ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University
Department of Electrical and Computer Engineering

Lecture 1

Hui Qun Deng, PhD

1. Administrative details
2. What are systems
3. What are signals
4. Continuous-time and discrete-time signals
5. Transformations of time variable
6. Properties of signals

Course Details (see syllabus)

Schedule

Lectures	Sep. 2- Dec.2	MWF	10:30 – 11:25	ENGTR 1100
	Dec. 2	T	10:30 – 11:25	ENGTR 1100
Tutorials	Sep. 8 – Dec. 7	W?	02:35 – 04:25?	ENGTR 0060?
		T?	04:35 – 06:25?	ENGTR 2120?

Required Text:

B. Boulet, ***Fundamentals of Signals and Systems***, Da Vinci Engineering Press, Charles River Media, 2006.

Suggested Text:

A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, ***Signals and Systems***, 2nd Eddition. Prentice Hall, New Jersey, 1997

Grading:

10 Assignments: 10%

Midterm 1 (Sep. 29): 20%

Midterm 2 (Nov. 5): 20%

Final exam (TBD): 50%

Acknowledgement

Thanks to professor Benoit Boulet for allowing me to adapt his slides for this course.

Thanks to professor Richard Rose for advices on teaching this course.

What is the course about

- Mathematical formulations of signals and systems
- Methods for describing, analyzing the responses of systems to input signals
  - Fourier Series
  - Fourier Transform
  - Laplace Transform
  - Z-Transform

How to be successful in the course

- Pay full attention to lectures and tutorials
- Carefully read textbooks
- Test your understanding and proficiency about concepts and methods by doing exercises and assignments
- Do your best in exams

What is a system

A system is a combination of components that act together to perform a function not possible with any of the individual parts

IEEE: Electronic Terms http://www.mame.mu.oz.au/mechatronics/imse/imse_ppt/01des.pdf.

Examples:

An RLC circuit

An algorithm for processing data

An equation describing input and output relationships

A car or an aircraft

What is a signal

A signal is a function of independent variables.

Examples:

- Electrical signals: voltages, currents, …
- Acoustic signals: music, speech, car noise, …
- Optical signals: pixel intensities in an image, …
- Biological signals: neuron firing signals, blood pressure, …
- Natural signals: earth quake signals, air pressure, wind speed, temperature, sea level, …
- Social signals: population, financial data, …

All signals contain information about their underlying systems.

The independent variables of signals

- Can be 1-D, 2-D, …, N-D
- Can be continuous
  e.g., the temperature of a day, the speed of a car, the air pressure …
- Can be discrete
  e.g., the instants of taking data, the population, …

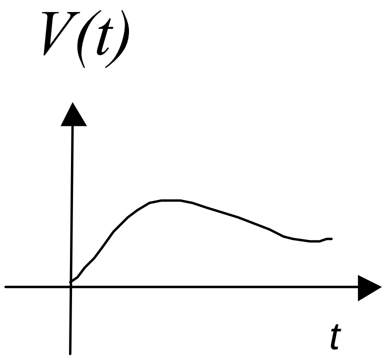
This course focuses on a 1-D independent variable denoted as "time".

Continuous-time (CT) signals

- Continuous-time signals are functions of a continuous variable (time).

Example:

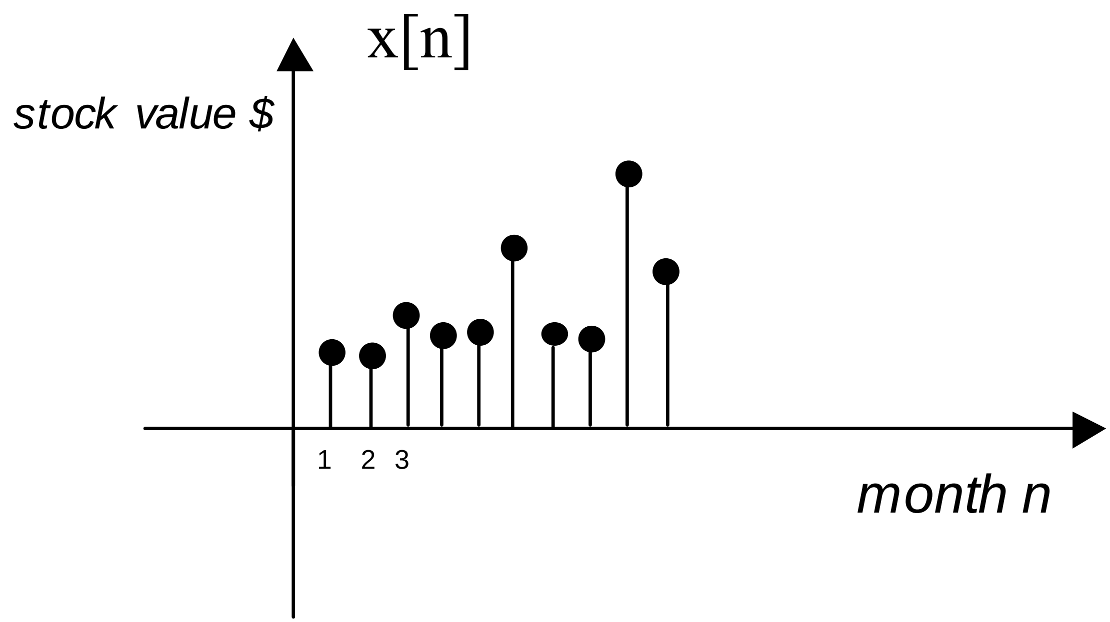
The speed of a car



Discrete-time (DT) signals

Discrete-time signals are functions of a discrete variable, i.e., they are defined only for discontinuous values of the independent variable (time instants, …).

Example: The value of a stock at the end of each month



Transformations of the time variable

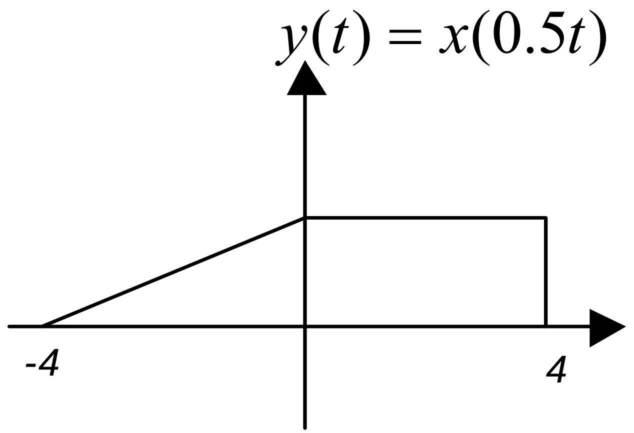
1. Time scaling
2. Time reversal
3. Time shift

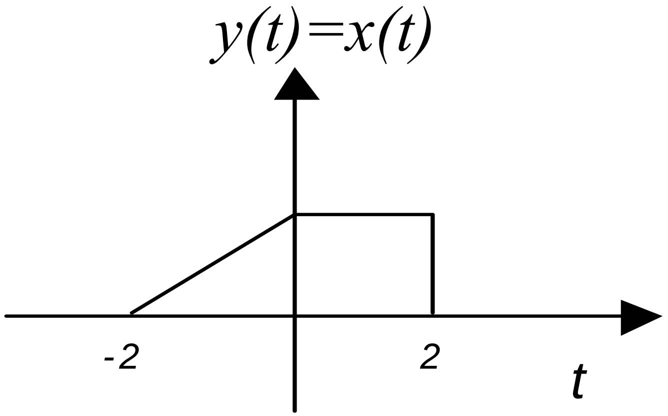
Time scaling: $y(t)=x(\alpha t)$

Multiplying the time variable by a real positive constant.

Case $0<\alpha<1$: The signal is "slowed down" or "expanded".

Example: $\alpha=0.5$

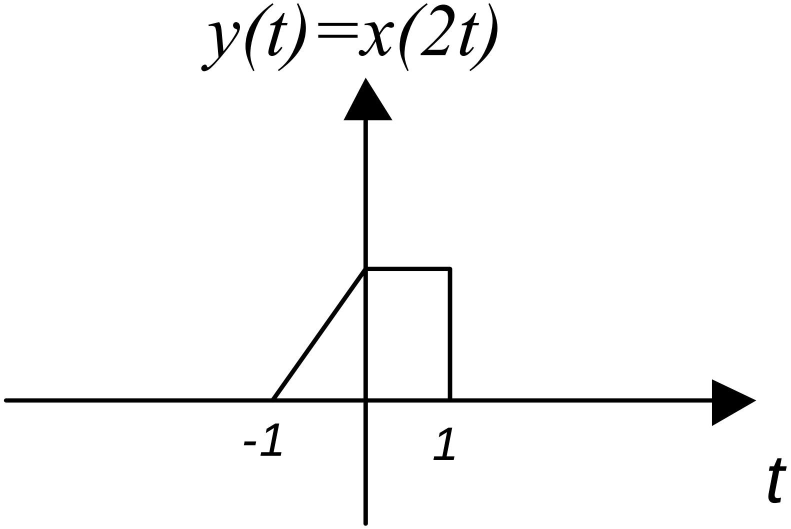


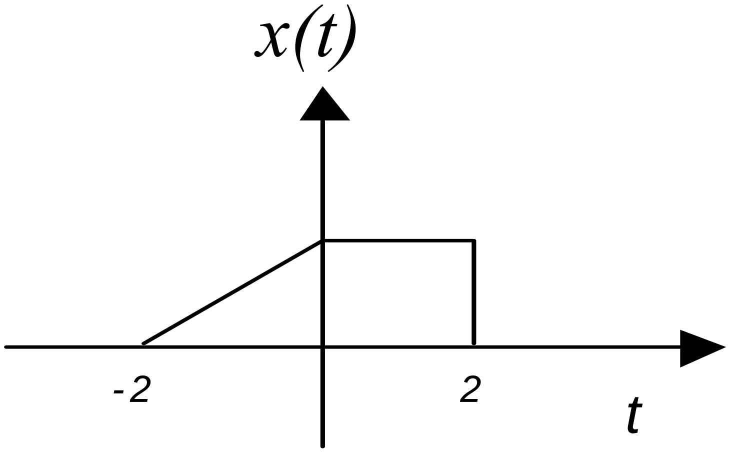


Time scaling case 2: time compression

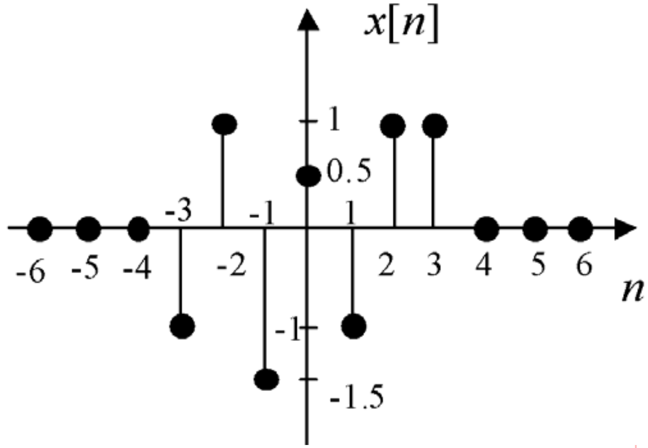
Case $a > 1$: The signal is "sped up" or "compressed"

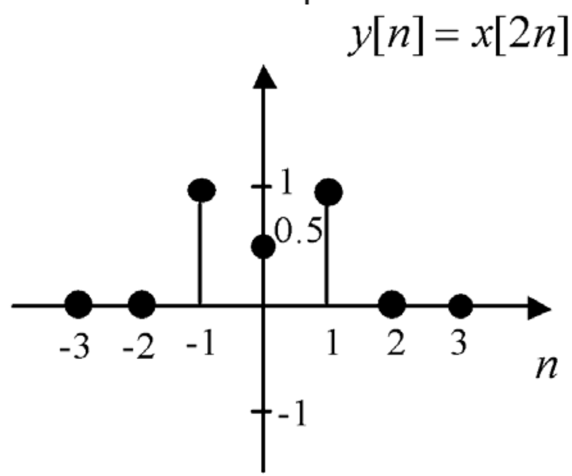
Example: $a = 2$





Time compression of DT signals (decimation, down-sampling)

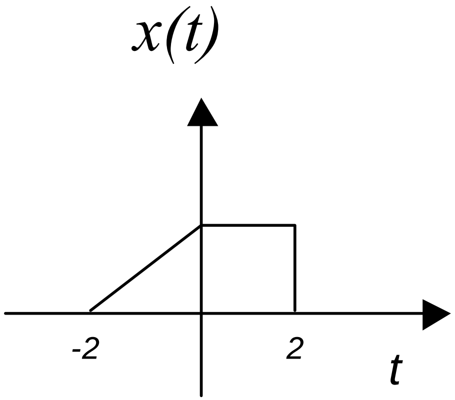


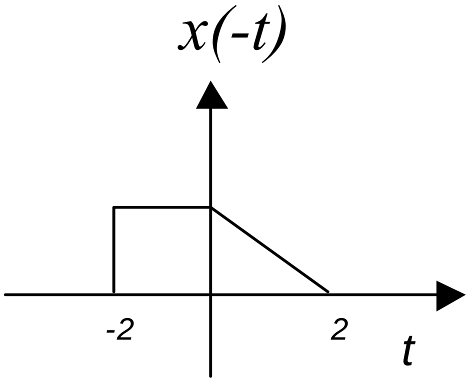


Be careful: down-sampling may introduce aliasing noise, as seen later in the course.

Time reversal: $y(t)=x(-t)$

A time reversal is achieved by multiplying the time variable by -1

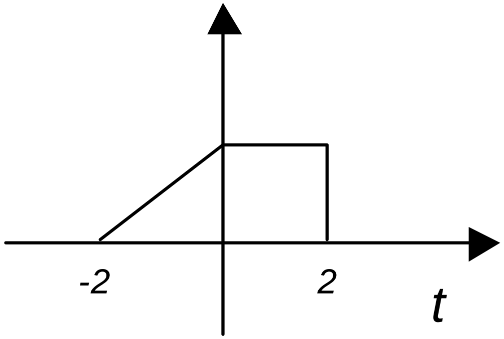




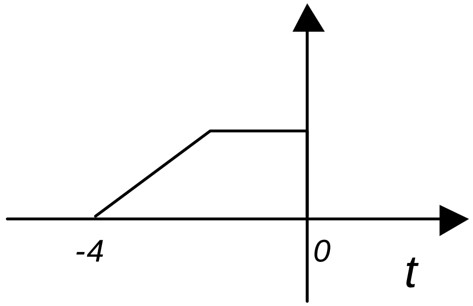
Time shift: $y(t)=x(t+T)$

A time shift delays or advances the signal in time by a time interval

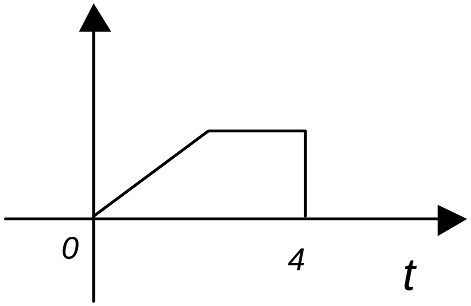
$y=x(t)$



$y(t)=x(t+2)$



$y(t)=x(t-2)$



Possible properties of signals

Periodic

Finite-energy

Finite-power

Even

Odd

Periodic signals

A continuous-time signal x(t) is periodic if there exists a positive real T for which:

$$x(t) = x(t+T) \tag{1.5}$$

A discrete-time signal $x[n]$ is periodic if there exists a positive integer N for which:

$$x[n] = x[n+N] \tag{1.6}$$

The smallest such T or N is called the *fundamental period*.

The power and energy of a CT signal

The average power of a CT signal is defined as:

$$P_\infty := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

It is the average power dissipated in a one-ohm resistor, assuming *x(t)* is the voltage of a one-ohm resistor.

The total energy of a CT signal is defined as:

$$E_\infty := \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

The power and energy of a DT signal

The average power of a DT signal is defined as:

$$P_\infty := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| x[n] \right|^2$$

The total energy of a DT signal is defined as:

$$E_\infty := \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

Finite-energy and finite-power signals

Finite-Energy Signals: signals with $E_\infty < \infty$.

Finite-Power Signals: signals for which $P_\infty < \infty$.

Examples:

$x(t)=4$ has infinite energy but an average power of 16.

$e^{j\omega t}$ has unit power over one period.

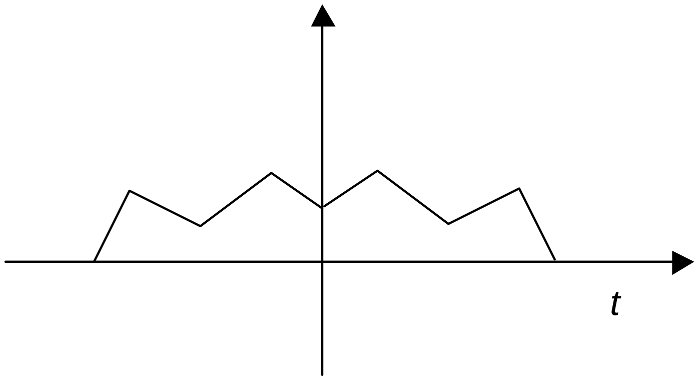
Check them yourself.

Even and odd signals

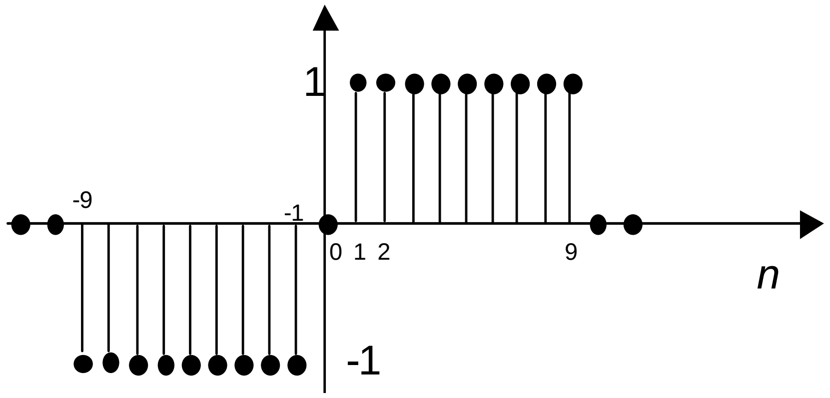
A signal is *even* if $$x(t) = x(-t), \quad x[n] = x[-n]$$

A signal is *odd* if $$x(t) = -x(-t), \quad x[n] = -x[-n]$$

$x(t)$ even



$x[n]$ odd



Even and odd components of a signal

A signal can be decomposed as:

$$x(t) = x_{even}(t) + x_{odd}(t) \qquad (1.45)$$

Even component:

$$x_{even}(t) = \frac{x(t) + x(-t)}{2} \qquad (1.46)$$

Odd component:

$$x_{odd}(t) = \frac{x(t) - x(-t)}{2} \qquad (1.47)$$

Derive and check them!

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Lecture 2

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1. Exponential and sinusoidal signals
2. DT and CT unit step signals
3. DT and CT unit impulse signals
4. System models

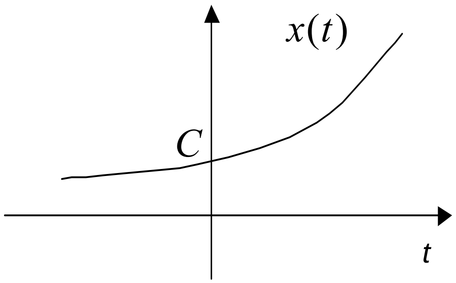
CT Exponential Signals

$$x(t) = Ce^{at}, \quad C, a \text{ real}$$

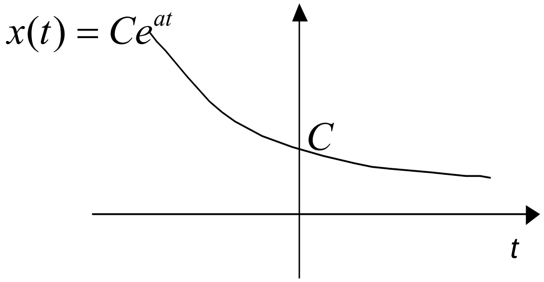
(1.10)

Case a=0: We get the constant signal $x(t) = C$.

Case a>0: The exponential tends to infinity as $t \to \infty$ (here C>0).



Case a<0: The exponential tends to zero as $t \to \infty$.



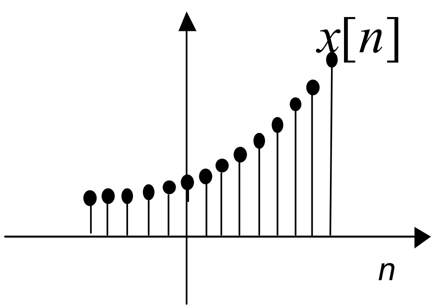
DT real exponential signals

$$x[n] = C\alpha^n, \quad C, \alpha \text{ real} \qquad (1.11)$$

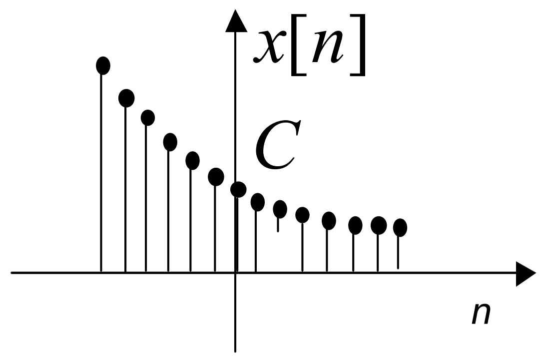
Six cases: $\alpha = 1$, $\alpha > 1$, $0 < \alpha < 1$, $\alpha < -1$, $\alpha = -1$ and $-1 < \alpha < 0$. Here we assume that C>0.

Case $\alpha = 1$: We get a constant signal $x[n] = C$.

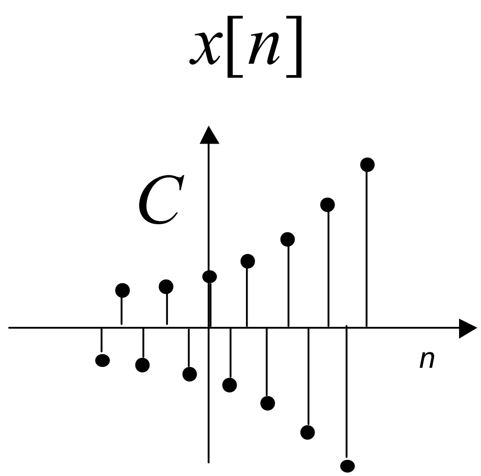
Case $\alpha > 1$: We get a positive signal that grows exponentially.



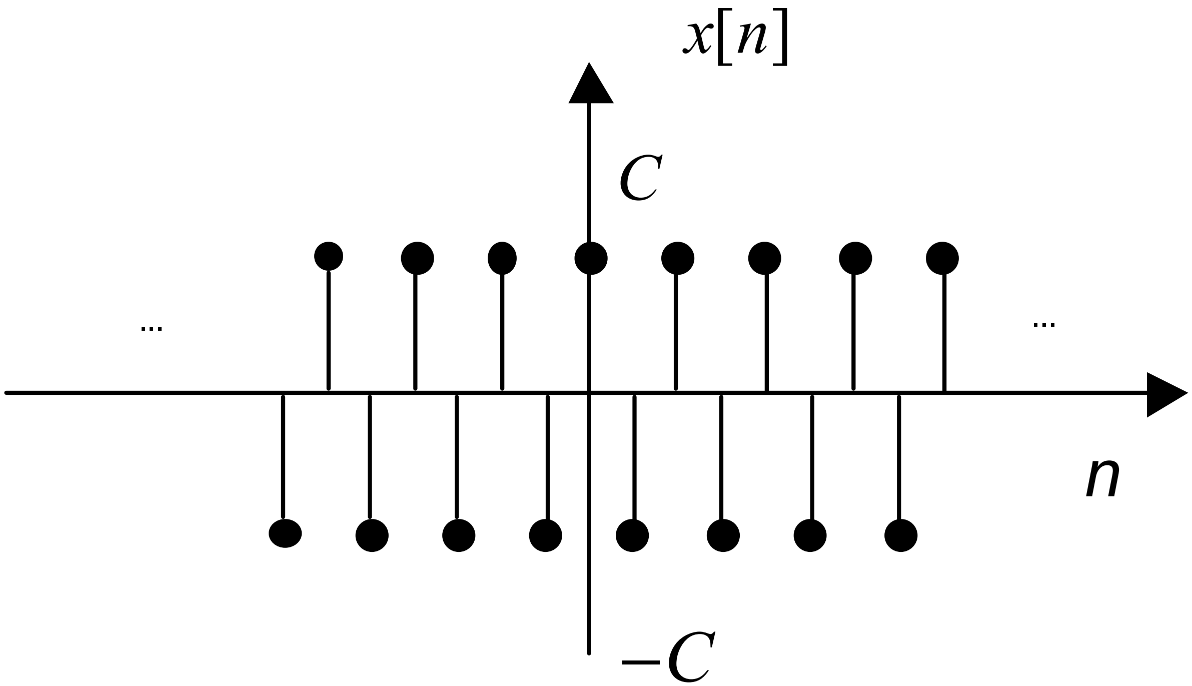
Case $0 < \alpha < 1$: The signal is positive and decays exponentially.



