## Proof of the Final Value Theorem for Z-Transforms

The final value theorem for z-transforms states that if $\lim_{k\to\infty} x(k)$ exists, then

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(z-1)X(z). \tag{1}$$

**Proof:** For no apparent reason, let's take the z-transform of the quantity $x(k+1) - x(k)$,

$$\begin{aligned}
\mathcal{Z}\left[x(k+1) - x(k)\right] &= \sum_{k=0}^{\infty} \left(x(k+1) - x(k)\right) z^{-k} && (2)\\
&= \lim_{k\to\infty} \left( \sum_{n=0}^{k} \left(x(n+1) - x(n)\right) z^{-n} \right) && (3)\\
&= \lim_{k\to\infty} \left( \sum_{n=0}^{k} x(n+1) z^{-n} - \sum_{n=0}^{k} x(n) z^{-n} \right) && (4)\\
&= \lim_{k\to\infty} \Big( -x(0) + x(1)(1-z^{-1}) + \cdots + x(\ell) z^{-(\ell-1)}(1-z^{-1}) + \\
&= \qquad \cdots + x(n) z^{-(n+1)}(1-z^{-1}) + x(n+1) z^{-n} \Big), && (5)
\end{aligned}$$

where we used the definition of the z-transform, then expressed the infinite sum as the limit of a sequence of finite sums, then rewrote the sum of the difference as the difference of sums, then expanded the sums and rearranged the terms.

Next let's take the limit of both sides as $z$ goes to 1 to obtain

$$\begin{aligned}
\lim_{z\to 1} \mathcal{Z}[x(k+1) - x(k)] &= \lim_{z\to 1} \lim_{k\to\infty} \Big( -x(0) + \cdots + x(\ell) z^{-(\ell+1)}(1-z^{-1}) + \cdots + x(n+1) z^{-n} \Big) \\
&= \lim_{k\to\infty} \lim_{z\to 1} \Big( -x(0) + \cdots + x(\ell) z^{-(\ell+1)}(1-z^{-1}) + \cdots + x(n+1) z^{-n} \Big) \\
&= -x(0) + \lim_{k\to\infty} x(k+1). && (6)
\end{aligned}$$

We were allowed to interchange the limits because we had required that the sum converge for the theorem to be applicable.

On the other hand, applying the real translation property, we have that

$$\begin{aligned}
\mathcal{Z}\left[x(k+1) - x(k)\right] &= z\left(X(z) - x(0)\right) - X(z) && (7)\\
&= (z-1)X(z) - zx(0) && (8)
\end{aligned}$$

so

$$\begin{aligned}
\lim_{z\to 1} \mathcal{Z}\left[x(k+1) - x(k)\right] &= \lim_{z\to 1} \left((z-1)X(z) - zx(0)\right) && (9)\\
&= \lim_{z\to 1}(z-1)X(z) - \lim_{z\to 1} zx(0) && (10)\\
&= -x(0) + \lim_{z\to 1}(z-1)X(z) && (11)
\end{aligned}$$

Thus, equating (6) and (11) we have

$$-x(0) + \lim_{k\to\infty} x(k) = -x(0) + \lim_{z\to 1}(z-1)X(z) \tag{12}$$

and the result follows.