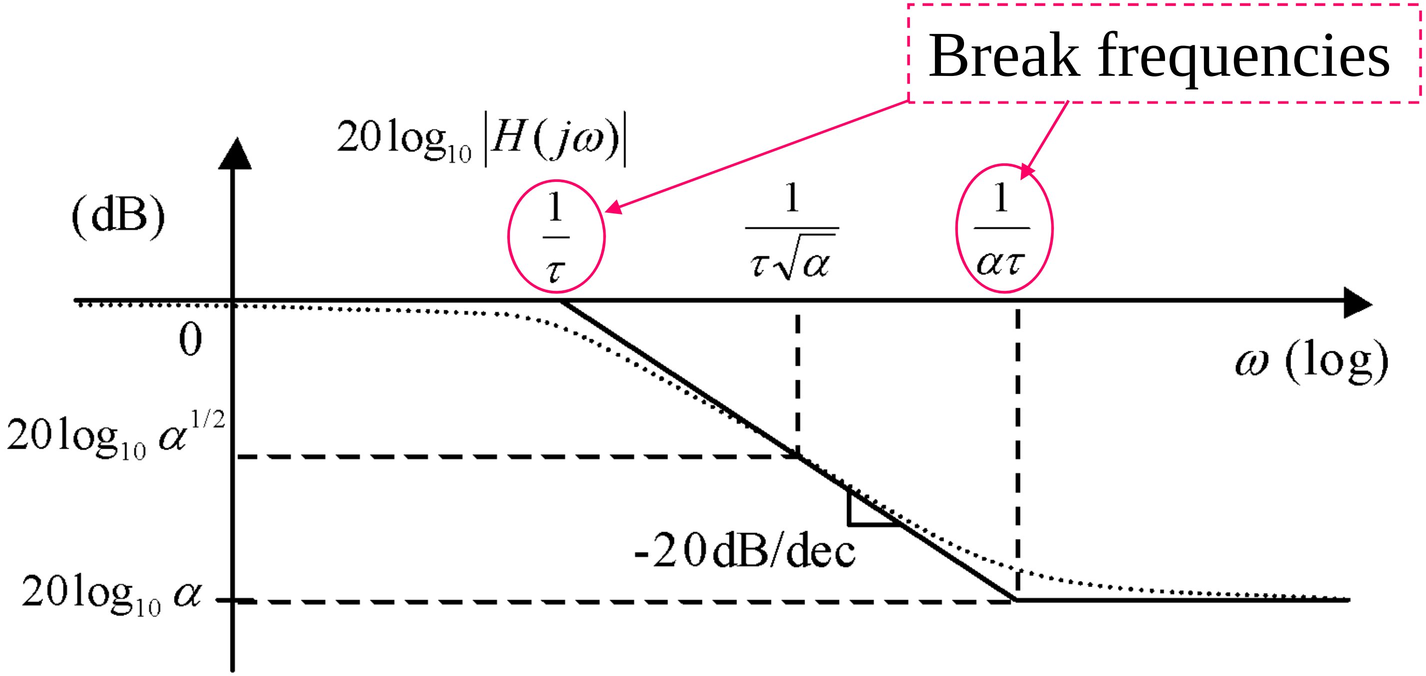
$$e^{-\tau s} = 1 - \tau s + \frac{1}{2}(\tau s)^2 - \frac{1}{3!}(\tau s)^3 + \ldots \approx 1 - \tau s$$

The Bode magnitude plot of the first-order lag system

Assuming that $0 < \alpha < 1$, the frequency response of the first-order lag system is

$$H(j\omega) = \frac{j\omega\alpha\tau + 1}{j\omega\tau + 1}$$

The Bode *magnitude* plot of the above system is shown below.