Case $\alpha < -1$: The signal alternates between positive and negative values and grows exponentially.



Case $\alpha = -1$: The signal alternates between +C and -C.



Case $-1 < \alpha < 0$: *assignment 1.1.*

CT complex exponential signals

$$x(t) = Ce^{at}$$

$$C, a \text{ complex}, \quad C = Ae^{j\theta}, \quad a = \alpha + j\omega_0 \qquad (1.12)$$

$$x(t) = Ce^{at} = Ae^{j\theta}e^{(\alpha + j\omega_0)t} = Ae^{\alpha t}e^{j(\omega_0 t + \theta)} \qquad (1.13)$$

Using Euler's relation, we get real part and imaginary part of the signal

$$x(t) = Ae^{\alpha t}\cos(\omega_0 t + \theta) + jAe^{\alpha t}\sin(\omega_0 t + \theta)$$

Sketch *x(t),* given different α values. See B.Boulet's p.12-15.

Harmonics of CT signals

Periodic exponential (or sinusoidal) signals are *harmonics* if their fundamental frequencies are integer multiples of a frequency.

$$e^{jk\omega_0 t}, \quad k = \ldots, -1, 0, 1, \ldots \qquad (1.20)$$

Fundamental frequency: $\omega_0 / 2\pi$ (Hertz)

Fundamental period: $2\pi / \omega_0$ (Second)

Fundamental angular frequency: ω_0 (radians/second)

There are infinite number of CT harmonics.

Orthogonal CT signals

Two signals x(t) and y(t) are orthogonal over an interval $[t_1, t_2]$ if their inner product is equal to zero:

$$\int_{t_1}^{t_2} x(t)^* y(t) dt = 0 \qquad (1.21)$$

$X^*(t)$ is the complex conjugate of $x(t)$.

Distinct harmonics are orthogonal over their fundamental period:

$$\int_0^{\frac{2\pi}{\omega_0}} e^{-jk\omega_0 t} e^{jm\omega_0 t} dt = \begin{cases} 0 & k \neq m \qquad (1.23) \\ \frac{2\pi}{\omega_0} & k = m \qquad (1.24) \end{cases}$$

DT complex exponential signals

$$e^{j\omega n}$$

It is NOT periodic, **unless**

$$\omega = \frac{m}{N} 2\pi \qquad (1.30)$$

If m and N have no common factors, then the *fundamental frequency is:* $\Omega_0 = 2\pi/N$ (in radians).

The *fundamental period* of the DT exponential signal is:

$$N = \frac{m}{\omega} 2\pi$$

Harmonics of DT signals

$$e^{jk\frac{2\pi}{N}n}, \qquad k=0,\dots,N-1$$

There are N distinct DT complex harmonics.

Harmonics of DT signals are also orthogonal:

$$\sum_{n=0}^{N-1} e^{-jk\frac{2\pi}{N}n} e^{jm\frac{2\pi}{N}n} = \begin{cases} 0, & k \neq m \\ N, & k = m \end{cases} \qquad (1.28)$$

finite sum formula

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \end{cases}$$

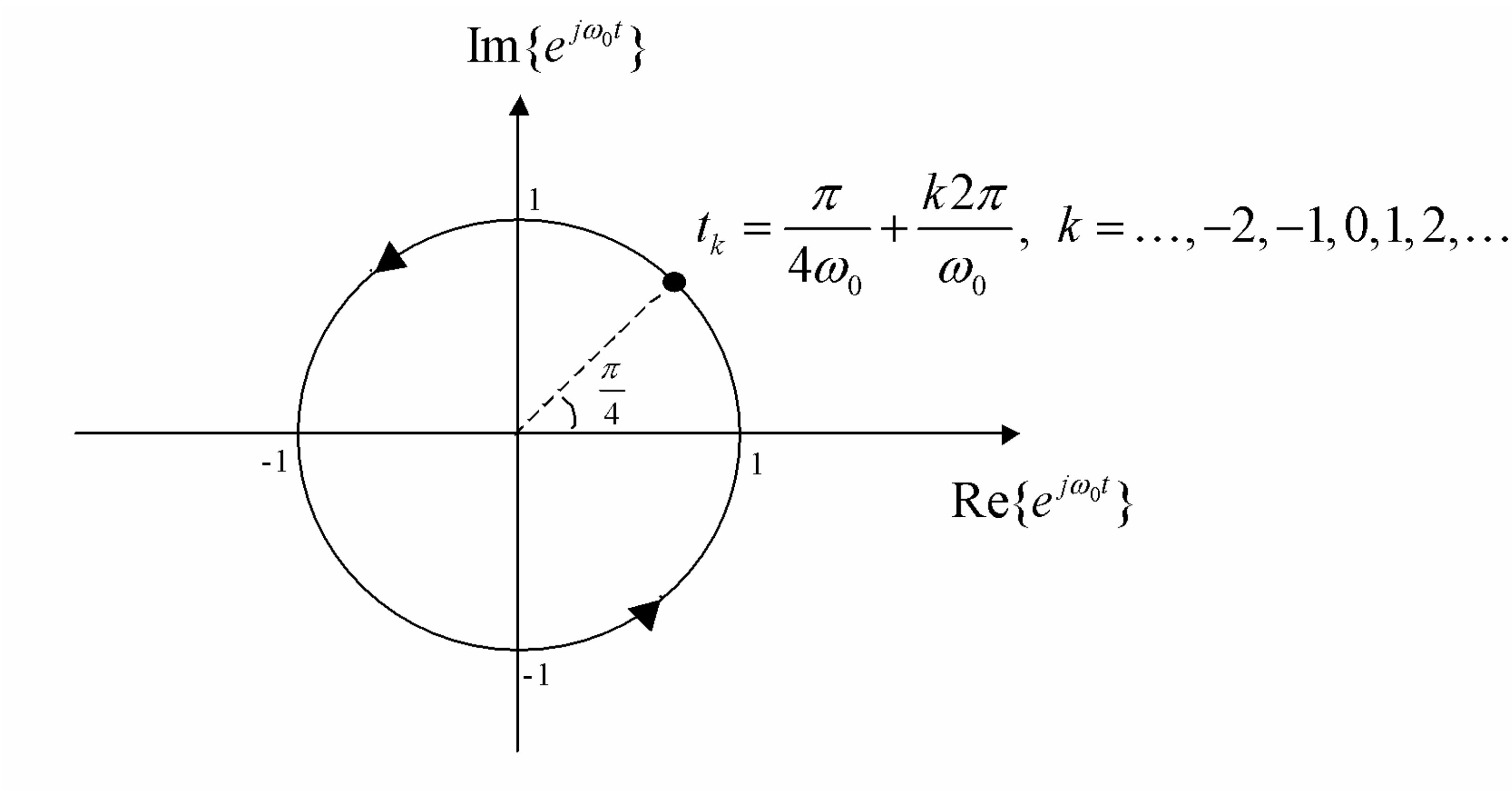
Note: Two DT signals are orthogonal if their inner product is zero:

$$\sum_{n=0}^{N-1} x[n]^* y[n] = 0$$

Compare $e^{j\omega t}$ and $e^{j\omega n}$

	$e^{j\omega t}$	$e^{j\omega n}$
Periodicity	Always periodic	Only if $\omega=2\pi m/N$
Fundamental period	$2\pi/\omega$	$N = 2\pi m/\omega$
Fundamental frequency	ω (radians/sec)	$2\pi/N$ (radians)

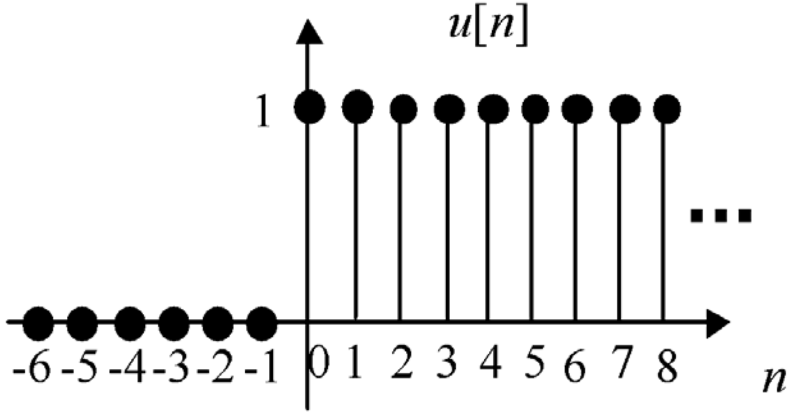
Representing complex signals in the complex plan



DT Unit Step Function (Signal)

The discrete-time *unit step* function is defined as:

$$u[n] := \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



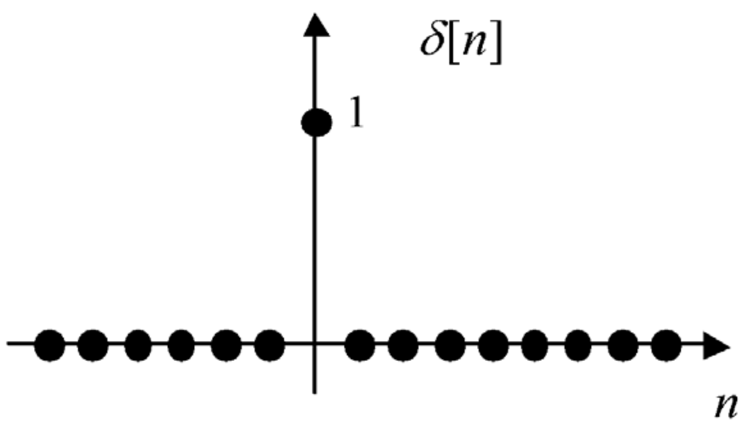
DT Unit Impulse Function (Signal)

The discrete-time *unit impulse* function is defined as:

$$\delta[n] := \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

Then, we have

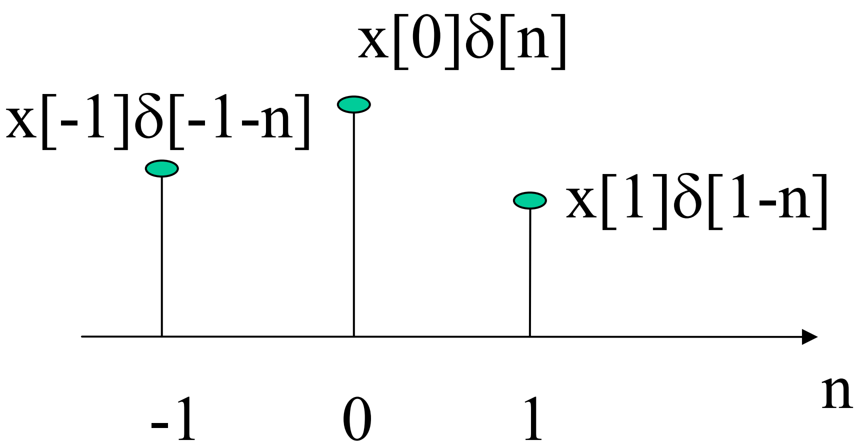
$$\delta(n) = u[n] - u[n-1]$$

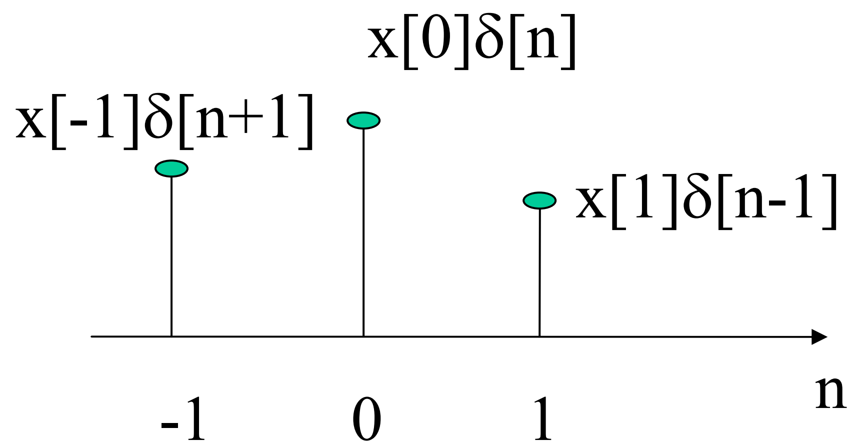


Representing DT signals in terms of impulse signals

$$u[n]=\sum_{k=-\infty}^{n}\delta[k]$$

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[k-n] \qquad x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$



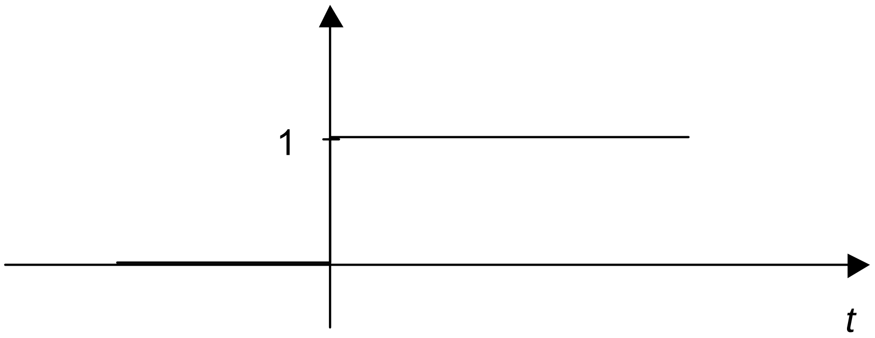


CT Unit Step Function (Signal)

The continuous-time *unit step* function is defined as:

$$u(t) := \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

$$u(t)$$

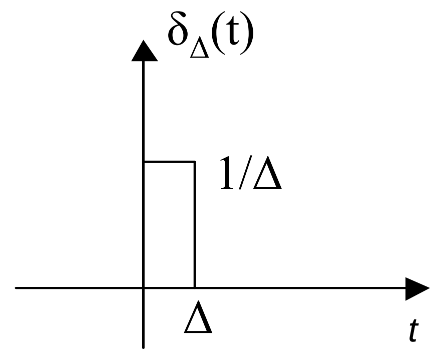


CT Unit Impulse Function (Signal)

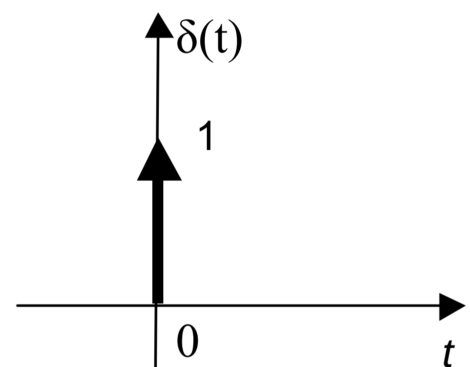
The continuous-time *unit impulse* function is defined as:

$$\delta(t) := \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$\delta_\Delta(t) := \begin{cases} \dfrac{1}{\Delta}, & 0 < t < \Delta \\ 0, & \textit{otherwise} \end{cases}$$



δ(t) is represented using an ***arrow*** at t=0, with height equal to the **area** of the impulse, "1".



Physical meaning of CT impulse signal δ(t)

An idealized signal, for which

- Energy is finite
- Amplitude is "infinite" large or very large
- Duration is "infinite short" or ***short enough*** to a system

The response of a system to the impulse signal is called ***impulse response***

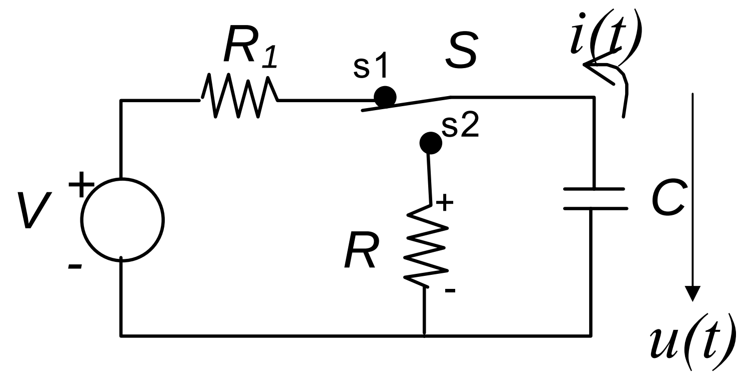
Examples of impulse signals

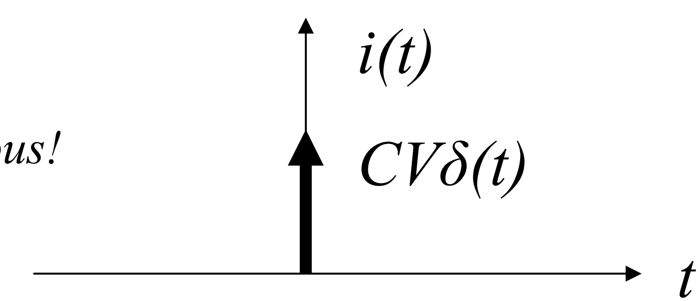
1. The sound of a gun shoot
2. The current *i(t)* of a capacitor discharging through a zero-ohm resistor R (s2 is on)

The current through the zero-ohm resistor R *i(t)* can be modeled using the impulse signal:

$$i(t) = \frac{u(t)}{R} = -CV\frac{du(t)}{dt} = CV\delta(t)$$

$$u(t) = Ve^{\frac{-t}{RC}}$$





Note: shorting a capacitor can get a huge current and burn the device, and is dangerous!

Properties of the unit impulse signal

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t) \qquad (1.62)$$

$$\frac{du(t)}{dt} = \delta(t)$$

Properties of the unit impulse signal

Sampling property:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \qquad (1.68)$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)x(t)dt = x(t_0)$$

Assuming x(t) is continuous at t_0.

Time scaling:

$$\int_{-\infty}^{\infty} x(t)\delta(\alpha t)dt = \frac{1}{|\alpha|}x(0) \qquad (1.71)$$

Time shifting:

$$\int_{-\infty}^{\infty} \delta(\tau-T)x(t-\tau)d\tau = x(t-T) \qquad (1.73)$$

Convolution of CT signals

$$x(t) * y(t) := \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau = \int_{-\infty}^{\infty} y(\tau) x(t-\tau) d\tau$$

Deductions:

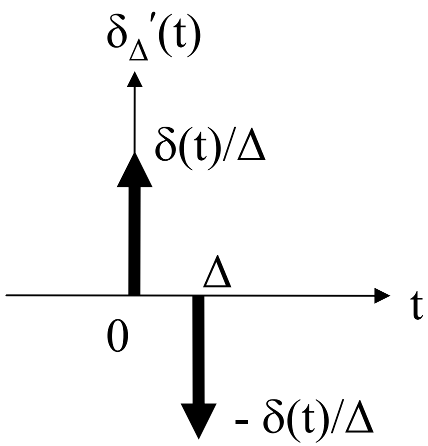
$$x(t) * y(t) = y(t) * x(t)$$

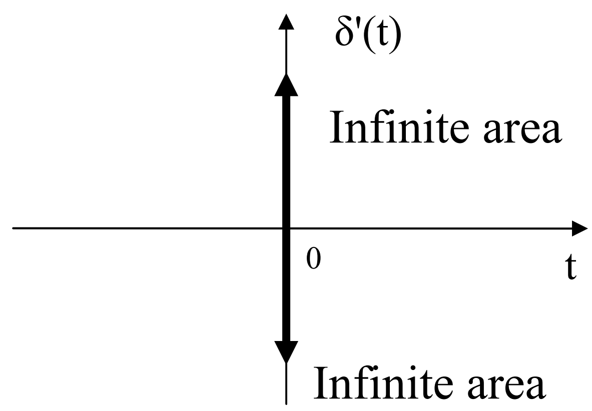
$$x(t) * \delta(t) = x(t)$$

Derivatives of the unit impulse signal

Unit doublet: $$\delta'(t) := \frac{d\delta(t)}{dt}$$

$$\frac{d\delta(t)}{dt} = \lim_{\Delta \to 0} \frac{d\delta_\Delta(t)}{dt}$$





Properties of unit doublet

$$\int_{-\infty}^{\infty} \delta'(t)dt = 0$$

$$\int_{-\infty}^{\infty} \delta'(t)x(t)dt = -x'(0) \qquad (1.74)$$

$$\boxed{\int udv = uv - \int vdu}$$

$$\int_{-\infty}^{\infty} \delta^{(k)}(t)x(t)dt = (-1)^k x^{(k)}(0) \qquad (1.75)$$

System Models

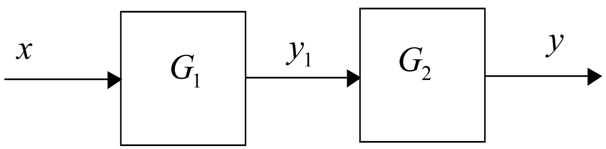
A system can be modeled as a mathematical relationship between its input signal and its output signal.

Different physical systems can have similar mathematical representations.

E.g., an RLC circuit and mass-spring system can be described using similar deferential equations.

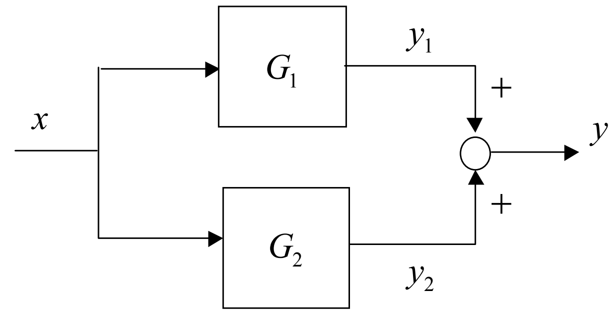
Input-Output Models of Systems

Cascade Interconnection



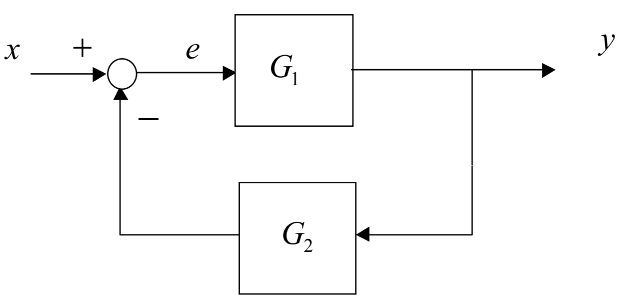
$$y = G_2 G_1 x$$

Parallel Interconnection



$$y = G_1 x + G_2 x$$

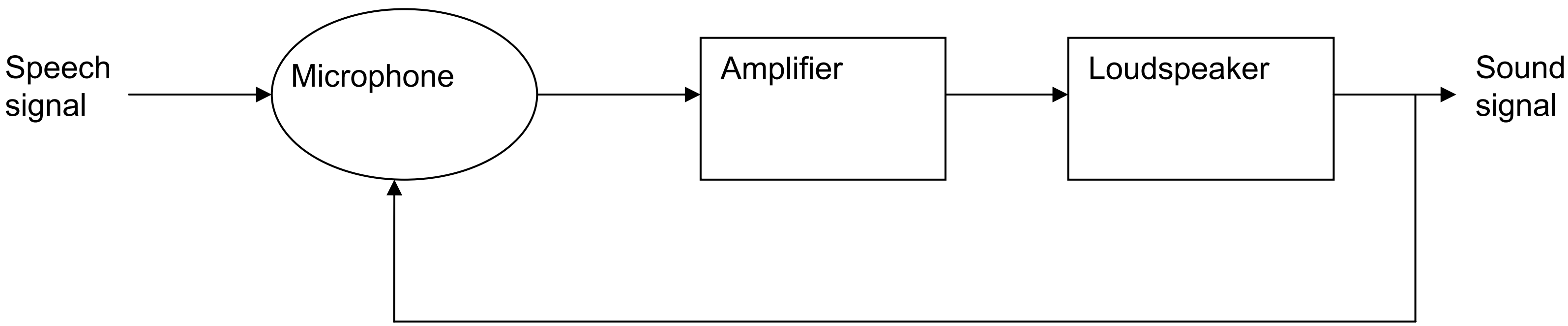
Feedback Interconnection



$$e = x - G_2 y$$

$$y = G_1 e$$

Examples



Knowing the feedback helps design a system to avoid instability.

Lecture 3

Hui Qun Deng, PhD

Possible Properties of Systems

- Causality
- Linearity
- Time-invariance
- Invertibility
- Bounded-input bounded-output stability
- Memory-less

Causality

A system is *causal* if its output at time t (or n) depends only on past or current values of the input.

Otherwise the system is said to be *non-causal*.

consequence:

If $y_1 = Sx_1,\ y_2 = Sx_2$ and $x_1(\tau) = x_2(\tau)\ \forall\ \tau \in (-\infty, t]$, then $y_1(\tau) = y_2(\tau),\ \forall\ \tau \in (-\infty, t]$.

Examples

<u>Causal?</u>

$$y[n] = \sum_{k=-\infty}^{n} x[n-k]$$

Non-causal.

The output for negative times depends on future values of the input up to .

Note: Non-causal system can be used for processing recorded signals, or applications allowing delays.

Causal? y(t)=x(t)cos(t+10)

Linearity

A system S is linear if it has the *additivity* property and the *homogeneity* property.

Additivity: The response of S to the sum of two signals $x_1 + x_2$ is the sum of the individual responses $y_1 = Sx_1$ and $y_2 = Sx_2$.

$$y_1 + y_2 = S(x_1 + x_2)$$

Homogeneity (scaling): $ay_1 = S(ax_1)$, a is complex

Note: the output of a linear system is zero if the input is zero.

Principle of Superposition of linear systems

The additivity and homogeneity of linear systems are summarized as the principle of superposition:

The response of a linear system to a linear combination of input signals is the same linear combination of the corresponding output signals .

$$x_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_1(t) \qquad\qquad x_2(t) \longrightarrow \boxed{\text{System}} \longrightarrow y_2(t)$$

$$Ax_1(t) + Bx_1(t) \longrightarrow \boxed{\text{System}} \longrightarrow Ay_1(t) + By_2(t)$$

A and B are complex numbers.

Examples

Linear? $Y[n]=2x[n]+4$

No additivity. No homogeneity.

Linear? $y(t)=\mathrm{Re}\{x(t)\}$

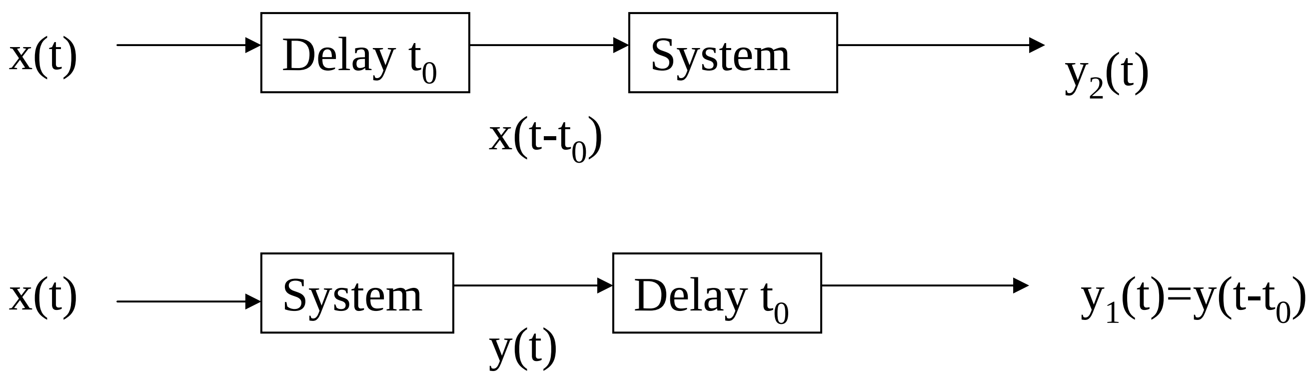
Additivity; but, no homogeneity for complex scaling.

Time-invariance

A system S is *time-invariant* if its response to a time-delayed input signal x[n-N] or $x(t-T)$ equals the time-delayed original response, i.e.,

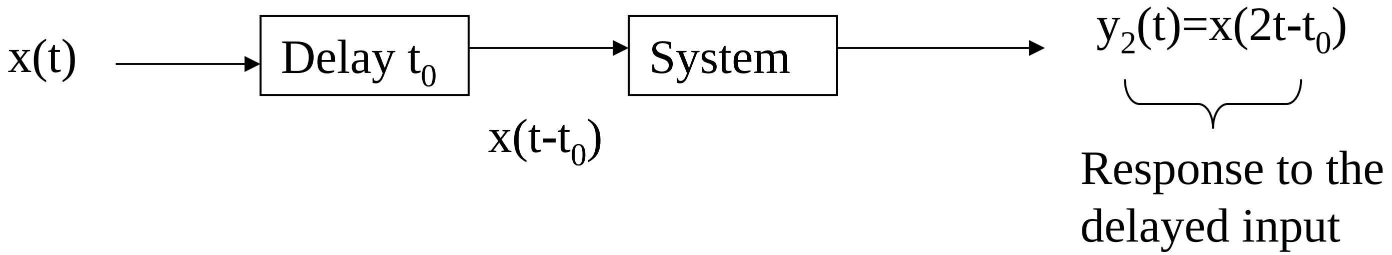
$$y_1[n] = Sx[n],\ y_2[n] = Sx[n-N],\ \text{then}\ y_2[n] = y_1[n-N]$$

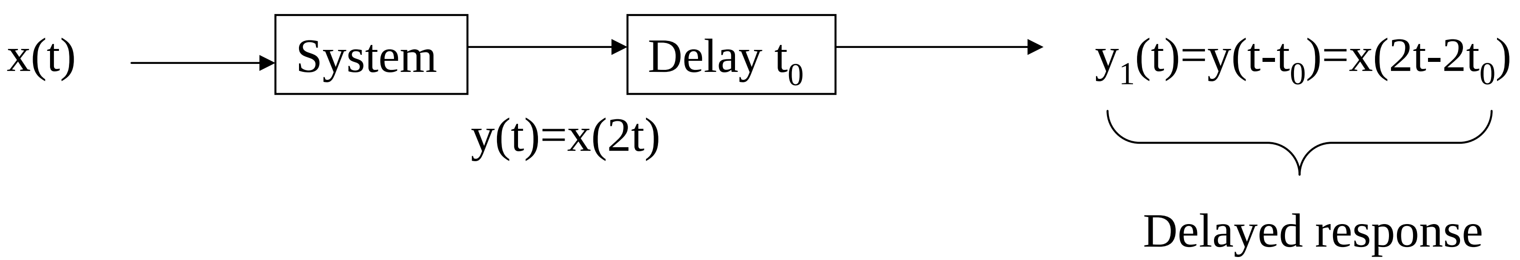
Illustrate above statements into diagrams of systems :



Example

Time-invariant? $y(t)=x(2t)$





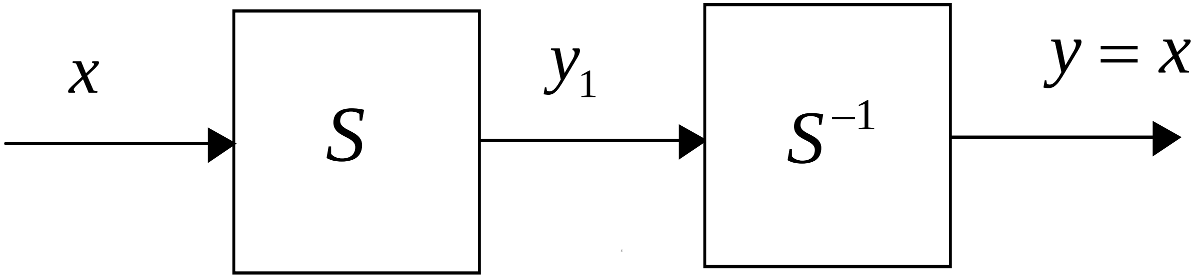
Because $y_1(t) \neq y_2(t)$, the system $y(t)=x(2t)$ is NOT time-invariant.

Invertibility

A system S is *invertible* if the input signal can be uniquely recovered from the output signal.

Mathematically, for $x_1 \neq x_2, \quad y_1 = Sx_1, \quad y_2 = Sx_2$, we have $y_1 \neq y_2$.

The inverse system S^{-1} is such that the following cascade interconnection is equivalent to the identity system.



Examples of Inverse Filters

1. The DT integrator and differentiator

x[n] → $y[n] = \sum_{k=-\infty}^{n} x[k]$ → $w[n] = y[n] - y[n-1]$ → w[n]=x[n]

2. The vocal-tract filter and its inverse filter

Glottal wave x[n] → $y[n] = \sum_{k=1}^{M} a_k y[n-k] + x[n]$ → $w[n] = y[n] - \sum_{k=1}^{M} a_k y[n-k]$ → w[n]=x[n]

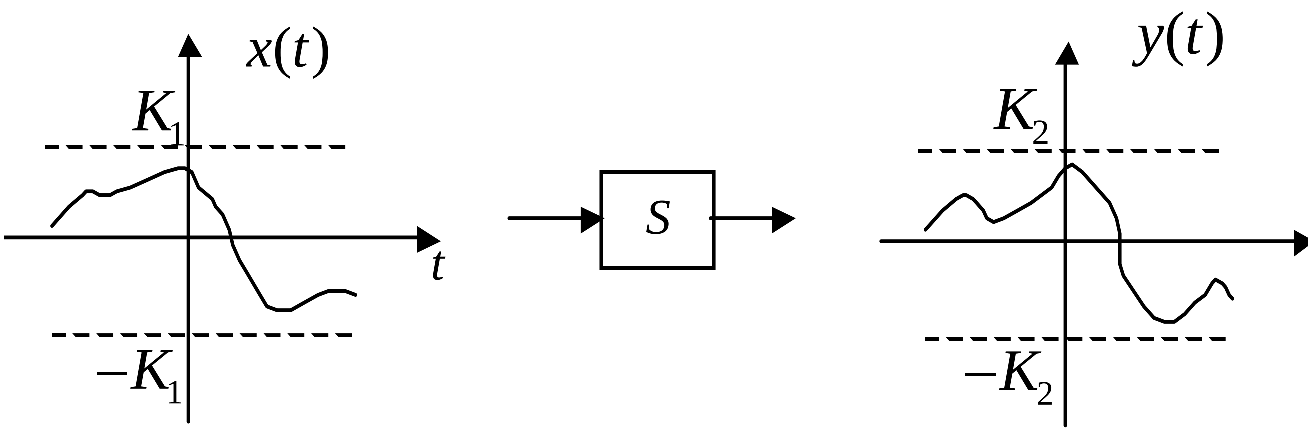
Speech signal y[n]

Bounded-Input Bounded-Output (BIBO) Stability

A system S is *BIBO stable* if for any bounded input x, the corresponding output y is also bounded.

$$| x(t) | < K_1, -\infty < t < \infty$$

$$\Rightarrow \quad | y(t) | < K_2, -\infty < t < \infty, \quad \text{where } y = Sx$$



BIBO stability is important to establish for feedback control systems, or filters.

Examples

1. The speed of a car and the force forms a BIBO system

2. $y(t)=tx(t)$

Stable?

NO.

Consider the input x(t) is bounded: $|x(t)|<B$.

For any large K, there exists t>T such that

$y(T)=|TB|>K$

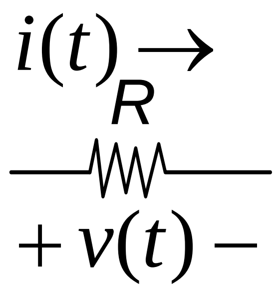
i.e., the output is unbounded.

Memory-less

A system is *memoy-less* if its output y at time t or n depends only on the input at that same time.

Examples: $y[n] = x[n]^2, \quad y(t) = \dfrac{x(t)}{1 + x(t)},$

Resistor



$v(t) = Ri(t)$

A system has memory if its output at time t or n depends on input values at some other times.

Examples

$y(t)=x^2(t)$ is memory-less

$y[n]=x[n]-x[n-1]$ has memory

McGill University
Department of Electrical and Computer Engineering

Lecture 4

Hui Qun Deng, PhD

From Sep. 15 to Dec. 2, the classroom is changed to ENGTR 2120

1. Linear time-invariant systems
2. Discrete-time convolution sum
3. Properties of convolution

Linear Time-Invariant Systems

- Systems with linearity and time-invariance are called linear time-invariant (LTI) systems
- LTI systems have properties:
  - Superposition
  - Time-invariance

The Impulse Response of a LTI System

The response a LTI system to an impulse signal is called ***impulse response***, *denoted as* $h(t)$, or $h[n]$

$\delta(t) \longrightarrow$ CT LTI System $\longrightarrow h(t)$

$\delta[n] \longrightarrow$ DT LTI System $\longrightarrow h[n]$

Impulse Responses Represent LTI Systems

h(t), or **h[n]** can represent a LTI system because

the response of a LTI system to an arbitrary input signal is a linear combination of time-shifted h(t) or h[n]

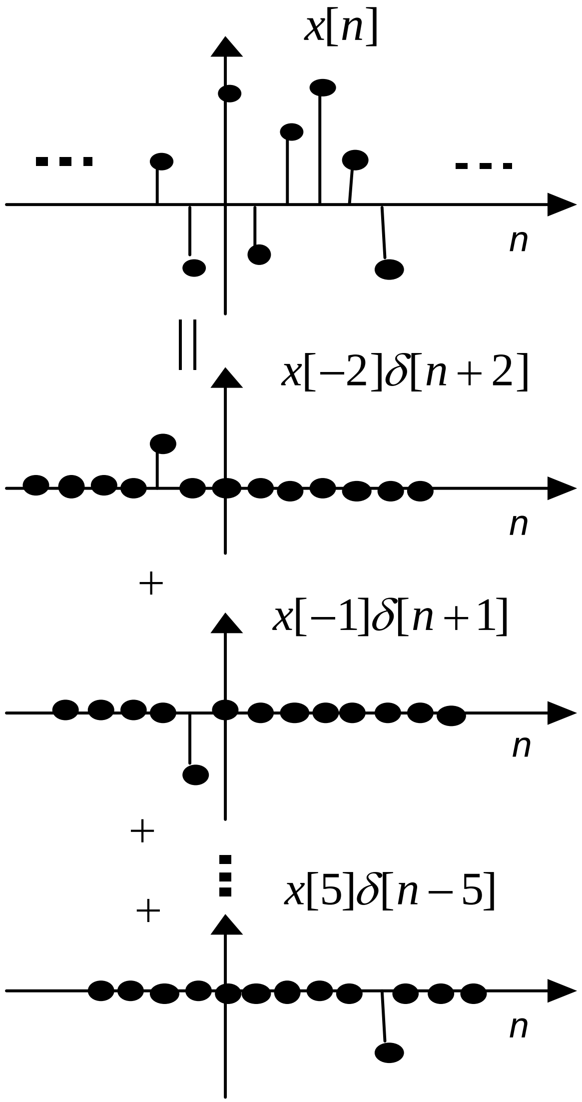
Decomposing DT Signals into Impulses

A DT signal is a linear combination of time-shifted impulses:

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \qquad (2.1)$$

$$x[n]=......+x[-2]\delta[n+2]+x[-1]\delta[n+1]+x[0]\delta[n]$$
$$+x[1]\delta[n-1]+x[2]\delta[n-2]+......$$

Example



The Response of a DT LTI System to x[n]

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow \mathbf{y[n]=?}$$

From ***time-invariance***, we have:

$$\delta[n-k] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow h_k[n]=h[n-k]$$

From ***linearity***, we have:

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow x[k]h[n-k]$$

From ***additivity***, we have:

$$x[n] \longrightarrow \boxed{\text{DT LTI System}} \longrightarrow y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

Compare LTI Systems With Time-Varying Systems

The impulse response h[n] of a DT LTI system characterizes the system for all times.

In contract, to characterize a linear *time-varying* system, different impulse responses are needed for different times: $h_k[n]$, k=…, -1, 0, 1, 2,…

The response of a linear time-varying system to x[n] is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h_k[n] \neq \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

The Convolution Sum

The convolution sum is the response of a DT LTI system to an arbitrary input x[n]:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

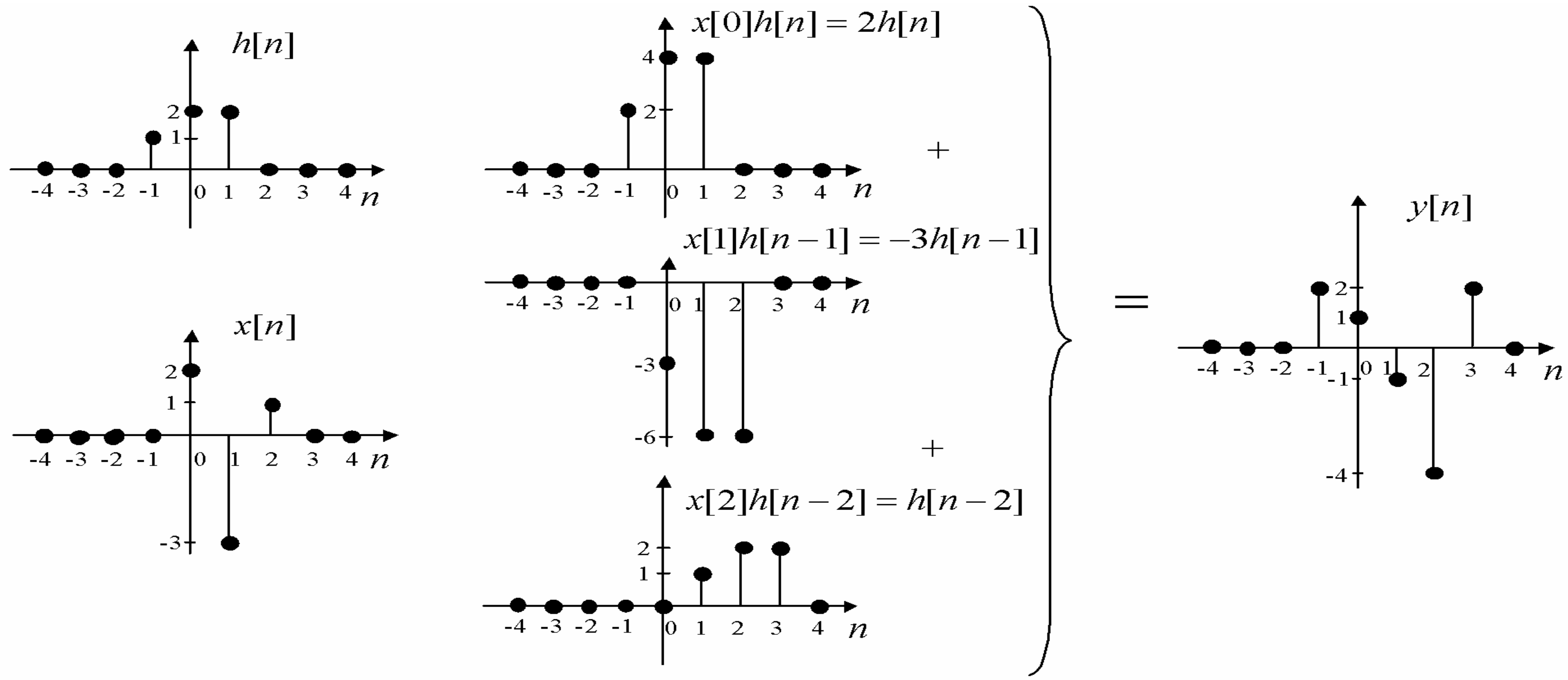
denoted as

$$y[n] = x[n] * h[n]$$

The summation runs over all entries of the input signal x[n] and of the impulse response h[n].

Method 1 of Calculating x[n]*h[n]: Use time-shifted h[n]

1. Plot h[n-k] *as functions of n*
2. For k=…,-1,0, 1, 2,…, calculate the weighted and shifted impulse responses x[k]h[n-k]
3. Sum x[k]h[n-k] over all k's



Method 2 of Calculating x[n]*h[n]: Use Time-reversed and Shifted h[k]

With *n* being fixed:

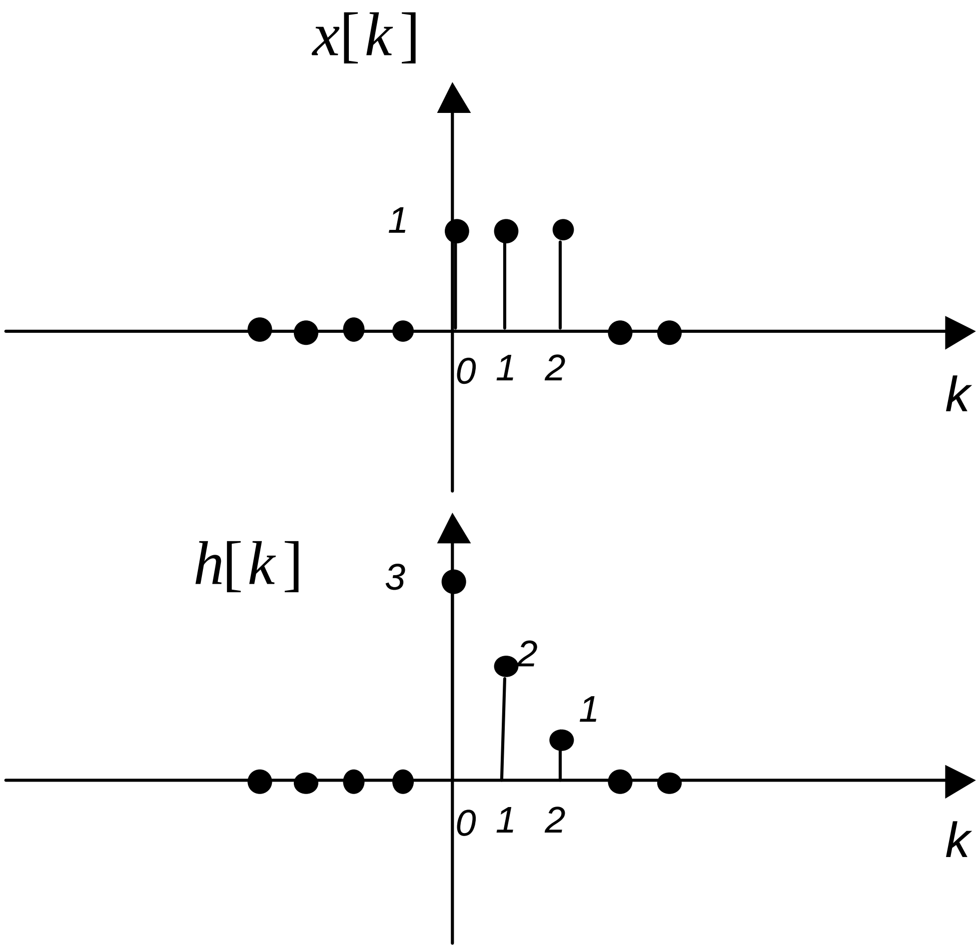
1. Plot h[n-k] *as a function of k (*time reversed and shifted h[k]):

   shift h[-k] to the right if n is positive or to the left if n is negative

2. For k=…,-2, -1, 0, 1, 2, ..., calculate x[k]h[n-k]
3. Sum x[k]h[n-k] over all k's

Example:

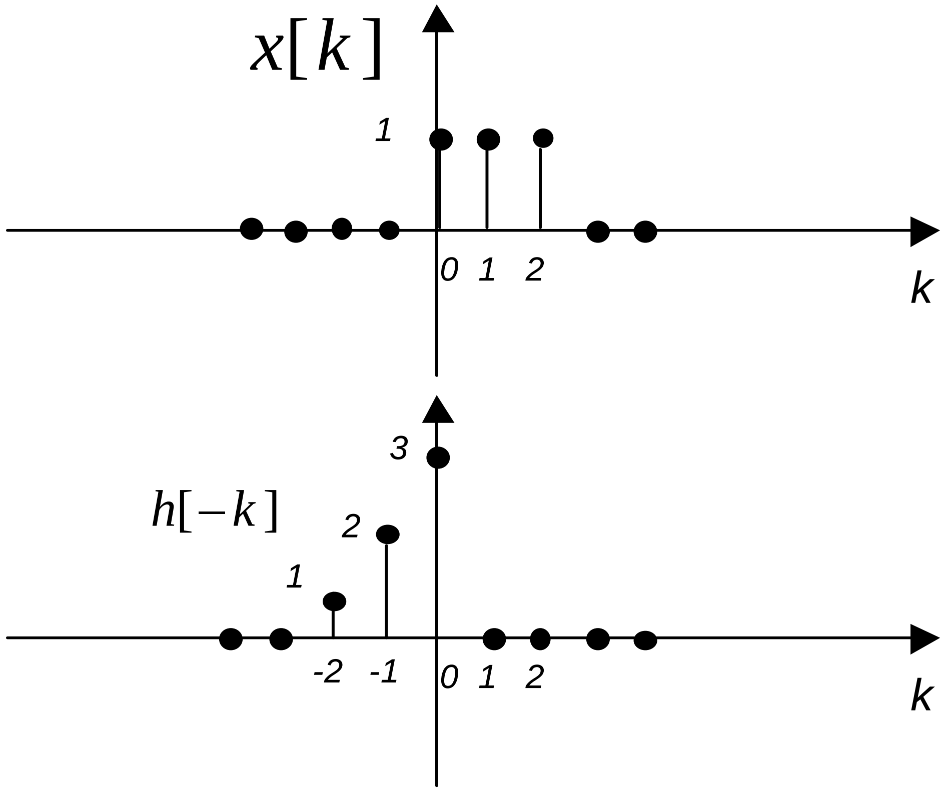
Compute y[0] and y[1] for the following x[n] and h[n].