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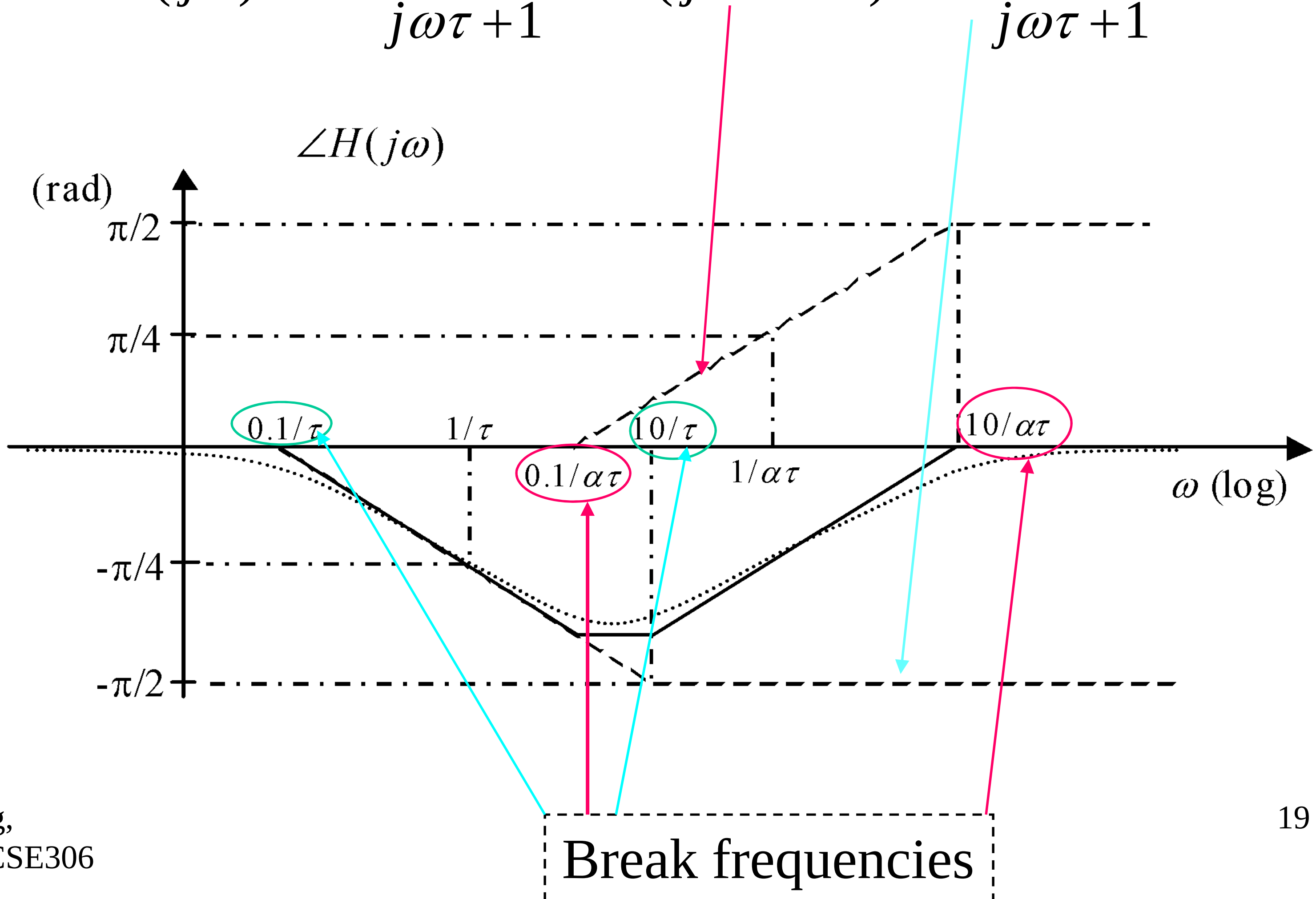
The phase Bode plot of the first-order lag system

$$H(s) = \frac{\alpha\tau s + 1}{\tau s + 1} \qquad \mathrm{Re}\{s\} > -\frac{1}{\tau}$$

$$\angle H(j\omega) = \angle \frac{j\omega\alpha\tau + 1}{j\omega\tau + 1} = \angle(j\omega\alpha\tau + 1) + \angle \frac{1}{j\omega\tau + 1}$$