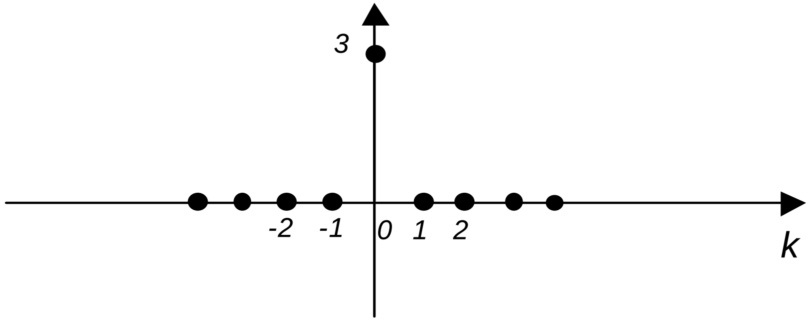
Calculate y[0] of x[n]*h[n] Using Method 2

With *n* being fixed as *n*=0

1: Plot x[k] and h[-k] for k=…, -1, 0, 1, 2,…



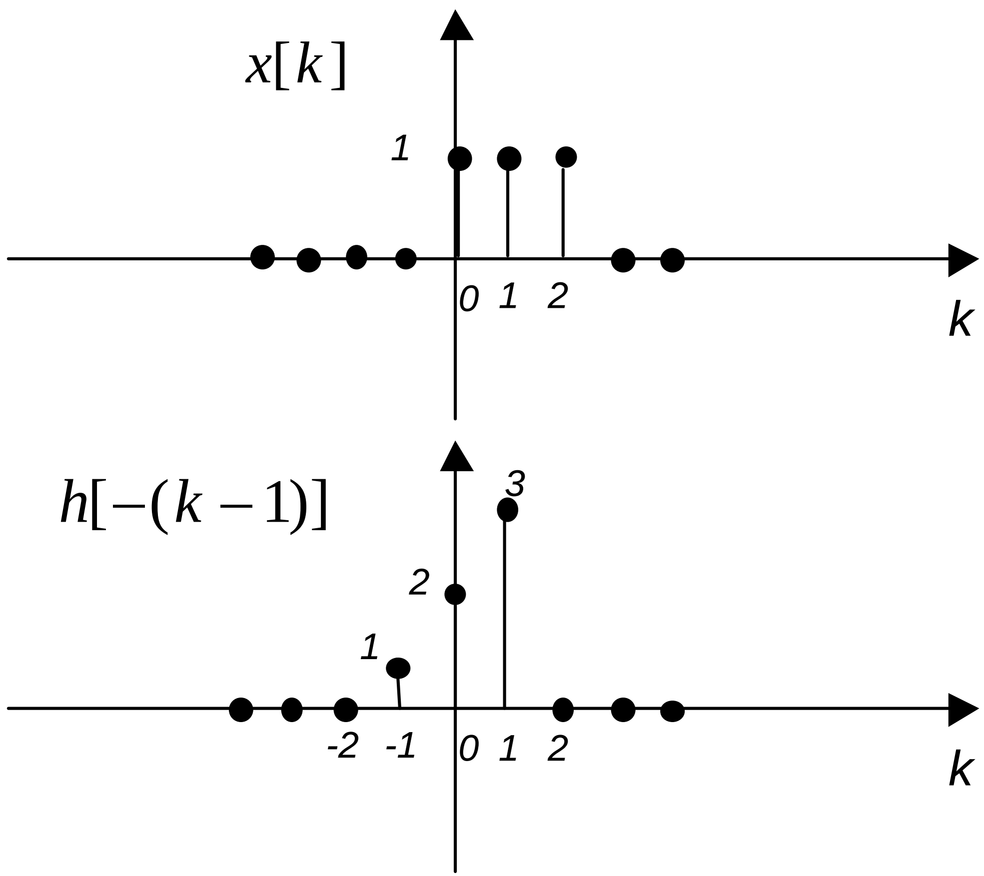
Step 2: Calculate x[k]h[-k] for all k's



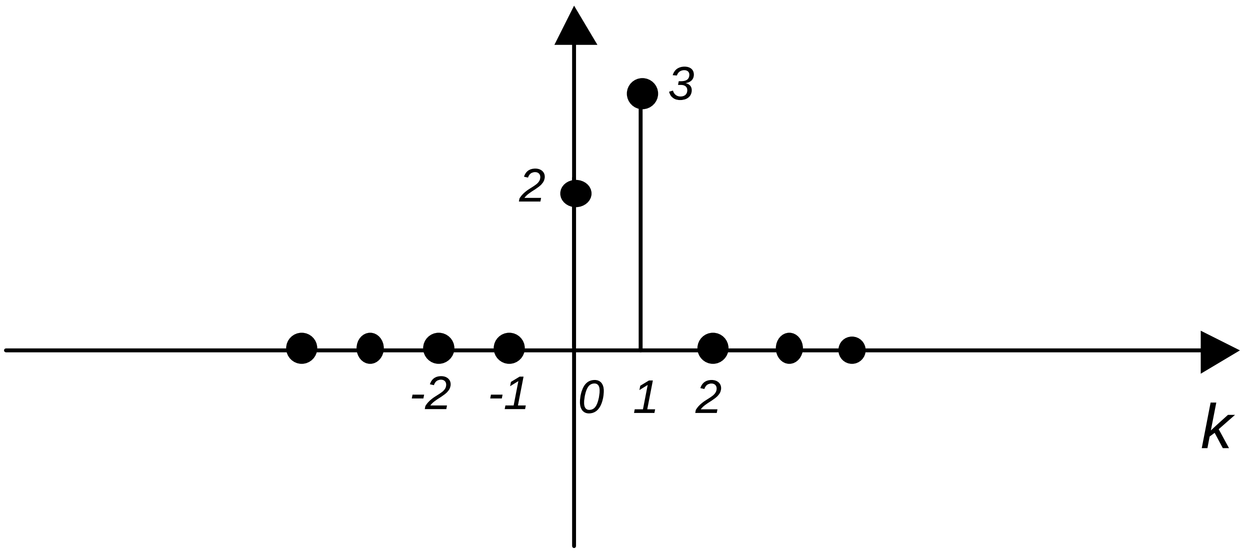
Step 3: Sum x[k]h[-k] from $k=-\infty$ to $+\infty$ to get y[0]=3

Calculating $y[1]$ *of* $x[n]*h[n]$

Step 1: Draw $x[k]$ and $h[-k+1]=h[-(k-1)]$ (i.e., the signal h[-k] delayed by 1)



Step 2: calculate x[k] h[1-k] for all k's



Step 3: Sum x[k]h[1-k] from $k=-\infty$ to $+\infty$ to get y[1] =2+3=5.

Properties of Convolution: Commutative

Commutative:

$$v[n] * w[n] = \sum_{k=-\infty}^{\infty} v[k]w[n-k] = \sum_{m=\infty}^{-\infty} v[n-m]w[m]$$

$$= \sum_{m=-\infty}^{+\infty} w[m]v[n-m] = w[n] * v[n]$$

Properties of Convolution: Associative

Associative

$$v[n]*(w[n]*y[n]) = v[n]*(y[n]*w[n]) = v[n]*\sum_{k=-\infty}^{+\infty} y[k]w[n-k]$$

$$= \sum_{m=-\infty}^{+\infty} v[m]\left(\sum_{k=-\infty}^{+\infty} y[k]w[n-m-k]\right) = \sum_{m=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} y[k]v[m]w[n-m-k]$$

$$= \sum_{k=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} y[k]v[m]w[n-m-k] = \sum_{k=-\infty}^{+\infty} y[k]\sum_{m=-\infty}^{+\infty} v[m]w[n-k-m]$$

$$= y[n]*\left(\sum_{m=-\infty}^{+\infty} v[m]w[n-m]\right) = y[n]*(v[n]*w[n])$$

$$= (v[n]*w[n])*y[n]$$

Properties of Convolution: Distributive

Distributive: x[n]*(v[n]+w[n])=x[n]*v[n]+x[n]*w[n]

$$x[n]*(v[n]+w[n]) = \sum_{k=-\infty}^{+\infty} x[k](v[n-k]+w[n-k])$$

$$= \sum_{k=-\infty}^{+\infty} x[k]v[n-k] + \sum_{k=-\infty}^{+\infty} x[k]w[n-k] = x[n]*v[n] + x[n]*w[n]$$

Properties of Convolution: Multiplication and Time-Shift

Commutative with respect to multiplication by a scalar

$$a(v[n]*w[n]) = (av[n])*w[n] = v[n]*(aw[n])$$

…check as an exercise.

Time-shifted when one of the two signals is time-shifted

$$v[n]*w[n-N] = \sum_{k=-\infty}^{\infty} v[k]w[n-N-k] = (v*w)[n-N]$$

Convoluting x[n] with δ[n]

$$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University
Department of Electrical and Computer Engineering

Lecture 5

Hui Qun Deng, PhD

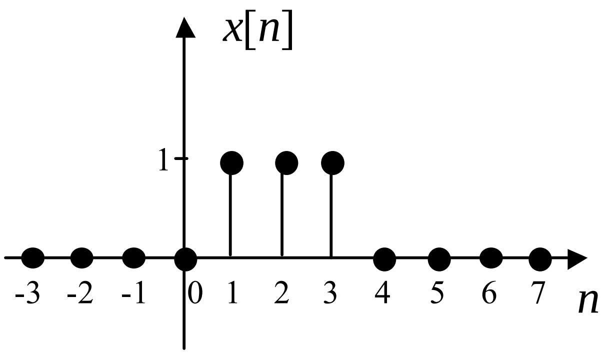
1. Example of Calculating Convolution Sum
2. Convolution Integral of CT Signals
3. Properties of Convolution Integral

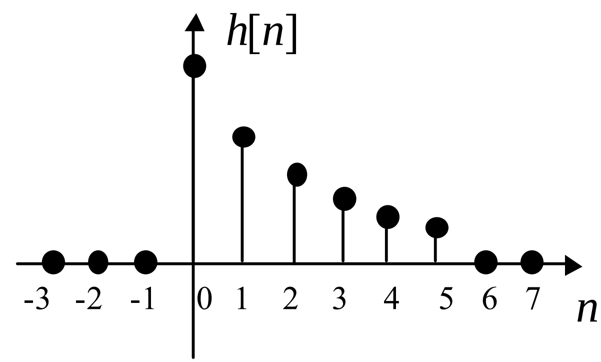
Example: calculating x[n]*y[n] using method 2

Let the impulse response of an LTI system be:

$$h[n] = \begin{cases} (0.8)^n, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

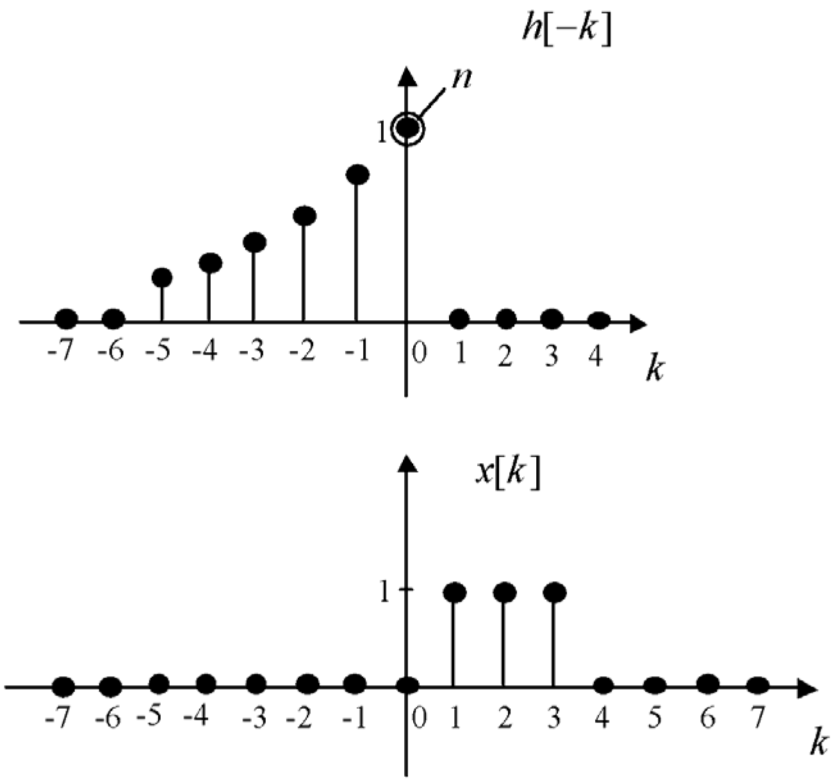
Obtain the response of the system to x[n] shown below:





*Method 2 is called numerical computation in Boulet's book.

Step 1: Sketch h[-k] and x[k] as **functions of *k***

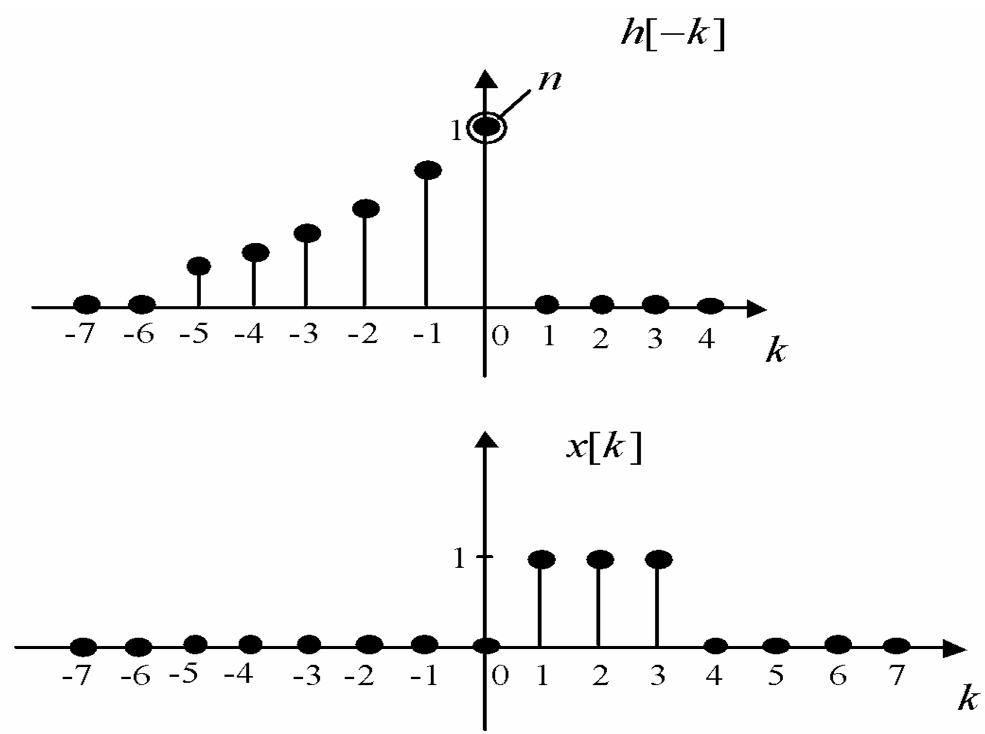


Step 2: Find ranges of *k* and *n* where h[n-k]x[k] $\neq 0$

for n>0, h[-(k-n)] means shifting h[-k] to the right

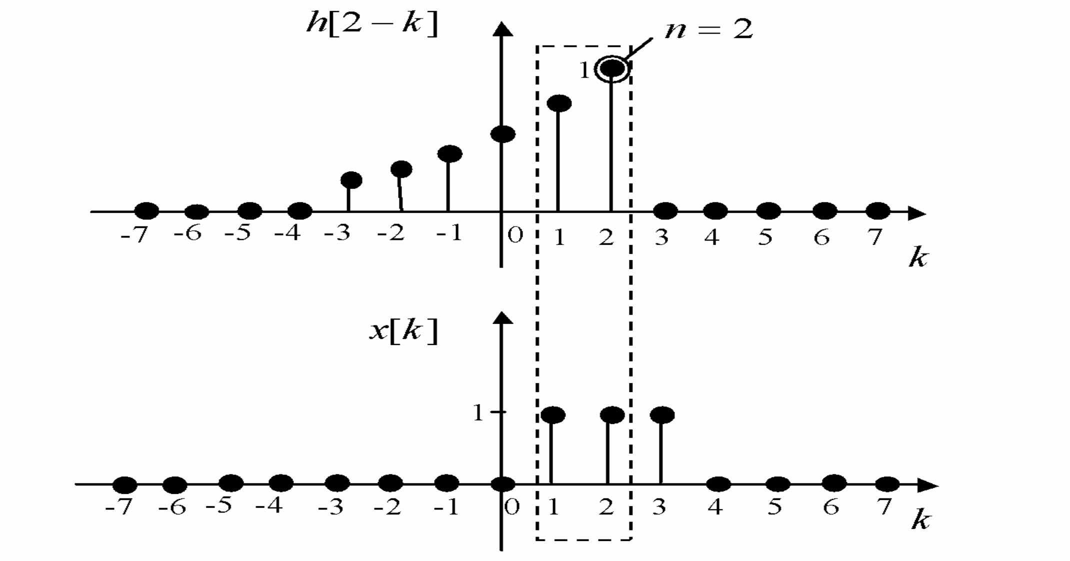
for n<0, h[-(k-n)] means shifting h[-k] to the left

For n<1: x[k] and h[n-k] don't overlap, so, x[k]h[n-k]=0 for all k hence y[n]=0.



For 1≤n ≤3: there is some overlap

Here is h[n-k] for n=2

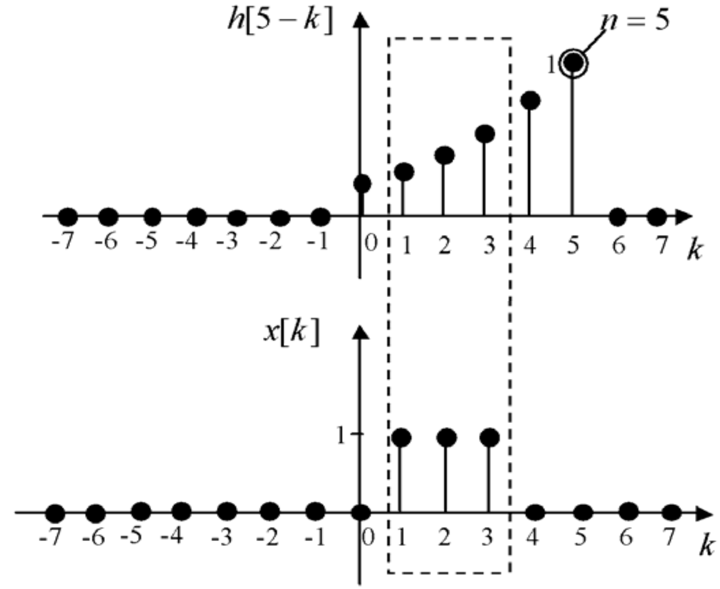


$$y[n] = \sum_{k=1}^{n} x[k]h[n-k] = \sum_{k=1}^{n} (0.8)^{n-k} = (0.8)^{n-1} \sum_{m=0}^{n-1} (0.8)^{-m}$$

$$= (0.8)^{n-1} \frac{1-(0.8)^{-n}}{1-(0.8)^{-1}} = \frac{(0.8)^{n}-1}{-0.2} = 5-5(0.8)^{n}$$

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha=1 \\ \frac{1-\alpha^N}{1-\alpha}, & \alpha\neq 1 \end{cases}$$

For 4 ≤n≤6: g[k]:=x[k]h[n-k]≠0, for k=1, 2, 3.



The case of n=5 is shown here.

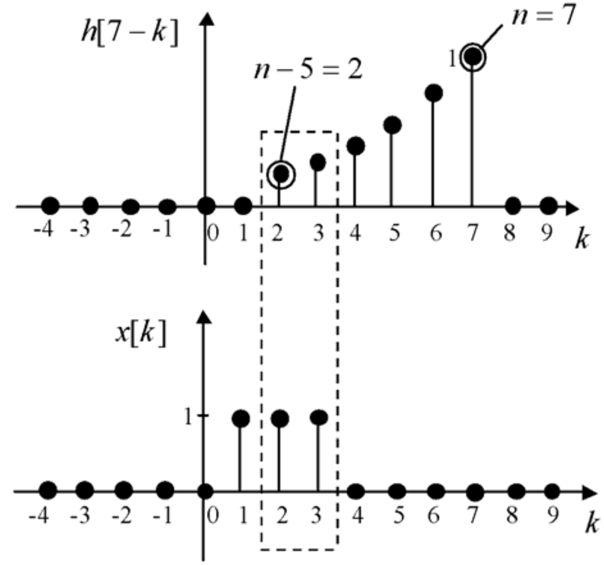
We get:

$$y[n]=\sum_{k=1}^{3}g[k]=\sum_{k=1}^{3}(0.8)^{n-k}=(0.8)^{n}\sum_{m=0}^{2}(0.8)^{-(m+1)}=(0.8)^{n-1}\sum_{m=0}^{2}(0.8)^{-m}$$

$$=(0.8)^{n-1}\left(\frac{1-(0.8^{-1})^{3}}{1-(0.8)^{-1}}\right)=\frac{(0.8)^{n}-(0.8)^{n-3}}{-0.2}=5(0.8)^{n-3}-5(0.8)^{n}=4.7656(0.8)^{n}$$

where we used the change of variables $m=k-1$.

For 7 ≤n≤8, g[k]=x[k]h[n-k]≠0 for n-5≤ k ≤ 3.



$$y[n] = \sum_{k=n-5}^{3} g[k] = \sum_{k=n-5}^{3} (0.8)^{n-k} = \sum_{m=0}^{8-n} (0.8)^{n-(m+n-5)} = (0.8)^5 \sum_{m=0}^{8-n} (0.8)^{-m}$$

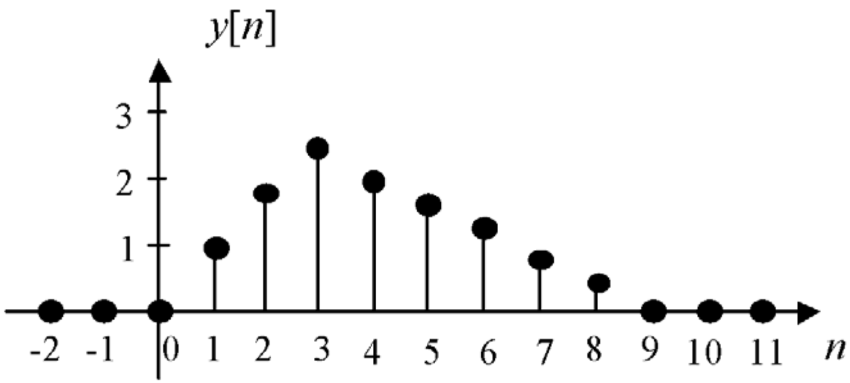
$$= (0.8)^5 \left(\frac{1-(0.8^{-1})^{9-n}}{1-(0.8)^{-1}} \right) = \frac{(0.8)^6 - (0.8)^{n-3}}{-0.2} = 5(0.8)^{n-3} - 5(0.8)^6$$

where we used the change of variables $m = k-(n-5)$.

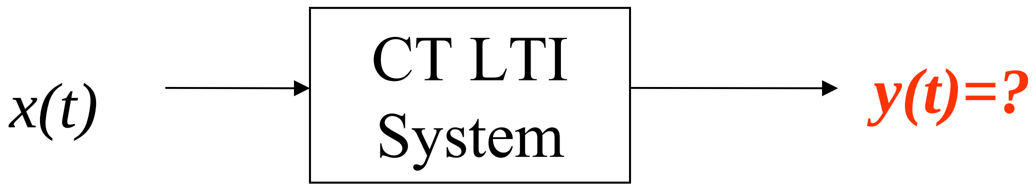
Finally for n>=9, the two signals do not overlap, so y[n]=0.

In summary,

$$y[n]=\begin{cases} 0, & n\le 0 \\ 5-5(0.8)^{n}, & 1\le n\le 3 \\ \left[5(0.8)^{-3}-5\right](0.8)^{n}, & 4\le n\le 6 \\ 5(0.8)^{n-3}-5(0.8)^{6}, & 7\le n\le 8 \\ 0, & n\ge 9 \end{cases}$$



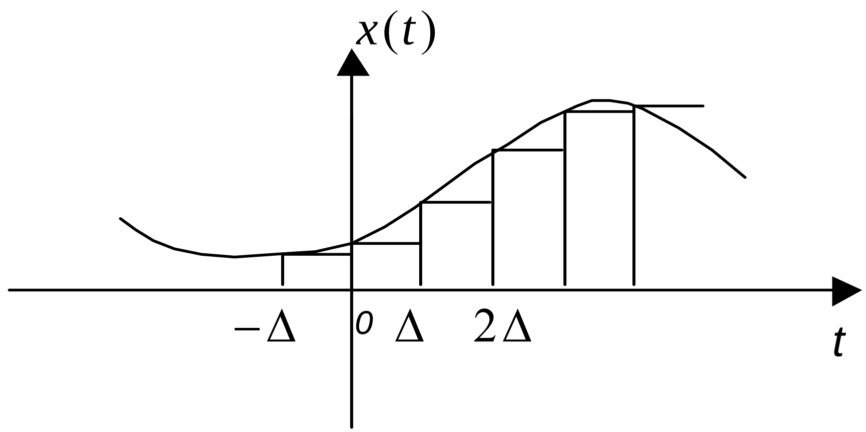
The Response of CT LTI System to x(t)



Given that *h(t)* is the impulse response of the CT LTI system:

Calculating The Response of CT System to *x(t)* "Divide and Conquer" Strategy

"Chopping up" the signal $x(t)$ into sections of width Δ .

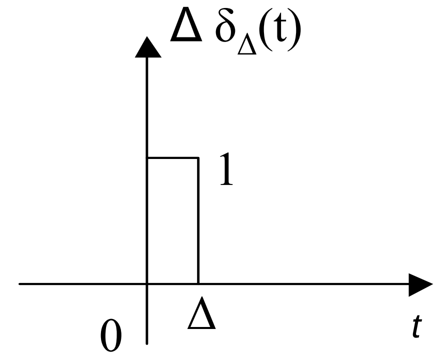


$$\hat{x}(t) := \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

Divide: Decompose $x(t)$ into $x(k\Delta)\delta_\Delta(t-k\Delta)$

Define a signal $\hat{x}(t)$ as

$$\hat{x}(t) := \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$



As $\Delta \to 0$,

(1) $k\Delta \to \tau$

(2) $x(k\Delta) \to x(\tau)$

(3) $\delta_\Delta(t-k\Delta) \to \delta(t-\tau)$

(4) $\Delta \to d\tau$

(5) The summation approaches an convolution integral $x(t)*\delta(t)=x(t)$

The Response of a CT System to x(kΔ)δ_Δ(t-kΔ)

Let $\hat{h}_{k\Delta}(t)$ be the responses of the linear *time-varying* system to $\delta_\Delta(t-k\Delta)$.

Then, from the Principle of Superposition, the response of S to $\hat{x}(t)$ is

$$\hat{y}(t) := \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

Conquer: Convert summation into Integral

As $\Delta \to 0$,

$$y(t) = \lim_{\Delta \to 0} \hat{y}(t) = \int_{-\infty}^{\infty} x(\tau) h_{\tau}(t) d\tau$$

For a LTI system, $h_{\tau}(t)=h(t-\tau)$, then y(t) becomes

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

The above integral is called ***convolution integral***. Denoted as

$$y(t)=x(t)*h(t)$$

Properties of Convolution Integral

1. Commutative:

$$x(t)*y(t)=y(t)*x(t)$$

2. Associative:

$$x(t)*[y(t)*z(t)]=[x(t)*y(t)]*z(t)$$

3. Distributive:

$$x(t)*[y(t)+z(t)]=x(t)*y(t)+x(t)*z(t)$$

Properties of Convolution Integral

4. Commutative with respect to multiplication by a scalar

$$\alpha[x(t)]*y(t) = x(t)*[\alpha y(t)]$$

5. Time-shifted by T, if one of the two signals is time-shifted by T:

Let $x(t)*y(t)=z(t)$, then

$$[x(t-T)]*y(t)=x(t)*y(t-T)=z(t-T)$$

6. $x(t)*\delta(t)=x(t)$

Method of Calculating x(t)*y(t)

With t being a fixed value

1. Sketch h($-\tau$) *as a* ***function of τ***
2. find ranges of τ and t where $x(\tau)h(t-\tau) \neq 0$

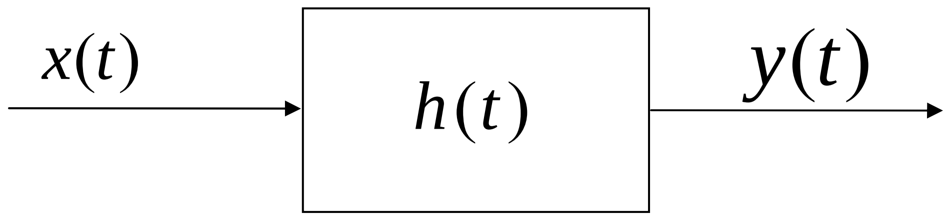
   for $t>0$, $h(t-\tau)$ means shifting $h(-\tau)$ to the right by t

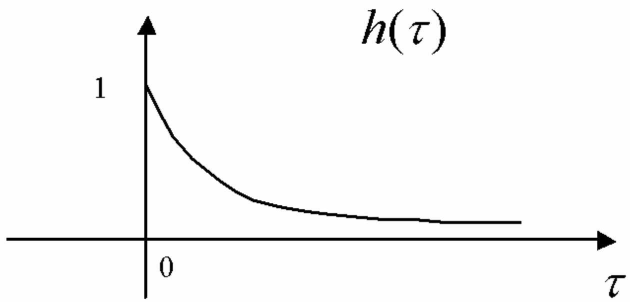
   for $t<0$, $h(t-\tau)$ means shifting $h(-\tau)$ to the left by $|t|$
3. Calculate $g(\tau)=h(t-\tau)x(\tau)$
4. ***Integrate*** $g(\tau)$ to get $y(t)$ for a given range of t

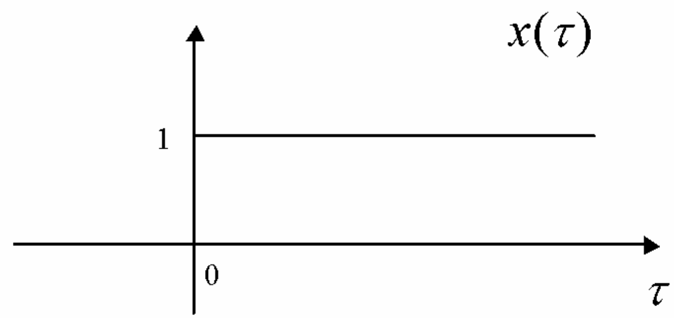
Example

Impulse response: $h(t) = e^{-at}u(t), a > 0$

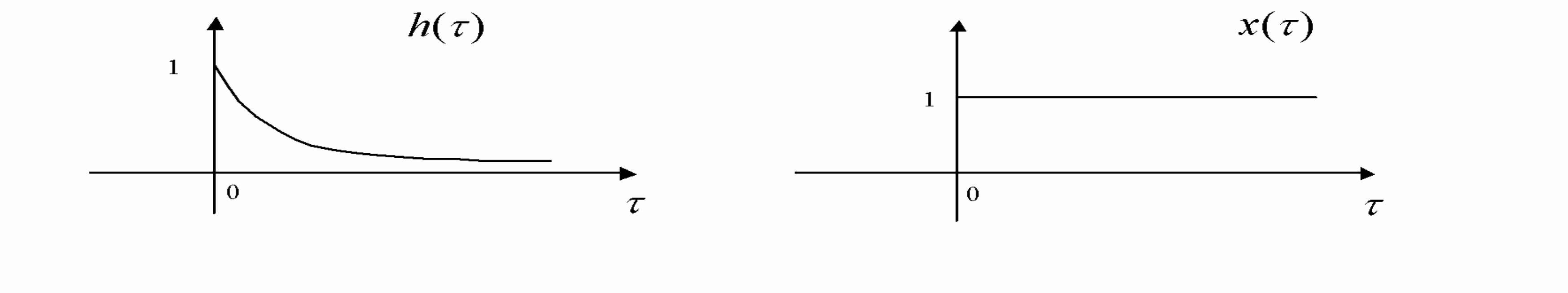
Input signal: $x(t) = u(t)$.

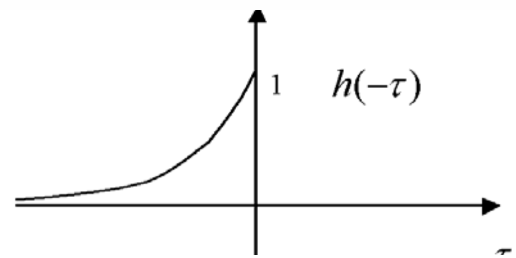




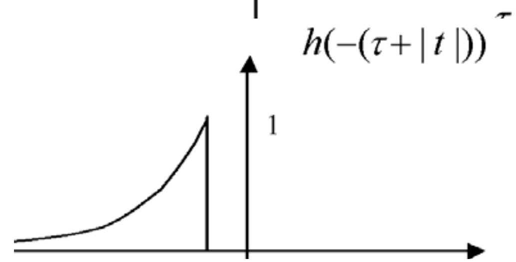


Step 1. Find overlap regions



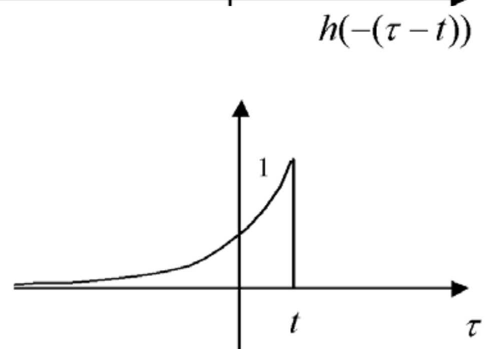


Case $t < 0$



No overlap
$y(t) = 0$

Case $t \geq 0$



Overlap
$0 \leq \tau \leq t$

Step 2. Integrate x(τ)h(t- τ) with respect to τ

Thus for $t \geq 0$, we have

$$y(t) = \int_{-\infty}^{+\infty} g(\tau) d\tau = \int_{0}^{t} g(\tau) d\tau$$

$$= \int_{0}^{t} x(\tau) h(t-\tau) d\tau$$

$$= \int_{0}^{t} e^{-a(t-\tau)} d\tau = \frac{1}{a}\left[1 - e^{-at}\right]$$

Combine the Results

Combining the results for $t < 0$ and $t \geq 0$, we get the response

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t), \quad -\infty < t < \infty$$

McGill University
Department of Electrical and Computer Engineering

Lecture 6

Hui Qun Deng, PhD

1. Techniques of calculating convolutions
2. Properties of LTI systems
3. The relationships between step response and impose response

Calculating Convolution Integral *y(t)=x(t)*h(t)*

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

The key issue of calculating the convolution integral is to determine the ranges of τ in terms of time shift "*t*", which depend on the *relative positions* of the two signals.

The best approach is to combine ***analytical method*** with ***graphical method*** to determine the bounds of τ in terms of "*t*".

Analytical Method: Determine upper and lower bounds of τ *(in terms of t)* where $x(\tau)h(t-\tau) \neq 0$

- If $h(t) \neq 0$ for $T_1 \le t \le T_2$, *then* $h(t-\tau) \neq 0$ for $T_1 \le t-\tau \le T_2$,

  i.e., **$h(t-\tau) \neq 0$ for $t-T_2 \le \tau \le t-T_1$**

- If $x(t) \neq 0$ for $T_3 \le t \le T_4$, *then*

  $x(\tau) \neq 0$ for $T_3 \le \tau \le T_4$

- *Hence, to have* $x(\tau)h(t-\tau) \neq 0$, *τ should be*

$$\max\{t-T_2, T_3\} \le \tau \le \min\{t-T_1, T_4\}$$

$$\max\{t-T_2, T_3\} = \begin{cases} t-T_2, & \text{if } t \ge T_2 + T_3 \\ T_3, & \text{if } t < T_2 + T_3 \end{cases} \qquad \min\{t-T_1, T_4\} = \begin{cases} t-T_1, & \text{if } t \le T_4 + T_1 \\ T_4, & \text{if } t > T_4 + T_1 \end{cases}$$

Graphical Method: Determine upper and lower bounds of τ (in terms of t) where $h(t-\tau)x(\tau)\neq 0$

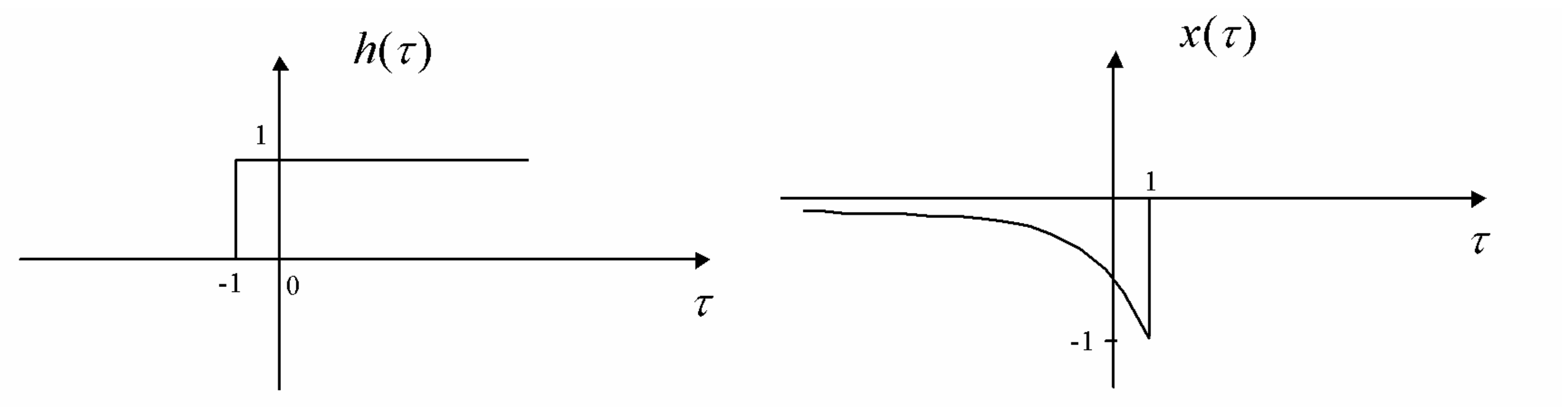
- For $t>0$, shift $h(-\tau)$ to the *right*, and determine the upper and lower bounds of τ where $h(t-\tau)x(\tau)\neq 0$
  - for different positive "t", $h(t-\tau)x(\tau)$ *may have different expressions*

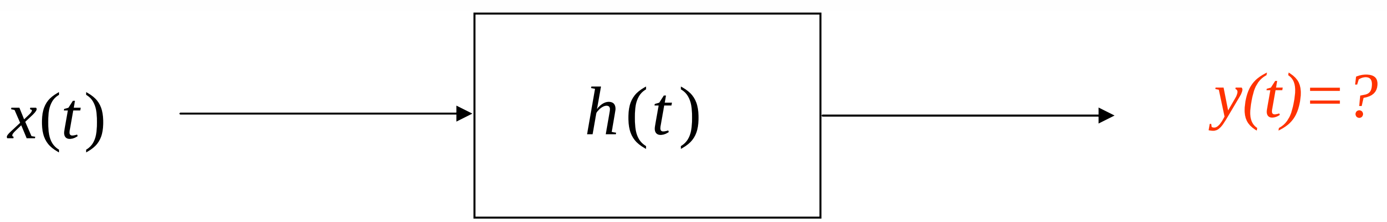
- For $t<0$, shift $h(-\tau)$ to the *left*, and determine the upper and lower bounds of τ where $h(t-\tau)x(\tau)\neq 0$
  - for different negative "t", $h(t-\tau)x(\tau)$ *may have different expressions*

Example

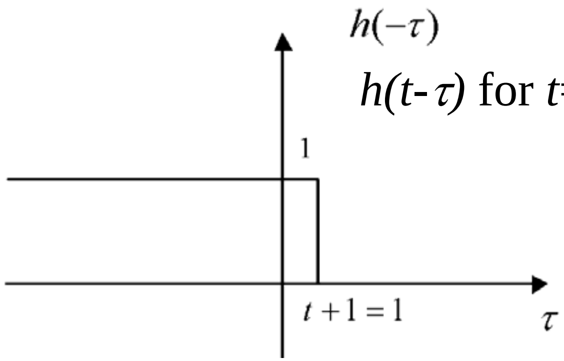
Impulse response: $h(t) = u(t+1)$

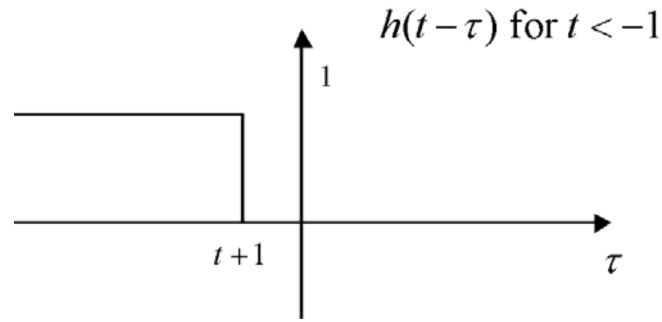
Input signal: $x(t) = -e^{2(t-1)}u(-(t-1))$

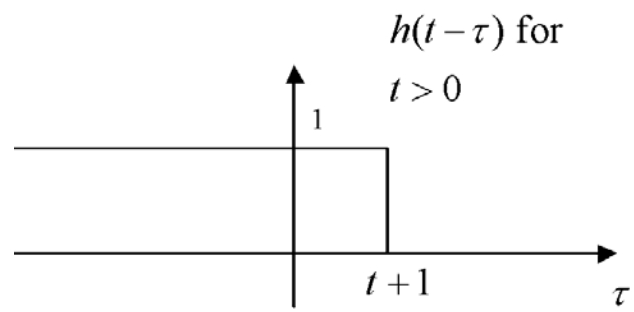




Determine the ranges of τ where $h(t-\tau) \neq 0$, in terms of the time-shift "t"







Because

$h(t) \neq 0$ for $t>-1$,

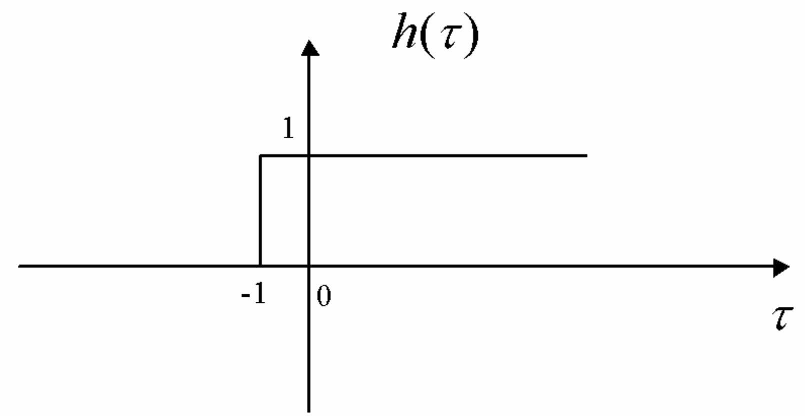
then

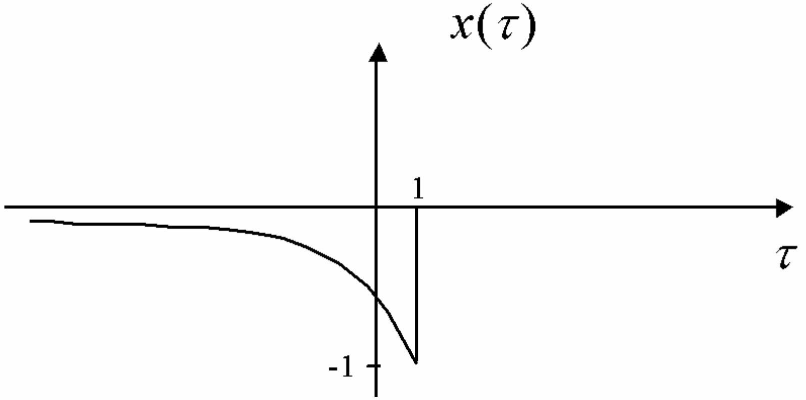
$h(t-\tau) \neq 0$ for $t-\tau>-1$

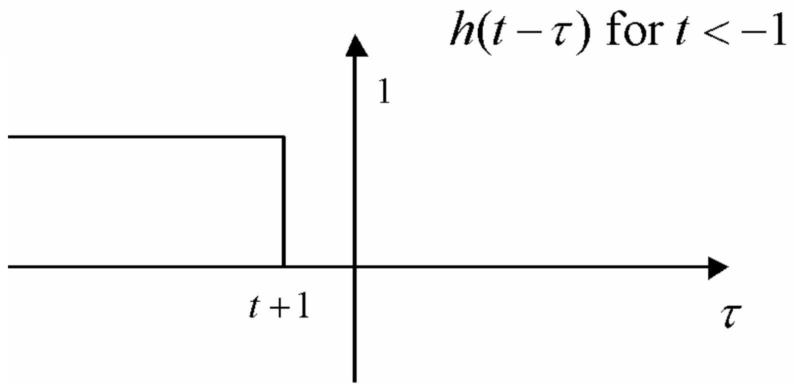
thus

$h(t-\tau) \neq 0$ for $\tau<t+1$

Determine the regions of τ where $h(t-\tau)x(\tau)\neq 0$ in terms of time-shift "t"