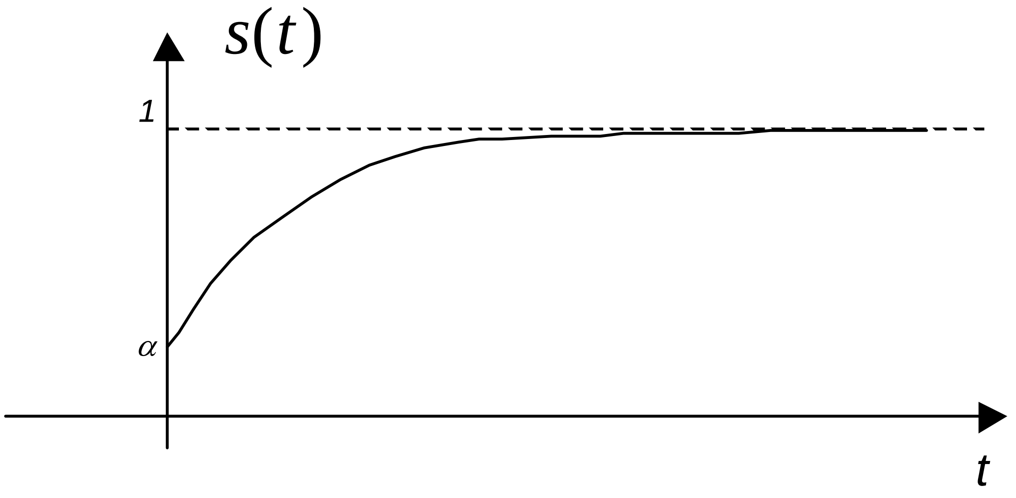
The step response of the first-order lag system

For the case $0<\alpha<1$ (lag) and time constant τ, i.e.,

$$H(s)=\frac{\alpha\tau s+1}{\tau s+1}=\alpha+\frac{1-\alpha}{\tau s+1},$$

the step response is

$$s(t)=\alpha u(t)+(1-\alpha)(1-e^{-\frac{t}{\tau}})u(t).$$



First-Order Lead

The transfer function of a first-order lead system has the same algebraic expression as that of first-order lag system:

$$H(s) = \frac{\alpha\tau s + 1}{\tau s + 1}$$

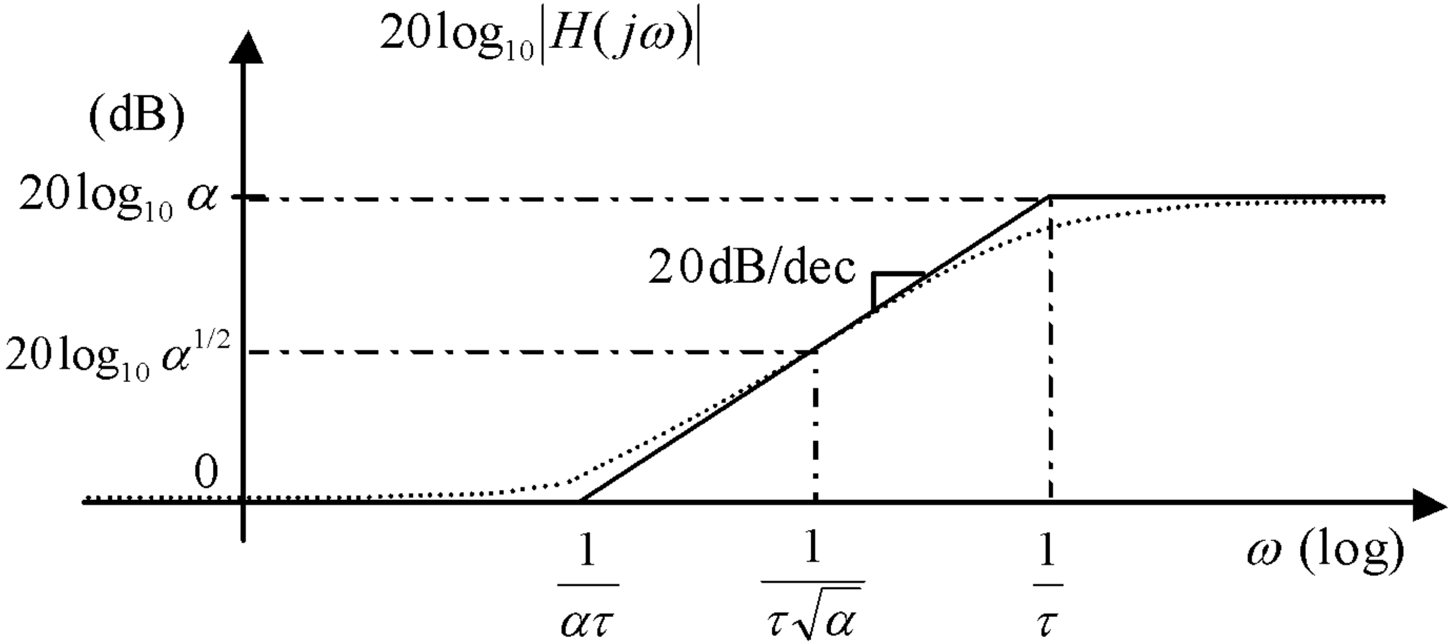
where $\tau > 0$, but $\alpha > 1$.