Because

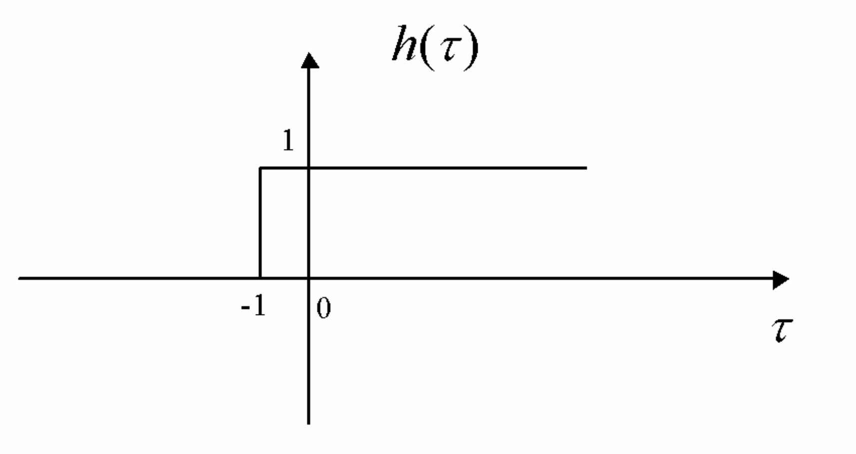
$x(\tau)\neq 0$ for $\tau<1$

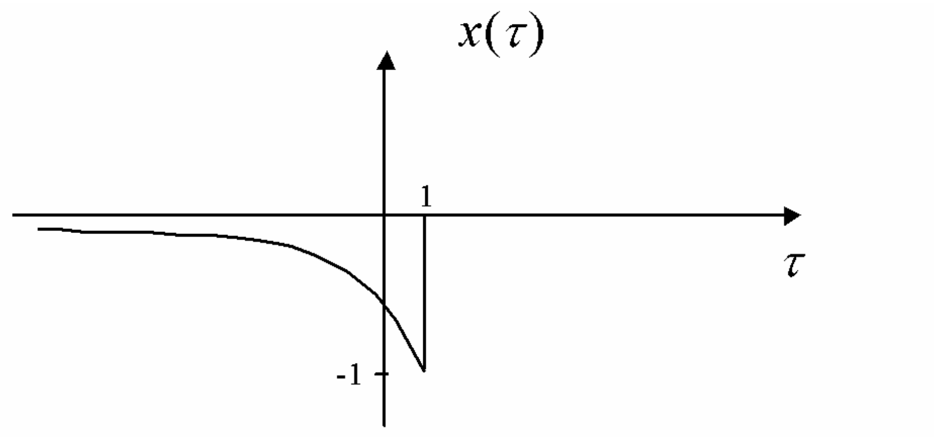
$h(t-\tau)\neq 0$ for $\tau<t+1$

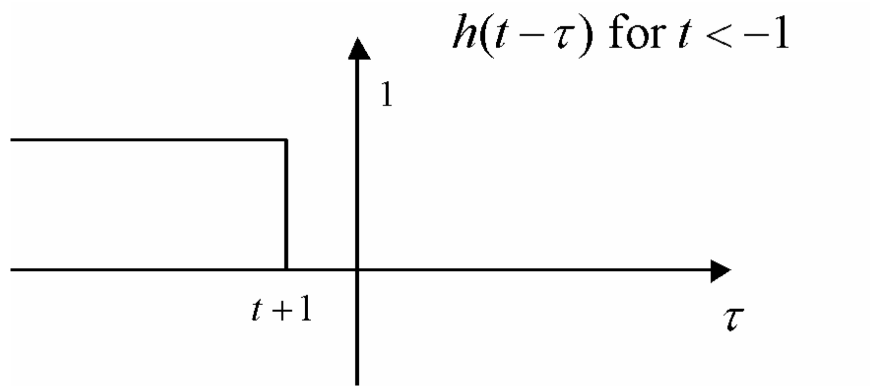
then

$x(\tau)h(t-\tau)\neq 0$ for $\mathbf{-\infty<\tau<\min(1, t+1)}$

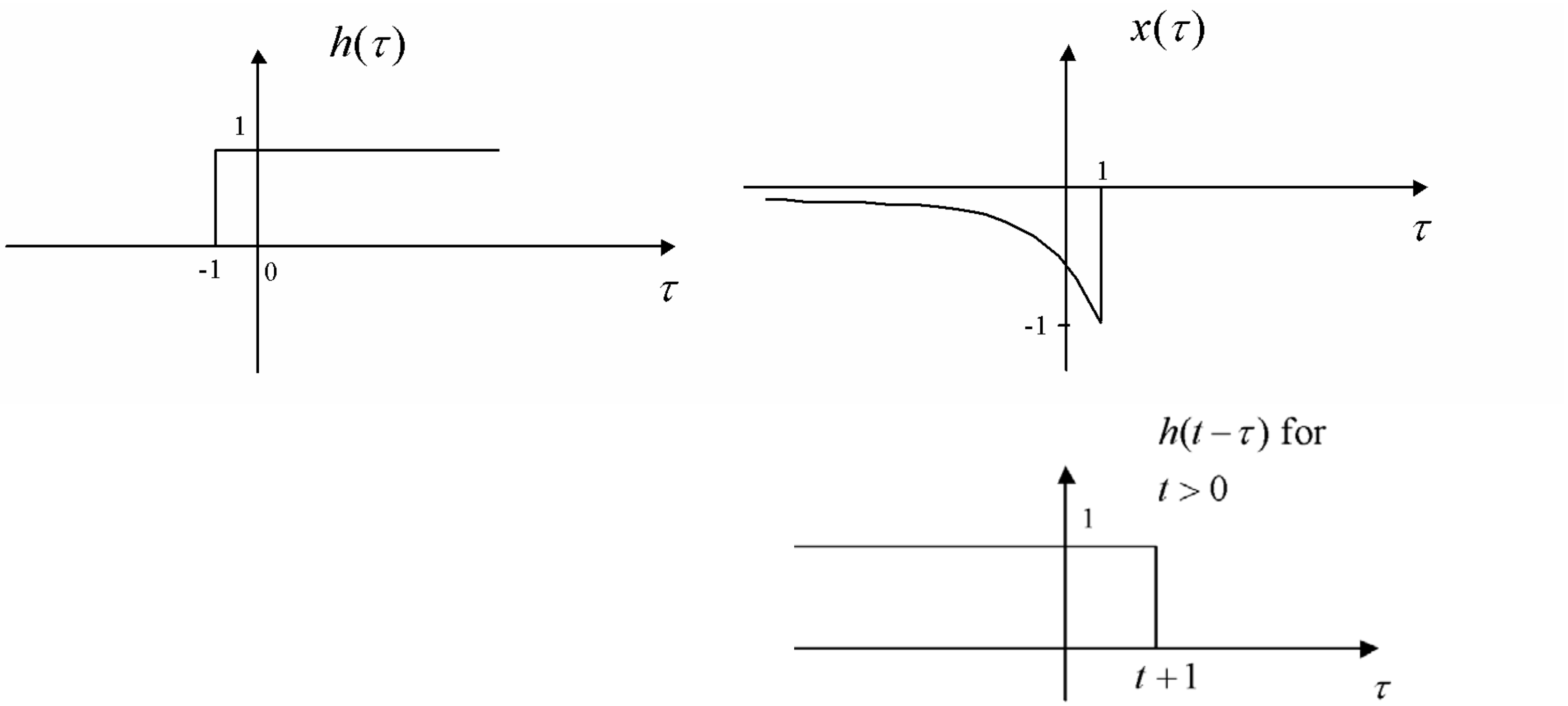
1. For t≤0, h(t-τ)x(τ)≠0 over the interval -∞τ<t+1







2. For t>0, $h(t-\tau)x(\tau)\neq 0$ over the interval $-\infty<\tau\leq 1$



Thus for $t \le 0$, we get

$$y(t) = \int_{-\infty}^{+\infty} g(\tau)d\tau = \int_{-\infty}^{t+1} g(\tau)d\tau$$

$$= \int_{-\infty}^{t+1} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{t+1} -e^{2(\tau-1)}d\tau = -\frac{1}{2}\left[e^{2t} - 0\right] = -\frac{1}{2}e^{2t}$$

Thus for $t > 0$, we get

$$y(t) = \int_{-\infty}^{+\infty} g(\tau)d\tau = \int_{-\infty}^{1} g(\tau)d\tau$$

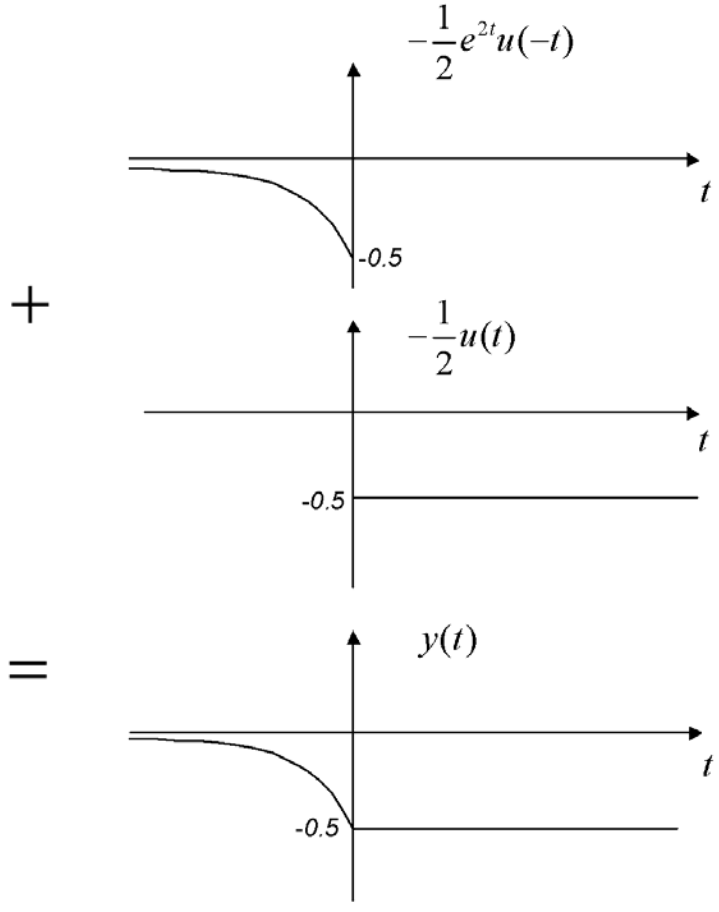
$$= \int_{-\infty}^{1} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{1} -e^{2(\tau-1)}d\tau = -\frac{1}{2}\left[e^{0} - 0\right] = -\frac{1}{2}$$

Combine the results

Piecing the two intervals together we obtain the response

$$y(t) = -\frac{1}{2}e^{2t}u(-t) - \frac{1}{2}u(t)$$



Techniques of Calculating h[n]*x[n] or h(t)*x(t)

- Apply the definition
  - Method 1: use time shifted h(t) or h[n] (or x(t) or x[n])
  - Method 2:
    1. Reverse $h(\tau)$; 2. Shift $h(-\tau)$ by "t"; 3. Multiply $h(t-\tau)x(\tau)$, 4. Integrate $h(t-\tau)x(\tau)$
- Divide and conquer based on properties of convolution
  - Decompose the input signal into multiple simpler signals:

$$x=x_1+x_2+x_3+\ldots$$

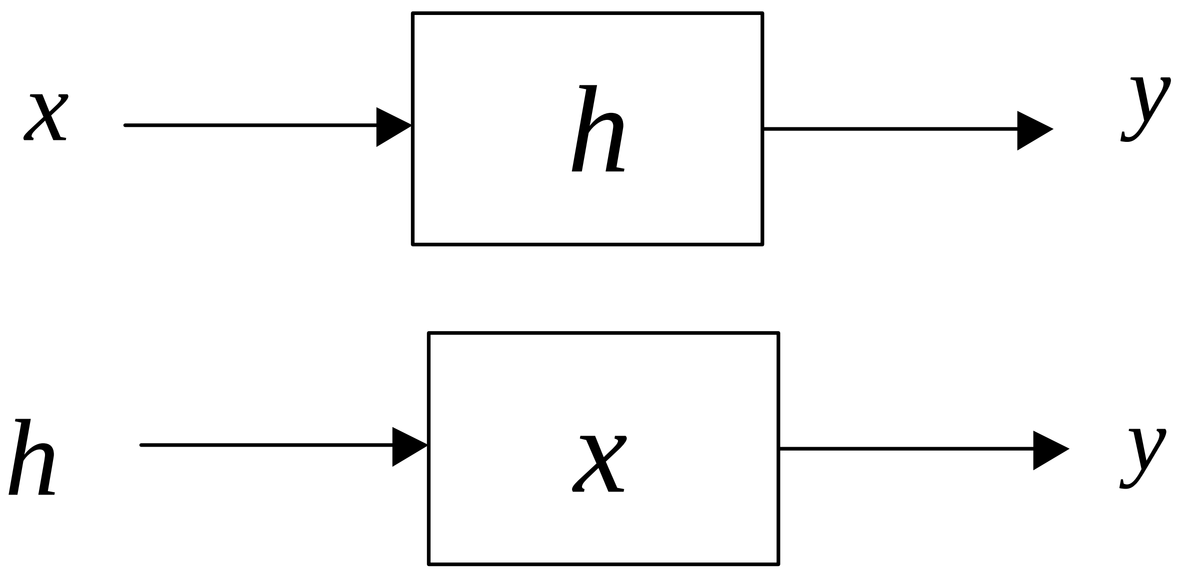
  - Calculate $y_1=x_1*h,\ y_2=x_2*h,\ y_3=x_3*h,\ \ldots.$
  - Sum up $y=y_1+y_2+y_3\ldots.$

The Commutative Properties of LTI System

This comes from the fact that a convolution is commutative

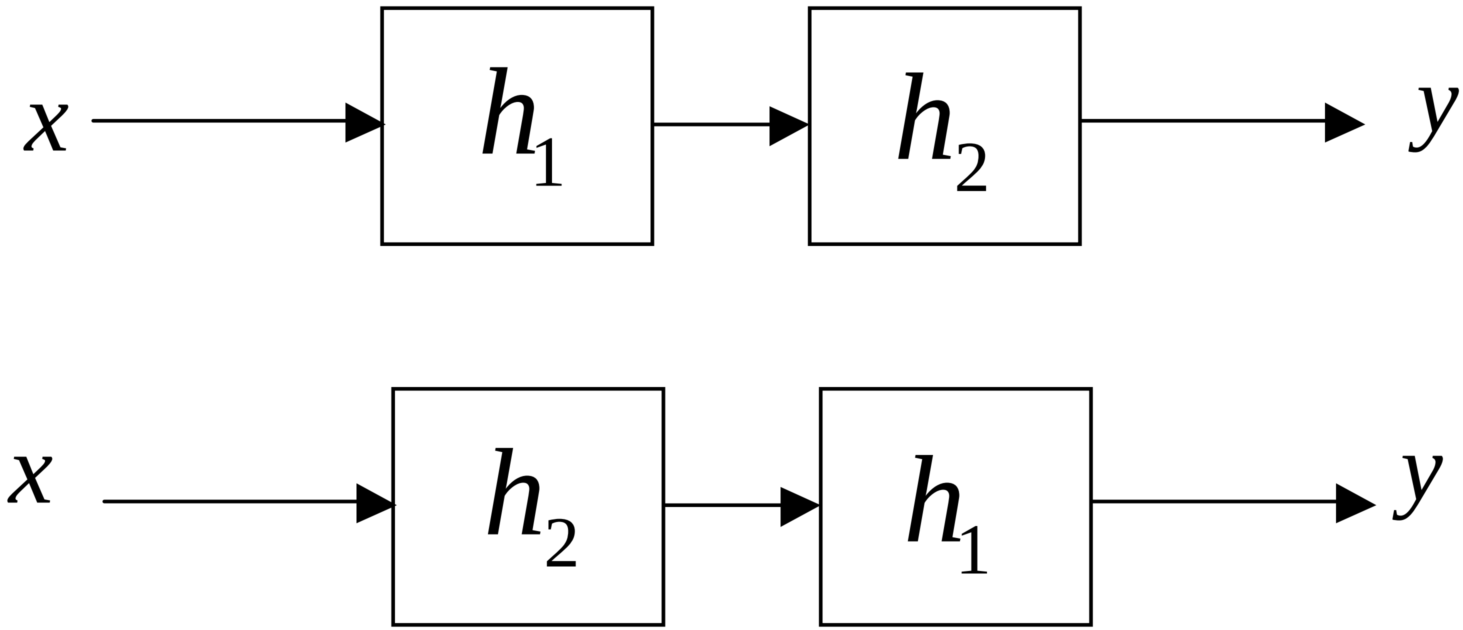
$$x(t)*h(t)=h(t)*x(t)$$

$$x[n]*h[n]=h[n]*x[n]$$



Commutative

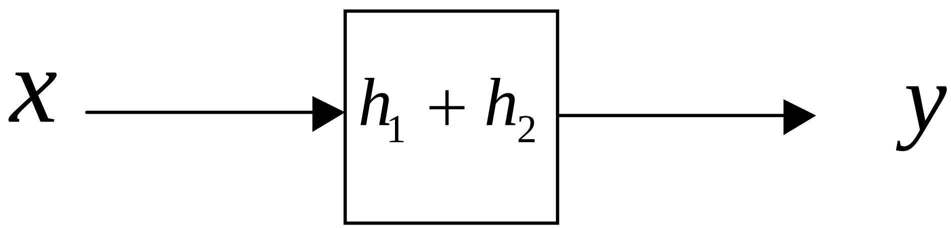
$$y=(x*h_1)*h_2=(x*h_2)*h_1$$

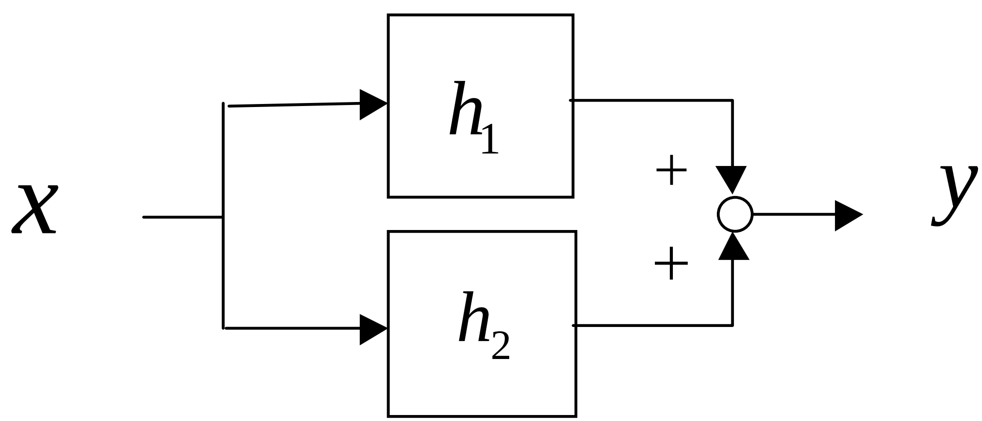


e.g., a low-pass filter and a high-pass filter can be connected one after the other in two different orders.

The Distributive Properties of LTI Systems

$$x*(h_1+h_2)=x*h_1+x*h_2$$

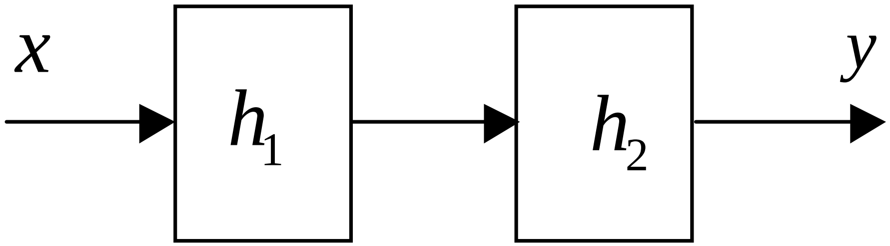


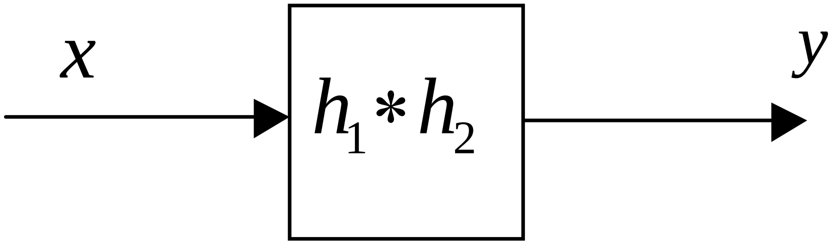


The Associative Property of LTI Systems

This property of LTI systems comes from the fact that the convolution operation is associative.

$$y = (x * h_1) * h_2 = x * (h_1 * h_2)$$





LTI Systems with and without Memory

A system is memoryless if its output at any time depends only on the value of its input at that same time.

To be memoryless, a CT LTI system must have an impulse response of the form ***$h(t)=A\delta(t)$***, and a DT LTI system must have an impulse response of the form ***$h[n]=A\delta[n]$***.

This can be seen from the convolution equations

$$y(t)=\int_{t^-}^{t^+} x(\tau)h(t-\tau)d\tau=\int_{t^-}^{t^+} x(\tau)A\delta(-(\tau-t))d\tau$$

$$=Ax(t)\int_{t^-}^{t^+}\delta(\tau-t)d\tau=Ax(t)$$

Invertibility of LTI Systems

Recall: a system S is invertible if and only if there exists an inverse system S^{-1} such that $S^{-1}S$ is the identity system $A\delta(t)$.

For an LTI system with impulse response *h*, invertibility is equivalent to the existence of another system with impulse response h_1 such that $h_1 * h = \delta$.

$$x \longrightarrow \boxed{h} \longrightarrow \boxed{h_1} \longrightarrow y = x$$

Causality of an LTI System

Recall: a system is causal if its output depends only on past and/or present values of the input signal.

For a DT LTI system to be causal, $y[n]$ should NOT depend on $x[k]$ *for* $k>n$. This means in the convolution sum, $h[n-k]=0$ *for* $k>n$, which is equivalent to $h[k]=0$ *for* $k<0$.

Thus, a DT LTI system is causal if its impulse response $h[n]=0$ *for* $t<0$.

A similar analysis for a CT LTI system leads to the same conclusion: a CT LTI system is causal if its impulse response $h(t)=0$ *for* $t<0$.

Sufficient Condition for BIBO Stability of DT LTI Systems

Recall: a system is BIBO stable if for every bounded input, the output is also bounded.

Let the impulse response of a DT system be *h[n]*. If the DT signal *x[n]* is bounded by B for all n, then the system output magnitude can be bounded as:

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$< B\sum_{k=-\infty}^{\infty} |h[k]|$$

Thus, $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$ is a sufficient condition for BIBO systems.

Necessary Condition for BIBO Stability of DT LTI Systems

Suppose that $\sum_{k=-\infty}^{+\infty} |h[k]| = +\infty$. Then we can construct an input signal $x[n] = sign(h[-k]) = \begin{cases} 1, & h[-k] > 0 \\ -1, & h[-k] < 0 \end{cases}$,

which is bounded by 1 and leads to an output that's unbounded at n=0.

$$|y[0]| = \left| \sum_{k=-\infty}^{\infty} x[k] h[-k] \right| = \left| \sum_{k=-\infty}^{\infty} \operatorname{sgn}(h[-k]) h[-k] \right| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

Sufficient and Necessary Condition for BIBO Stability of LTI Systems

In summary, a discrete-time LTI system is BIBO stable if and only if $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$, i.e., the impulse response is *absolutely summable*.

The same analysis applies to continuous-time LTI systems for which stability is equivalent to $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$, i.e., the impulse response is *absolutely integrable*.

Unit Step Responses of LTI Systems

$$u[n] \longrightarrow \boxed{h[n]} \longrightarrow s[n] \qquad u(t) \longrightarrow \boxed{h(t)} \longrightarrow s(t)$$

$$s[n]=\sum_{k=0}^{\infty}u[k]h[n-k]=\sum_{k=0}^{\infty}h[n-k]=\sum_{m=n}^{-\infty}h[m]=\sum_{m=-\infty}^{n}h[m] \qquad (2.41)$$

$$s(t)=\int_{0}^{\infty}u(\tau)h(t-\tau)d\tau=\int_{0}^{\infty}h(t-\tau)d\tau=-\int_{t}^{-\infty}h(l)dl=\int_{-\infty}^{t}h(l)dl \qquad (2.43)$$

Thus, the unit step response of a DT LTI system is the ***running sum*** of its impulse response h[n]; the unit step response of a CT LTI system is the ***running integral*** of its impulse response h(t).

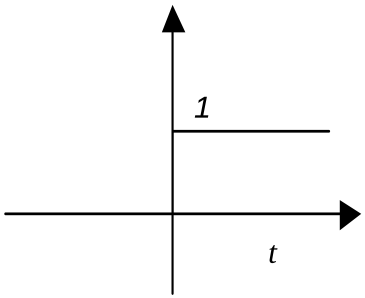
Derive Impulse Responses from Step Responses

From (2.41): $$h[n] = s[n] - s[n-1] \quad (2.42)$$

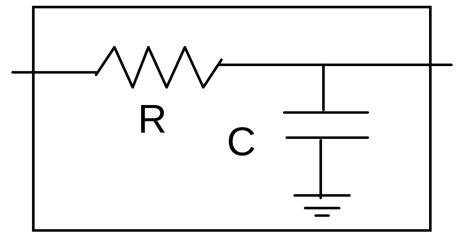
From (2.43): $$h(t) = \frac{d}{dt}\int_{-\infty}^{t} h(l)\,dl = \frac{d}{dt}s(t) \quad (2.44)$$

Thus, the impulse response of an LTI system is the ***first-order derivative*** of its unit step response.

Example: The unit step response and the impulse response of an RC circuit

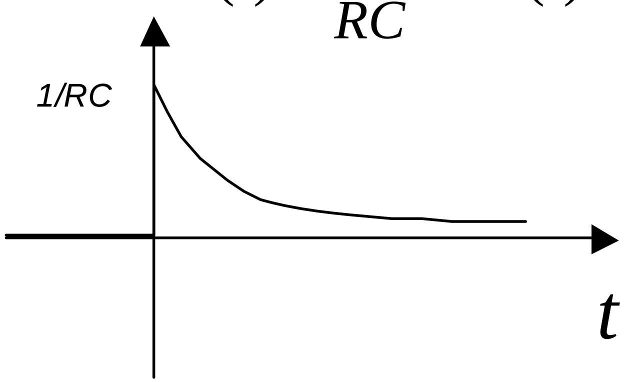


$$v_{in}(t) = u(t)$$

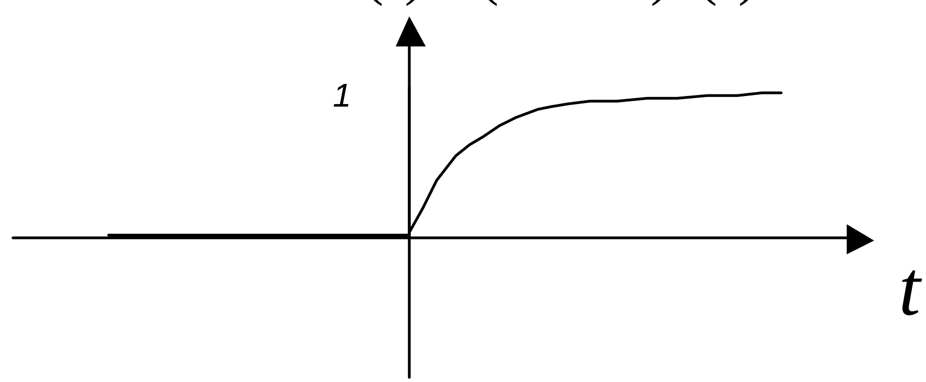


$$v_{out}(t) = s(t)$$

$$h(t) = \frac{1}{RC} e^{\frac{-t}{RC}} u(t)$$



$$s(t) = (1 - e^{\frac{-t}{RC}})u(t)$$



Lecture 7

Hui Qun Deng

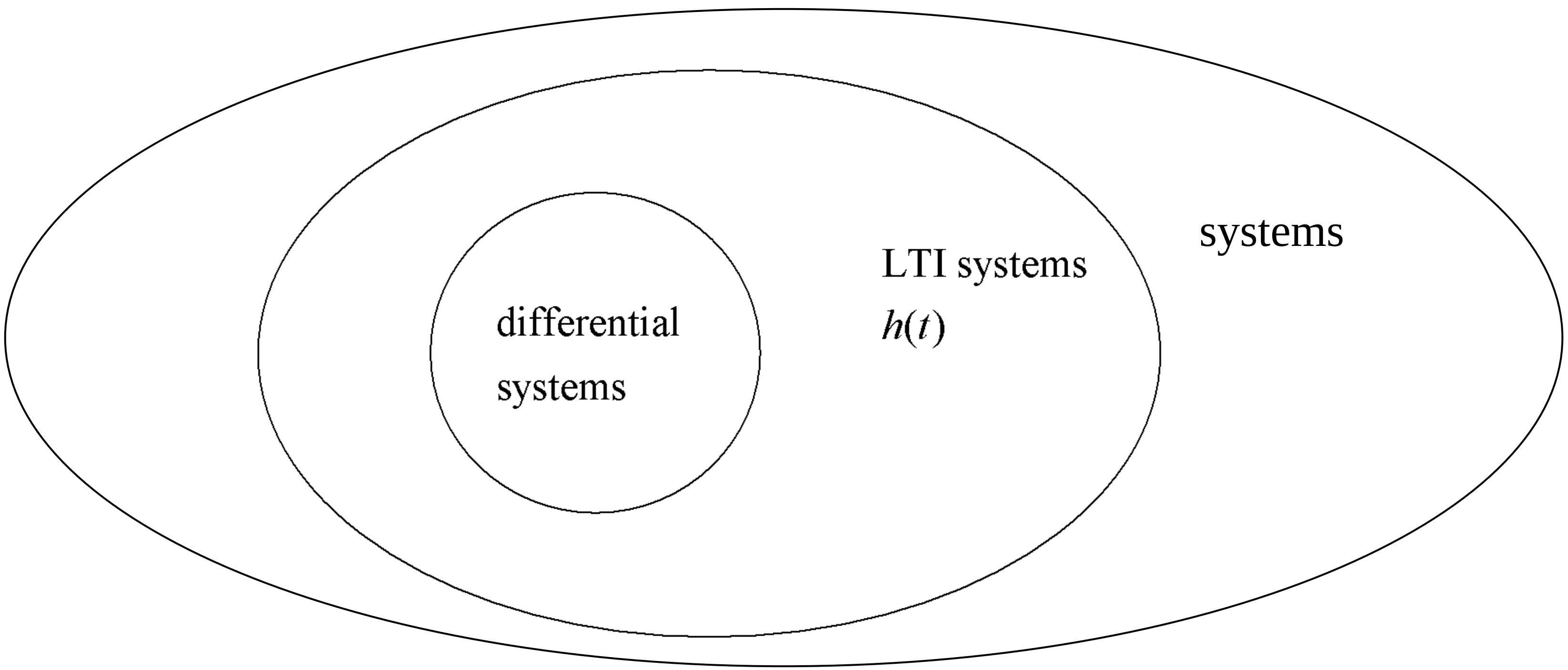
1. Solutions of CT differential equations
2. Solutions of DT difference equations
3. Recursive solution for DT difference equations
4. An Matlab script

Differential and Difference LTI Systems

Differential and difference LTI systems constitute an extremely important class of systems in engineering. They are used for:

- circuit analysis,
- filter design,
- controller design,
- process modeling, etc.

Differential Systems: A Subset of LTI Systems



This is a class of LTI systems, for which the input and output signals are related *implicitly* through a *linear constant coefficient differential equation.*

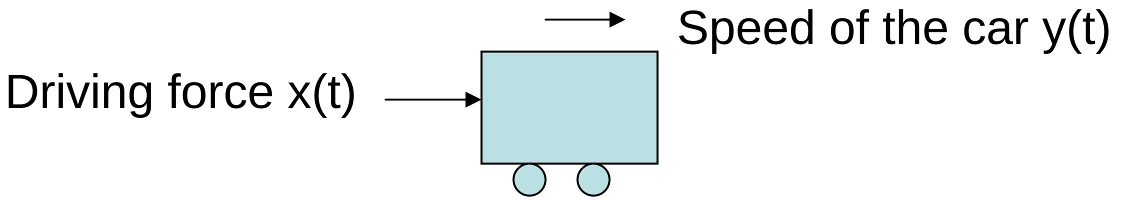
CT LTI Systems Described Using Differential Equations

Example: First-order differential equation relating the input $x(t)$ to the output $y(t)$

$$1000\frac{dy(t)}{dt} + 300y(t) = x(t)$$

where *y(t)* is the speed of a car, which is subjected to a friction force proportional to the speed, and the tractive force *x(t)*.

We have to solve the differential equation to obtain the speed of the car, an output signal of the system.



A general form of differential equations

In general, an *N^{th}-order linear constant coefficient differential equation* has the form:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

which can be expanded to

$$a_N \frac{d^N y(t)}{dt^N} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

where x(t) is input signal, y(t) is output signal. To find a solution, we need N initial conditions on the output variable and its derivatives to be able to fully determine a solution.

The solutions of differential equations

Recall from previous math courses:

The complete solution

= ***the homogeneous solution + the particular solution***

- The *homogeneous solution* is a solution of the differential equation with the input signal set to zero
  - Homogeneous solution is also called *natural response* of the system, and depends on initial conditions and forced response
- The *particular solution* is a function that satisfies the differential equation.
  - Particular solution is also called *forced response* of the system, and usually has the form of the input signal

Example:

Consider the system

$$1000\frac{dy(t)}{dt}+300y(t)=x(t) \quad (**)$$

We will calculate the output of this system to the input signal $x(t)=5000e^{-2t}u(t)$ N .

Let

$$y(t)=y_h(t)+y_p(t)$$

where the particular solution satisfies (**), and the homogeneous solution $y_h(t)$ satisfies the zero-input equation:

$$1000\frac{dy_h(t)}{dt}+300y_h(t)=0 \quad .$$

The particular solution

For the particular solution (forced response) for $t > 0$, we look at a signal $y_p(t)$ of the same form as $x(t)$ for $t > 0$:

$$y_p(t) = Ce^{-2t}.$$

Substituting the exponentials for $x(t)$ and $y_p(t)$ in the differential equation, we get

$$-2000Ce^{-2t} + 300Ce^{-2t} = 5000e^{-2t},$$

which yields $C = -{}^{5000}\!/_{1700} = -2.941$ and

$$y_p(t) = -2.941e^{-2t}, \;\; t > 0.$$

The Natural Response

Now we want to determine $y_h(t)$, the natural response of the system. We hypothesize a solution of the form of an exponential:

$$y_h(t) = Ae^{st}.$$

Substituting this exponential in the homogeneous differential equation, we get

$$1000Ase^{st} + 300Ae^{st} = Ae^{st}(1000s + 300) = 0,$$

which holds for $s = -0.3$. Also with this value for s,

$$y_h(t) = Ae^{-0.3t}$$

is a solution to the homogeneous equation for any A.

Combining the natural response (homogenous solution) and the forced response (particular solution)

The solution to the differential equation is:

$$y(t) = y_h(t) + y_p(t) = Ae^{-0.3t} - 2.941e^{-2t}, \quad t > 0\,.$$

As the value of A is still unknown due to an unknown initial condition on $y(t)$,this response is not completely determined.

Initial conditions for the solution

For causal LTI systems defined by Nth-order linear constant coefficient differential equations, the initial conditions are always

$$y(0) = \frac{dy(0)}{dt} = \cdots = \frac{d^{N-1}y(0)}{dt^{N-1}} = 0$$

for what's called "initial rest".

For our car example, initial rest implies that $y(0) = 0$, so that

$$y(0) = Ae^0 - 2.941e^0 = A - 2.941 = 0$$

and we get $A = 2.941$. Thus, for $t > 0$, the solution (output signal) is:

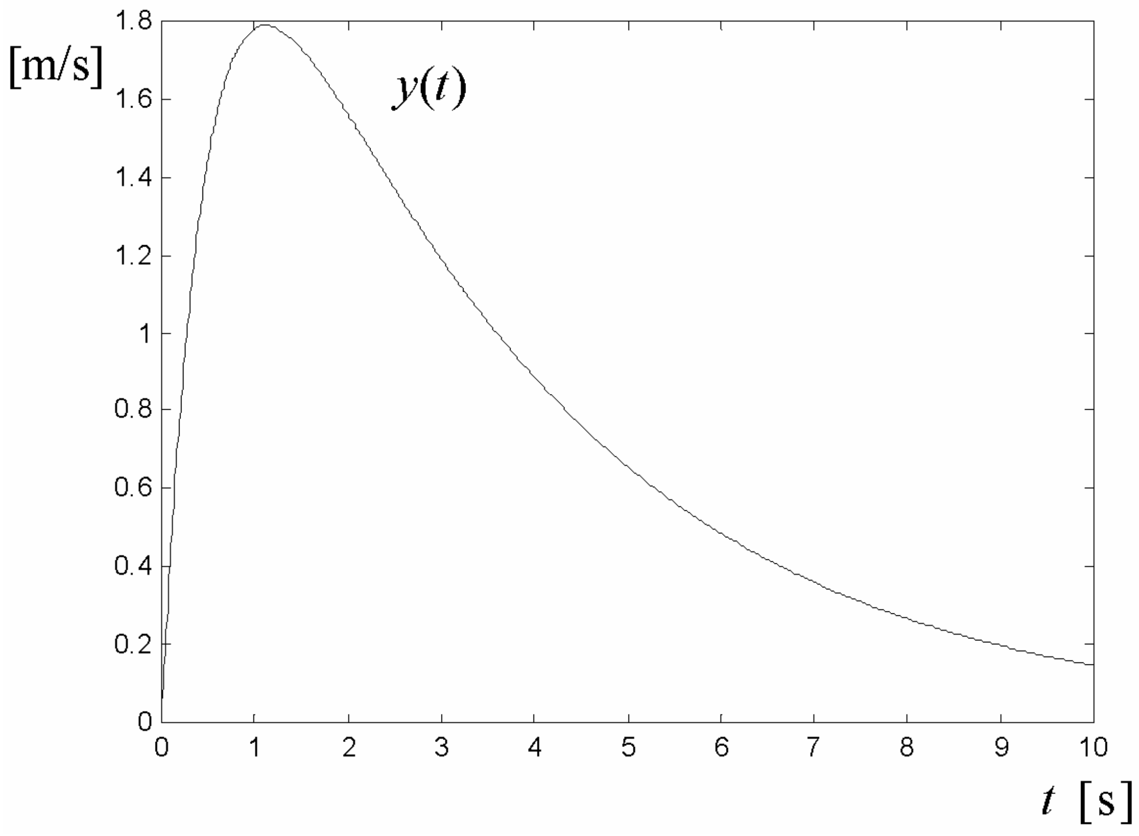
$$y(t) = 2.941(e^{-0.3t} - e^{-2t}), \quad t > 0 .$$

What about the negative times?

The condition of initial rest and causality of the system implies that *y(t)=0 for t<0* since *x(t)=0 for t<0*. This is true in general for causal LTI systems.

For causal differential systems, the condition of initial rest means that the output of the system is zero until the time when the input becomes nonzero.

Speed of the car



DT LTI Described by Difference Equations

In a DT causal LTI difference system, the discrete-time input x[n] and output y[n] signals are related *implicitly* through a *linear constant-coefficient difference equation.*

In general, an N^{th}*-order linear constant coefficient difference equation* has the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

which can be expanded to

$$a_N y[n-N] + \cdots + a_1 y[n-1] + a_0 y[n] = b_M x[n-M] + \cdots + b_1 x[n-1] + b_0 x[n]$$

We need N initial conditions on the output variable (its N past values) to be able to compute a specific solution.

General Solution

A general solution can be expressed as the sum of a homogeneous solution (natural response) to $\sum_{k=0}^{N} a_k y[n-k]=0$ and a particular solution (forced response), in a manner analogous to the continuous-time case.

$$y[n]=y_h[n]+y_p[n].$$

The concept of *Initial rest* of the LTI causal system described by the difference equation here means that $x[n]=0,\ n<n_0$ implies $y[n]=0,\ n<n_0$.

Example

Consider the first-order difference equation initially at rest:

$$y[n] + 0.5y[n-1] = (-0.8)^n u[n] \qquad (\&)$$

The solution is composed of a homogeneous response, and a particular solution of the system:

$$y[n] = y_h[n] + y_p[n],$$

where the particular solution satisfies (&) for $n \geq 0$, and the homogeneous solution satisfies:

$$y_h[n] + 0.5y_h[n-1] = 0.$$

The particular (forced) solution

for $n \geq 0$, we look at a signal $y_p[n]$ of the same form as $x[n]$:

$y_p[n] = Y(-0.8)^n$. Then, we get

$$Y(-0.8)^n + 0.5Y(-0.8)^{n-1} = (-0.8)^n$$

$$\Leftrightarrow$$

$$Y\left[1 + 0.5(-0.8)^{-1}\right] = 1$$

$$Y = \frac{8}{3}$$

which yields $\quad y_p[n] = \frac{8}{3}(-0.8)^n$.

The homogenous (natural) solution

Now we want to determine $y_h[n]$, the natural response of the system. We hypothesize a solution of the form of an exponential signal: $y_h[n] = Az^n$. Substituting this exponential in the difference equation, we get

$$Az^n + 0.5Az^{n-1} = 0$$

$$\Leftrightarrow$$

$$1 + 0.5z^{-1} = 0$$

$$z = -0.5$$

With this value for z, $y_h[n] = A(-0.5)^n$ is a solution to the homogeneous equation for any choice of A.

Combining the natural response and the forced response

The solution for n>=0 is :

$$y[n] = y_h[n] + y_p[n] = A(-0.5)^n + \frac{8}{3}(-0.8)^n.$$

The assumption of initial rest implies y[-1]=0. But we need to use an initial condition at a time "*n*" when the particular solution exists, e.g. y[0]=? y[0] can be found by recursion:

$$y[n] = -0.5y[n-1] + (-0.8)^n u[n]$$

$$n = 0: \quad y[0] = -0.5y[-1] + (-0.8)^0 = 0 + 1 = 1$$

Then we can compute the value of A:

$$1 = y[0] = A(-0.5)^0 + \frac{8}{3}(-0.8)^0 \Rightarrow A = -\frac{5}{3}$$

Thus, the complete solution is (check as an exercise):

$$y[n] = -\frac{5}{3}(-0.5)^n u[n] + \frac{8}{3}(-0.8)^n u[n].$$

Recursive Solution

For DT systems output y[n], we can compute it *recursively,* derive the current output from the input and previous outputs:

$$y[n] = \frac{1}{a_0}\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\}$$

Suppose that the system is initially at rest. Then, the condition of initial rest means that

$$y[-1]=y[-2]=\ldots=y[-N]=0$$

One can start computing recursively. This is often how digital filters are implemented on a computer or a DSP board.

Example

Consider the difference equation:

$$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=3x[n]-2x[n-1].$$

Rearranging, we obtain the recursive form:

$$y[n]=\frac{5}{6}y[n-1]-\frac{1}{6}y[n-2]+3x[n]-2x[n-1].$$

Assuming initial rest and that the input is an impulse x[n]=δ[n]. We have y[-2]=y[-1], and the recursion can be started as follows.

$$y[0]=\frac{5}{6}y[-1]-\frac{1}{6}y[-2]+3x[0]-2x[-1]$$

$$=\frac{5}{6}(0)-\frac{1}{6}(0)+3(1)-2(0)=3$$

$$y[1]=\frac{5}{6}y[0]-\frac{1}{6}y[-1]+3x[1]-2x[0]$$

$$=\frac{5}{6}(3)-\frac{1}{6}(0)+3(0)-2(1)=\frac{1}{2}$$

$$y[2]=\frac{5}{6}y[1]-\frac{1}{6}y[0]+3x[2]-2x[1]$$

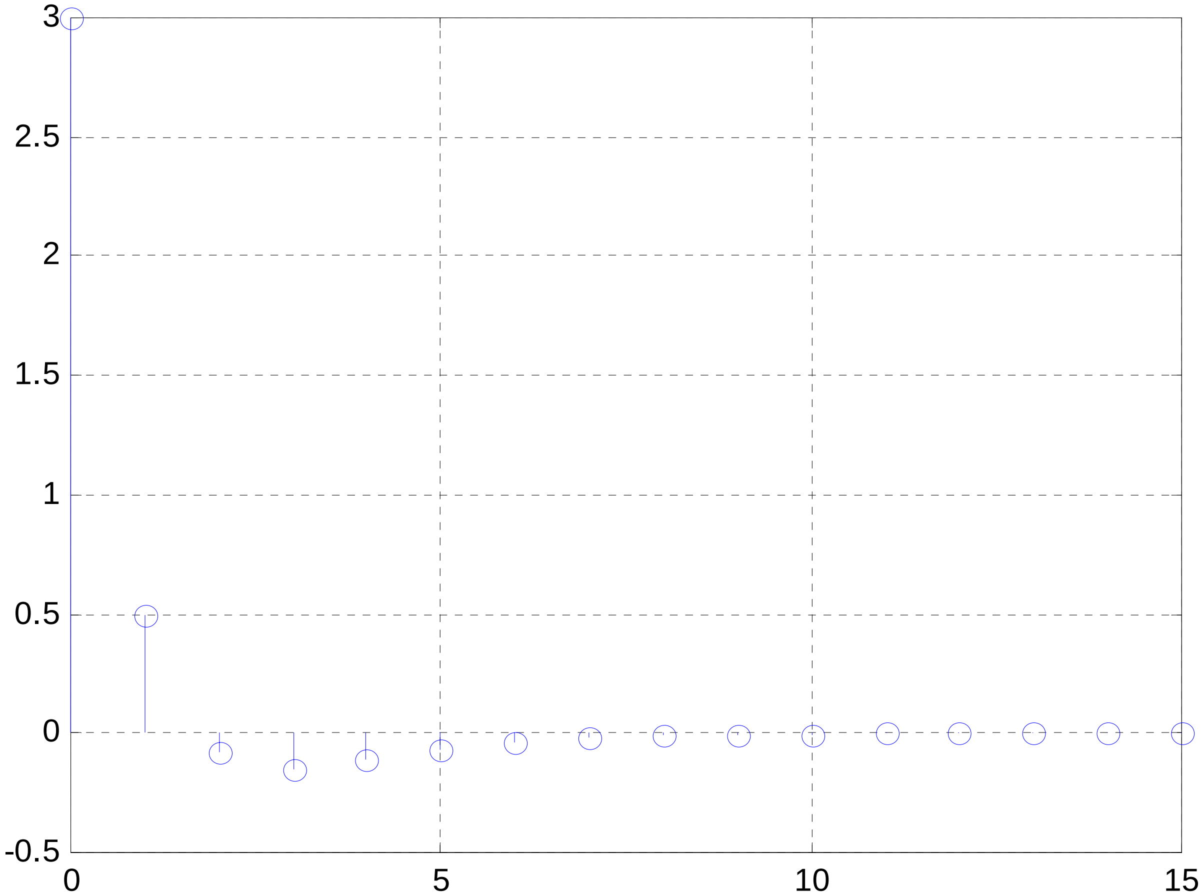
$$=\frac{5}{6}(\frac{1}{2})-\frac{1}{6}(3)+3(0)-2(0)=-\frac{1}{12}$$

$$\vdots$$

Obtaining recursive solution using Matlab

```
% computes the response of a difference system recursively
% time vector
n=0:1:15;
% define the input signal: delta function
x=[1 zeros(1,length(n)-1)];
y=zeros(1,length(n));
% initial conditions
yn_1=0;
yn_2=0;
xn_1=0;
xn=0;
```

```
% recursion
for k=1:length(n)
    xn=x(k);
    yn=(5/6)*yn_1-(1/6)*yn_2+3*xn-2*xn_1;
    y(k)=yn;
    yn_2=yn_1;
    yn_1=yn;
    xn_1=xn;
end
% plot output
stem(n,y)
```



3
2.5
2
1.5
1
0.5
0
-0.5
0
5
10
15

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Department of Electrical and Computer Engineering

Lecture 8

Hui Qun Deng, PhD

Obtaining the impulse response given a differential equation

1. Deriving the impulse response $h_a(t)$ of a homogeneous differential equation
2. Linearly combining $h_a(t)$ and its derivatives

The Impulse Response of a Differential LTI System

The general form of a causal LTI differential system with **input x(t)** and **output y(t)**:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \qquad (3.1)$$

which can be expanded to

$$a_N \frac{d^N y(t)}{dt^N} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$x(t) = \delta(t)$

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}}$$

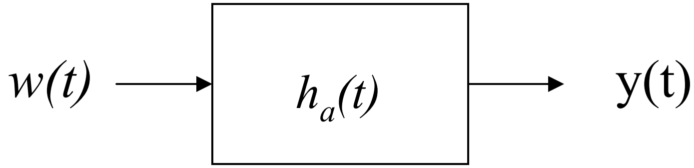
y(t)=h(t)=?