For $\alpha > 1$, the break frequency ($1/\alpha\tau$) given by the numerator is smaller than that ($1/\tau$) given by the denominator.

Recall: in a first-order lag system, $\alpha < 1$, and the break frequency ($1/\alpha\tau$) given by the numerator is greater than that ($1/\tau$) given by the denominator.

The Bode magnitude plot

For a first-order lead system, in which $\alpha > 1$, the magnitude increase from a level to a higher level as frequency increases.

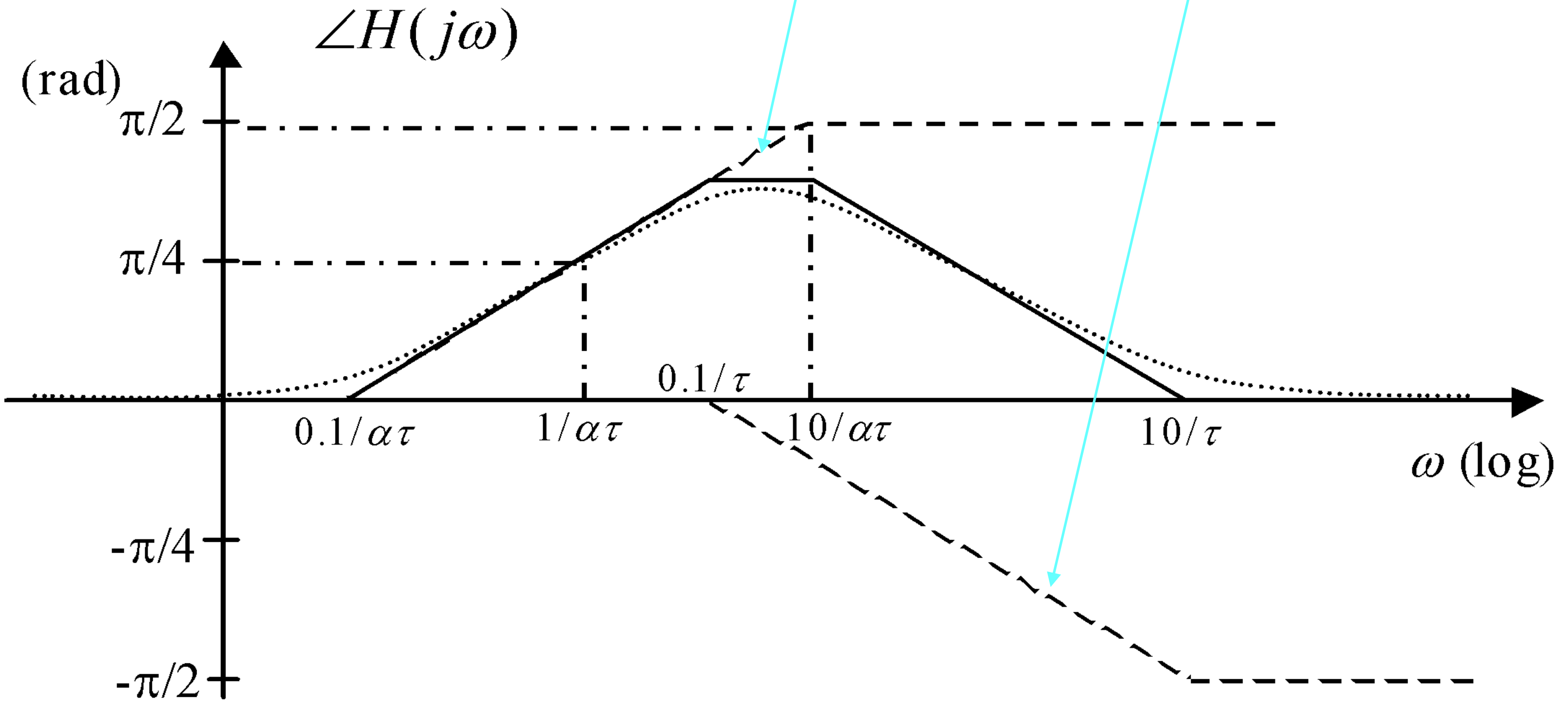


Compare with the first-order lag system, in which $\alpha < 1$, the magnitude decrease from a level to a lower level for a system.

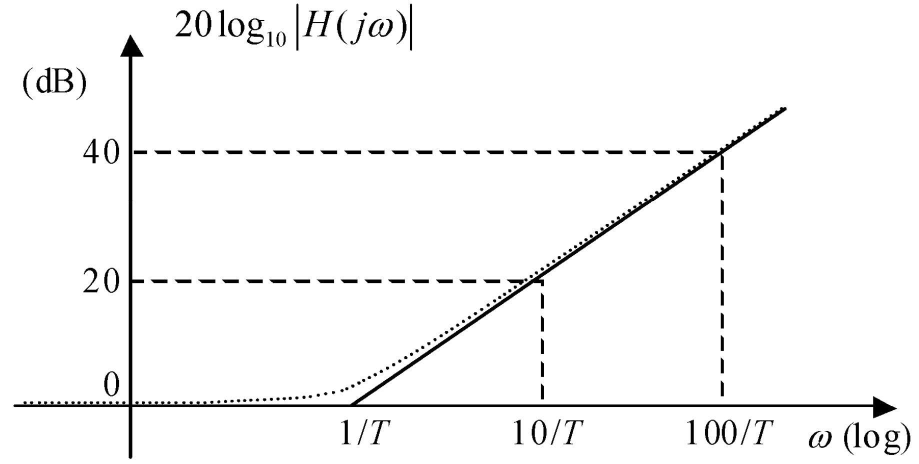
The Bode phase plot

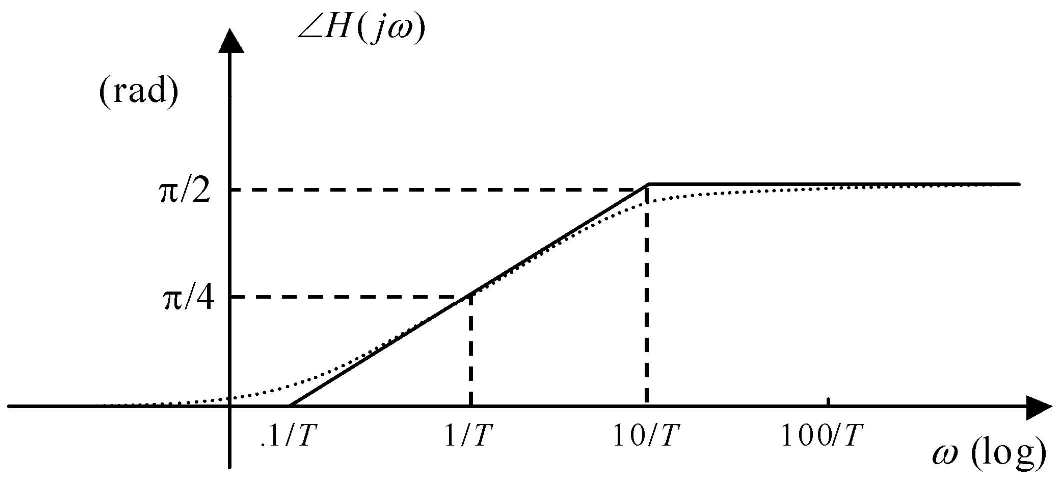
$$H(s)=\frac{\alpha\tau s+1}{\tau s+1}\qquad \mathrm{Re}\{s\}>-\frac{1}{\tau}$$

$$\angle H(j\omega)=\angle\frac{j\omega\alpha\tau+1}{j\omega\tau+1}=\angle(j\omega\alpha\tau+1)+\angle\frac{1}{j\omega\tau+1}$$



For the case where $\tau \to 0,\ \alpha\tau \to T$, then the first-order lead is equivalent to a differentiator with gain T in parallel with the identity system: $H(s) = Ts + 1$





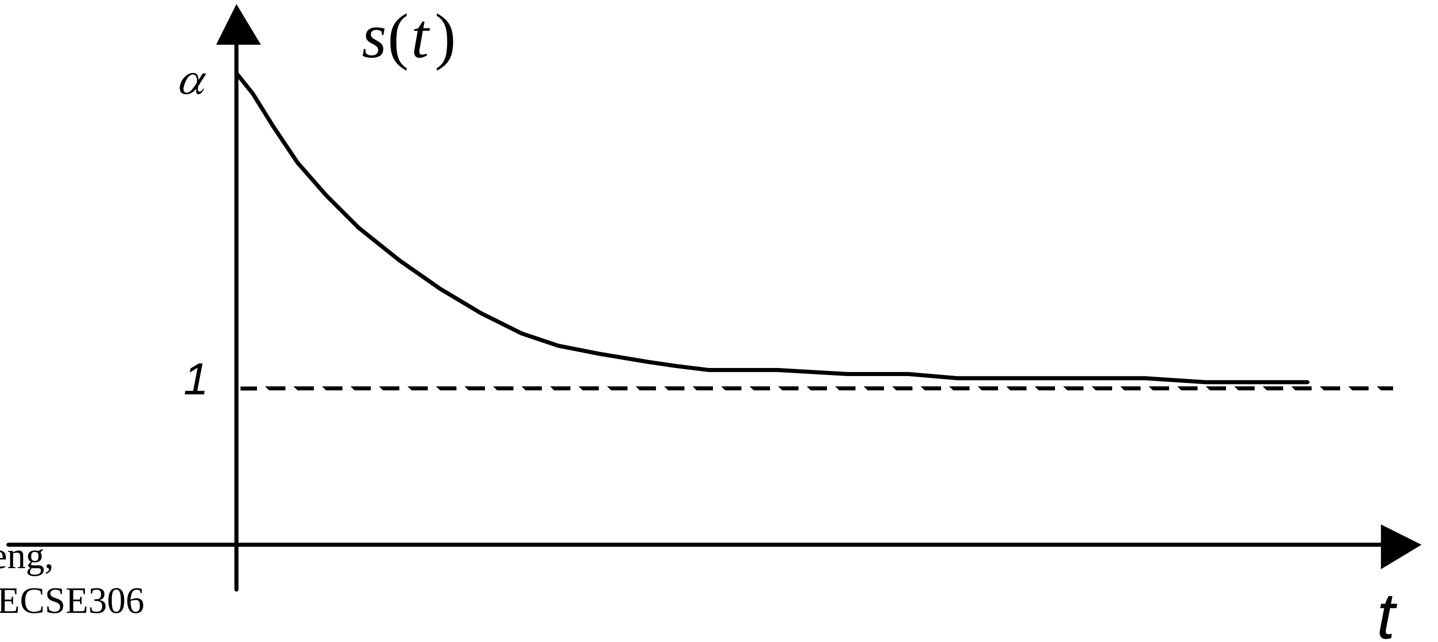
The step response of the first-order lead system

For the case $\alpha > 1$ (lead) and time constant τ, i.e.,

$$H(s) = \frac{\alpha\tau s + 1}{\tau s + 1} = \alpha + \frac{1-\alpha}{\tau s + 1},$$

the step response is

$$s(t) = \alpha u(t) + (1-\alpha)(1 - e^{-\frac{t}{\tau}})u(t).$$



Applications of first-order lead systems

The first-order lead may be used

- To "differentiate" signals at frequencies higher than $(\alpha\tau)^{-1}$ but lower than τ^{-1}.
- To "reshape" pulses that could have been distorted by a communication channel with a lowpass frequency response.
- As a controller because it adds positive phase to the overall loop transfer function .