The equivalent system of the differential equation

Look at a system defined by:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = w(t) \qquad \text{(a)}$$

Eq. (a) can be view as a system with input $w(t)$ and output $y(t)$. Let $h_a(t)$ be the impulse response of the system defined by Eq. (a). Then system (a) can be represented using the diagram:



Now, the input $w(t)$ is:

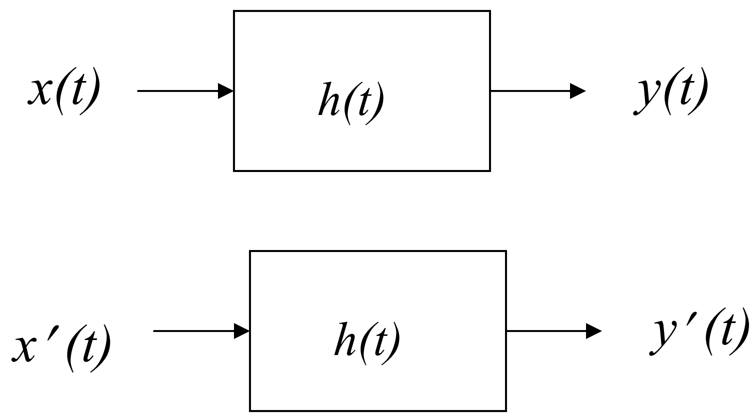
$$w(t) = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \qquad \text{(b)}$$

Let $y_m(t)$ be the response of $h_a(t)$ to the m^{th}-order derivative of $x(t)$. From the principle of superposition of LTI systems, we then have:

$$y(t) = \sum_{m=0}^{M} b_m y_m(t)$$

The response of an LTI system to the derivative of a signal

Let the response of an LTI system to an input signal *x(t)* be *y(t)*. Then, the response of the LTI system to the *derivative of x(t)* is the *derivative of y(t).*



This property comes from the properties of superposition and the time-invariance of LTI systems.

Method 1: obtain *h(t)* from a differential equation according to the *principle of superposition*

1. Replace the whole right-hand side of the differential equation (3.1) by $\delta(t)$,
2. Integrate from $t = 0^-$ to $t = 0^+$ to find a set of initial conditions at $t = 0^+$,
3. Calculate the impulse response $h_a(t)$ of the homogeneous equation given these initial conditions,
4. Finally, differentiate the impulse response $h_a(t)$ of the homogeneous equation, and use linear superposition to form the overall impose response of the system (3.1)

Step 1

First replace the right-hand side of the differential equation by a single unit impulse:

$$\sum_{k=0}^{N} a_k \frac{d^k h_a(t)}{dt^k} = \delta(t).$$

Under the assumption that $N \geq M$ and the system is initially at rest, then

$$y(0^-) = \frac{dy(0^-)}{dt} = \cdots = \frac{d^{N-1}y(0^-)}{dt^{N-1}} = 0$$

Step 2: determine initial conditions of the homogeneous impulse response $h_a(t)$

With $\delta(t)$ being the right hand of Eq. (a), the impulse on the left-hand side can only be generated by the highest-order (i.e., N^{th}-order) derivative of $h_a(t)$. Otherwise, there would be $\delta^{(k)}(t)$ for k>1 on the left-hand side of Eq. (3.1).

Thus, $h_a(t), \dfrac{dh_a(t)}{dt}, \cdots, \dfrac{d^{N-1}h_a(t)}{dt^{N-1}}$ have finite discontinuities at worst, and their integrals over an infinitely small interval of time are zeros:

$$\int_{0^-}^{0^+} \frac{d^k h_a(t)}{dt^k} dt = \frac{d^{k-1} h_a(0^+)}{dt} = 0, \qquad k = 1, \ldots, N-1$$

Step 3: obtain the homogeneous solution of the deferential equation

Given the initial conditions at time $t=0^+$ found in Step 2, find the homogeneous solution of the equation

$$\sum_{k=0}^{N} a_k \frac{d^k h_a(t)}{dt^k} = 0. \qquad \text{(c)}$$

Assuming that the solution has the form of a complex exponential Ae^{st} for t>0, substitute $h_a(t)$ with Ae^{st} in Eq. (c), we get:

$$Ae^{st} \sum_{k=0}^{N} a_k s^k = 0$$

This last equation holds if and only if the characteristic polynomial below equals zero:

$$p(s) := \sum_{k=0}^{N} a_k s^k = a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0 = 0$$

This equation has N roots. Assume that the N roots are distinct, then there are N distinct functions $A_k e^{s_k t}$ that satisfy the homogeneous equation. Then, the homogeneous solution can be written as:

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t}.$$

Determine the coefficients of the homogeneous solution $h_a(t)$

The complex coefficients A_k's can be computed using the initial conditions:

$$0 = h_a(0^+) = \sum_{k=1}^{N} A_k e^{s_k 0^+} = \sum_{k=1}^{N} A_k$$

$$0 = \frac{dh_a(0^+)}{dt} = \sum_{k=1}^{N} A_k s_k$$

$$\vdots$$

$$\frac{1}{a_N} = \frac{d^{N-1} h_a(0^+)}{dt^{N-1}} = \sum_{k=1}^{N} A_k s_k^{\,N-1}$$

This set of linear equations can be written as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1/a_N \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_N \\ s_1^{\,2} & s_2^{\,2} & \cdots & s_N^{\,2} \\ \vdots & \vdots & \vdots & \vdots \\ s_1^{\,N-1} & s_2^{\,N-1} & \cdots & s_N^{\,N-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_N \end{bmatrix}$$

The matrix in this equation is called a *Vandermonde matrix* and it can be shown to be nonsingular (invertible). So a unique solution always exists for the A_k's which gives us the unique solution $h_a(t)$

Step 4: linear combination of responses

According to the linearity of the differential system defined in Eq. (3.1), its impulse response is a linear combination of $h_a(t)$ and its derivatives.

Thus, the impulse response of the system defined by Eq. (3.1) is:

$$h(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} h_a(t) \qquad (3.42)$$

Example

Consider the first-order system initially at rest with *time constant* τ_0

$$\tau_0 \frac{dy(t)}{dt} + y(t) = \frac{d}{dt}x(t) + x(t).$$

Step 1: calculate the impulse response of the left-hand side of the differential equation.

$$\tau_0 \frac{dh_a(t)}{dt} + h_a(t) = \delta(t)$$

Step 2: Find the initial condition of the homogeneous solution at $t = 0^+$ by integrating from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\tau_0 \dfrac{dh_a(t)}{dt}$, so $h_a(t)$ will have a finite jump at most. Thus we have

$$\tau_0 \int_{0^-}^{0^-} \frac{dh_a(t)}{dt} dt = \tau_0 h_a(0^+) = 1$$

Hence $\quad h_a(0^+) = \dfrac{1}{\tau_0}$

is our initial condition for solving

$$\tau_0 \frac{dh_a(t)}{dt} + h_a(t) = 0.$$

Step 3:

Let the homogeneous solution be $h_a(t)=Ae^{st}$. The characteristic polynomial is $p(s)=\tau_0 s+1$ and it has one root at $s=\dfrac{-1}{\tau_0}$.

Thus, $h_a(t)=Ae^{\frac{-t}{\tau_0}}$ for $t>0$. The initial condition allows us to determine the constant A:

$$h_a(0^+)=A=\frac{1}{\tau_0},$$

and
$$h_a(t)=\frac{1}{\tau_0}e^{\frac{-t}{\tau_0}}u(t).$$

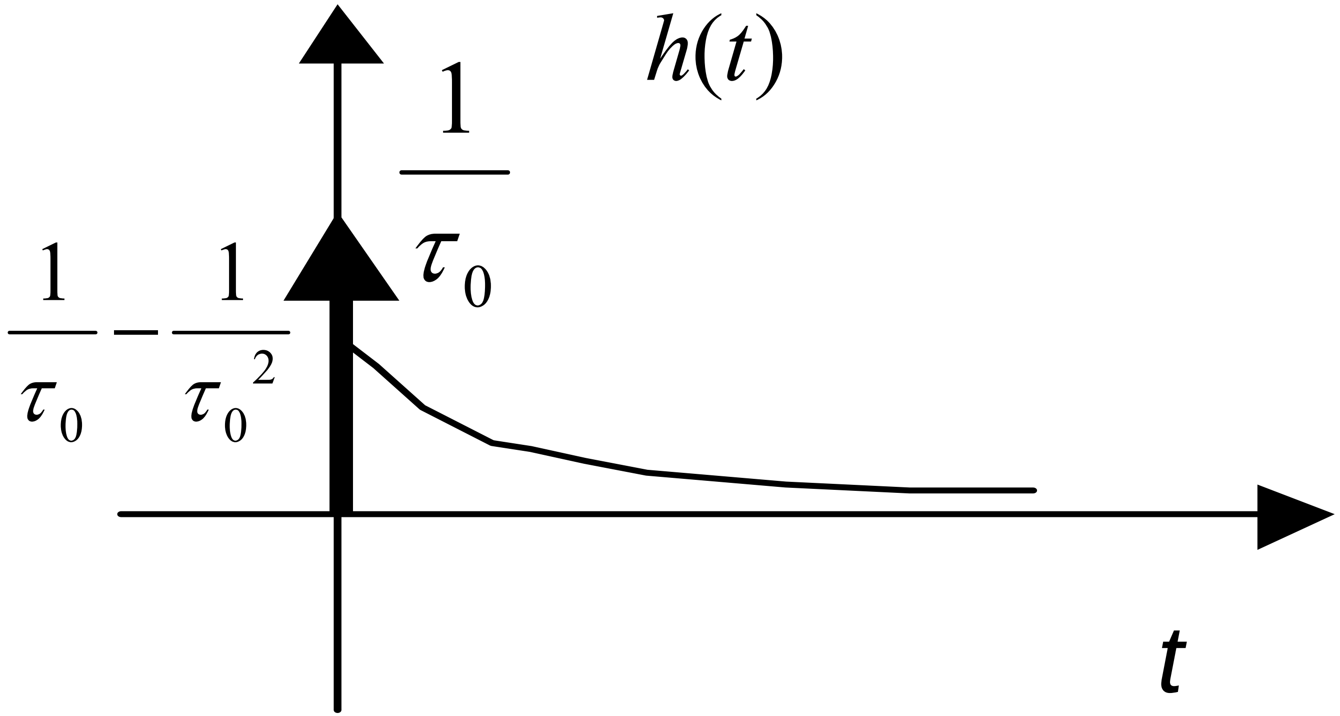
Step 4: Finally, the impulse response of the differential system is:

$$h(t) = \frac{dh_a(t)}{dt} + h_a(t)$$

$$= \frac{d}{dt}\{\frac{1}{\tau_0} e^{-\frac{t}{\tau_0}} u(t)\} + \frac{1}{\tau_0} e^{-\frac{t}{\tau_0}} u(t)$$

$$= -\frac{1}{\tau_0{}^2} e^{-\frac{t}{\tau_0}} u(t) + \frac{1}{\tau_0} e^{-\frac{t}{\tau_0}} \delta(t) + \frac{1}{\tau_0} e^{-\frac{t}{\tau_0}} u(t)$$

$$= \{\frac{1}{\tau_0} - \frac{1}{\tau_0{}^2}\} u(t) + \frac{1}{\tau_0} e^{-\frac{t}{\tau_0}} \delta(t)$$


$h(t)$
$\frac{1}{\tau_0}$
$\frac{1}{\tau_0} - \frac{1}{\tau_0^2}$
t

Method 2: by differentiating the step response

We have seen that $h(t) = \dfrac{ds(t)}{dt}$.

Thus we can obtain the impulse response of an LTI differential system by first calculating its step response, and then differentiating it.

Example

Consider the following second-order causal LTI differential system initially at rest.

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$$

Let $x(t) = u(t)$. The ***characteristic polynomial*** of this system is

$$p(s) = s^2 + 3s + 2 = (s+2)(s+1),$$

and its *roots* (i.e., values of s for which $p(s) = 0$) are $s = -2$ and $s = -1$. Hence ***the homogeneous solution has the form***

$$y_h(t) = A_1 e^{-2t} + A_2 e^{-t}.$$

Denote a particular solution of the form $y_p(t) = K$ for $t > 0$ when input is $x(t) = 1$. Substituting in the differential equation, we find K=1/2. So,

$$y_p(t) = \frac{1}{2}.$$

Adding the homogeneous and particular solutions, we obtain the **overall step response** for $t > 0$:

$$s(t) = A_1 e^{-2t} + A_2 e^{-t} + \frac{1}{2}.$$

The initial conditions at $t = 0^-$ are $\dot{y}(0^-) = y(0^-) = 0$. Because the input signal *u(t)* has a finite jump at $t > 0$, this jump will be included in $\ddot{y}(t)$ only and $\dot{y}(t), y(t)$ will be ***continuous***. Hence $\dot{y}(0^-) = \dot{y}(0^+) = 0,\ y(0^-) = y(0^+) = 0$ and

$$y(0^+) = A_1 + A_2 + \frac{1}{2} = 0$$

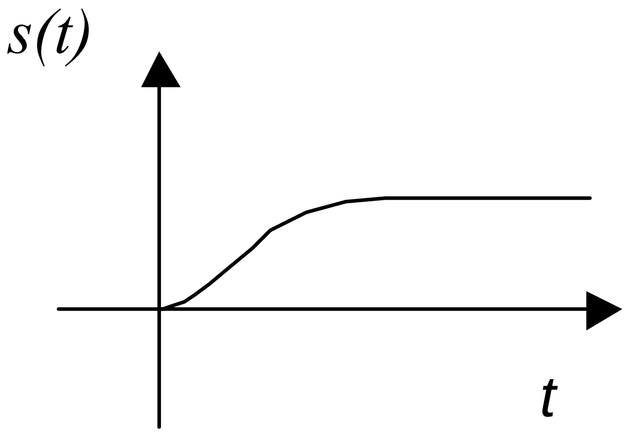
$$\dot{y}(0^+) = -2A_1 - A_2 = 0$$

The solution to these two linear algebraic equations is

$$A_1 = \frac{1}{2}, A_2 = -1$$

Thus, the step response of the system is:

$$s(t) = [\frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}]u(t)$$



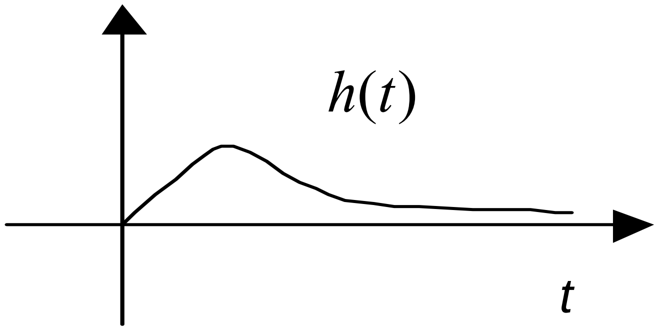
Finally, the impulse response of the second-order system is obtained by differentiating the step response.

$$h(t) = \frac{d}{dt}s(t) = \frac{d}{dt}\left[\left(\frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}\right)u(t)\right]$$

$$= \left(e^{-t} - e^{-2t}\right)u(t) + \left(\frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}\right)\delta(t)$$

$$= \left(e^{-t} - e^{-2t}\right)u(t)$$

which evaluates to 0 at time $t = 0^{+}$. Hence there is no jump in the impulse response.



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Lecture 9

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1. Obtaining the impulse response given a difference equation
   - Deriving the impulse response $h_a[n]$ of the homogeneous difference equation
   - Linearly combining $h_a[n]$ and its delayed versions
2. Characteristic polynomials of differential and difference equations
3. Stability of differential and difference systems

Obtaining the impulse response h[n] given a difference equation

The given difference equation has the following form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \qquad (3.60)$$

The main idea is first to derive the solution $h_a[n]$ to the following equation:

$$\sum_{k=0}^{N} a_k h_a[n-k] = \delta[n] \qquad (3.61)$$

From the principle of superposition of LTI systems, the impulse response of the system defined by Eq. (3.60) can then be obtained by linearly combining $h_a[n]$ and its delayed versions:

$$h[n] = \sum_{m=0}^{M} b_m h_a[n-m] \qquad (3.68)$$

Step 1: determine the impulse response for the homogeneous equation

Consider the system with input being the unit impulse signal:

$$\sum_{k=0}^{N} a_k h_a[n-k] = \delta[n] \qquad (3.61)$$

For n=0, $a_0 h_a[0]=1$, i.e., $h_a[0]=1/a_0$. This is due to the causality and initial rest conditions $h_a[-1]=\ldots= h_a[-N+1]=0$.

For n>=1, the input is zero, and thus $h_a[n]$ is the solution to the homogeneous equation:

$$\sum_{k=0}^{N} a_k h_a[n-k] = 0 \qquad (3.62)$$

The solution of the homogeneous equation

For $n>=1$, solve for $h_a[n]$ from the homogeneous equation:

$$\sum_{k=0}^{N} a_k h_a[n-k] = 0$$

$h_a[n]$ has the *complex exponential* form $\boldsymbol{h_a[n]=Cz^n}$. Substituting $h_a[n]$, we get:

$$\sum_{k=0}^{N} a_k C z^{n-k} = 0 \qquad (3.63)$$

i.e.,

$$Cz^n (a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}) = 0$$

$$p(z) := \sum_{k=0}^{N} a_k z^{N-k}$$

p(z) is called the ***characteristic polynomial*** of Eq. (3.60).

The solution of the homogeneous equation

Eq. (3.63) holds if and only if the *characteristic polynomial* *p(z)* vanishes at the complex number *z*:

$$p(z) := \sum_{k=0}^{N} a_k z^{N-k} = a_0 z^N + a_1 z^{N-1} + \cdots + a_N = 0$$

p(z) has *N* zeros, z_k, $k=1, 2, \ldots, N$. If they are distinct, then there are N distinct functions $C_k z_k$ that satisfy the homogeneous equation Eq. (3.62).

Then the solution to the homogeneous equation is a linear combination of these complex exponentials:

$$h_a[n] = \sum_{k=1}^{N} C_k {z_k}^n \quad (3.65)$$

Determine the coefficients of the homogeneous solution

The complex coefficients C_k can be computed using the initial conditions:

$$\frac{1}{a_0} = h_a[0] = \sum_{k=1}^{N} C_k$$

$$0 = h_a[-1] = \sum_{k=1}^{N} C_k z_k{}^{-1}$$

$$\vdots$$

$$0 = h_a[-N+1] = \sum_{k=1}^{N} C_k z_k{}^{-N+1}$$

Determine the impulse response for the general difference equation

Finally, by the properties of the LTI difference system, the response of the left-hand side of the difference equation to its right-hand side inputs is a linear combination of $h_a[n]$ and its delayed versions.

Combining Eq. (3.68) and Eq. (3.65), we get the impulse response of the general causal LTI system described by Eq. (3.60):

$$h[n] = \sum_{m=0}^{M} b_m h_a[n-m] = \sum_{m=0}^{M} b_m \left(\sum_{k=1}^{N} C_k z_k^{\,n-m}\right)$$

Example

Consider the following second-order, causal difference LTI system initially at rest:

$$y[n]-0.5y[n-1]+0.06y[n-2]=x[n-1]$$

The characteristic polynomial is given by:

$$p(z)=z^2-0.5z+0.06=(z-0.2)(z-0.3)$$

and its zeros are $z_1=0.2,\ \ z_2=0.3$. The homogeneous response is given by:

$$h_a[n]=A(0.2)^n+B(0.3)^n,\ \ n>0\,.$$

The initial conditions for the homogeneous solution are $h_a[-1]=0$ and $h_a[0]=\delta[0]=1$.

Now we can compute the coefficient A and B:

$$h_a[-1] = A(0.2)^{-1} + B(0.3)^{-1} = 5A + \frac{10}{3}B = 0$$

$$h_a[0] = A + B = 1$$

Hence, $A = -2,\ B = 3$, and the impulse response is obtained:

$$h[n] = h_a[n-1] = \left[-2(0.2)^{n-1} + 3(0.3)^{n-1}\right]u[n-1]$$

Characteristic polynomials of differential systems

The characteristic polynomial of a causal differential LTI system is given by

$$p(s) = a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0 .$$

p(s) depends only on the coefficients of the left-hand side of the differential equation, doesn't depend on the input, and thus characterizes the intrinsic properties of the differential system

The zeros s_k of p(s) are the exponents of the exponentia e^{st} forming the homogeneous response. So, s_k's indicate system properties, such as stability.

Stability of an LTI differential system

Recall: an LTI system is BIBO stable if and only if its impulse response is absolutely integrable.

Assume N>=M. The impulse response h(t) will have at worst a single impulse, which integrates to a finite value when integrated from $t=-\infty$ to $t=0^+$. Then we examine the integral from t=0$^+$ to ∞:

$$\int_{0^+}^{\infty}|h(t)|\,dt = \int_{0^+}^{\infty}\left|\sum_{k=0}^{M} b_k \frac{d^k}{dt^k}\left(\sum_{m=1}^{N} A_m e^{s_m t}\right)\right| dt \le \int_{0^+}^{\infty}\sum_{k=0}^{M}|b_k|\left|\frac{d^k}{dt^k}\left(\sum_{m=1}^{N} A_m e^{s_m t}\right)\right| dt$$

$$= \sum_{k=0}^{M}|b_k|\int_{0^+}^{\infty}\left|\frac{d^k}{dt^k}\left(\sum_{m=1}^{N} A_m e^{s_m t}\right)\right| dt = \sum_{k=0}^{M}|b_k|\int_{0^+}^{\infty}\left|\sum_{m=1}^{N} A_m s_m{}^k e^{s_m t}\right| dt$$

$$\le \sum_{k=0}^{M}|b_k|\int_{0^+}^{\infty}\sum_{m=1}^{N}\left|A_m s_m{}^k e^{s_m t}\right| dt \le \sum_{k=0}^{M}|b_k|\sum_{m=1}^{N}\left|A_m s_m{}^k\right|\int_{0^+}^{\infty} e^{\mathrm{Re}\{s_m\}t}\,dt$$

The last upper bound will be finite if and only if $\mathrm{Re}\{s_m\}<0$.

To summarize:

A causal LTI differential system is BIBO stable if and only if the real part of all of the zeros of its characteristic polynomial are negative

(we say that they are in the open left-half of the complex plane).

Example

Let's assess the stability of

$$\frac{dy(t)}{dt} - y(t) = \frac{d}{dt}x(t) + x(t).$$

The characteristic polynomial is $p(s) = s - 1$ which has its zero at $s = 1$.

This system is therefore unstable, which is easy to see with an impulse response of the form $Ae^{t}u(t)$ (a growing exponential.)

Characteristic polynomials of deference systems

Recall: the characteristic polynomial of a causal difference LTI system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

is given by $p(z) := \sum_{k=0}^{N} a_k z^{N-k} = a_0 z^N + a_1 z^{N-1} + \cdots + a_N = 0$

The zeros z_k (assumed to be distinct) of the characteristic polynomial are the arguments of the exponentials forming the homogeneous response, so they indicate system properties, such as stability.

Stability of an LTI difference system

Recall: an LTI difference system is stable if and only if its impulse response is absolutely summable.

For the causal difference system above, this leads to the upper bound

$$\sum_{n=0}^{+\infty}\left|h[n]\right| = \sum_{n=0}^{+\infty}\left|\sum_{k=0}^{M} b_k h_a[n-k]\right| \le \sum_{n=0}^{+\infty}\sum_{k=0}^{M}\left|b_k\right|\left|h_a[n-k]\right|$$

$$= \sum_{k=0}^{M}\sum_{n=k}^{+\infty}\left|b_k\right|\left|h_a[n-k]\right| = \sum_{k=0}^{M}\sum_{n=k}^{+\infty}\left|b_k\right|\left|\sum_{m=1}^{N} C_m z_m{}^{n-k}\right|$$

$$\le \sum_{k=0}^{M}\sum_{n=k}^{+\infty}\left|b_k\right|\sum_{m=1}^{N}\left|C_m\right|\left|z_m{}^{n-k}\right| = \sum_{k=0}^{M}\left|b_k\right|\sum_{m=1}^{N}\left|C_m\right|\sum_{n=k}^{+\infty}\left|z_m\right|^{n-k}$$

$$= \sum_{k=0}^{M}\left|b_k\right|\sum_{m=1}^{N}\left|C_m\right|\sum_{r=0}^{+\infty}\left|z_m\right|^{r}$$

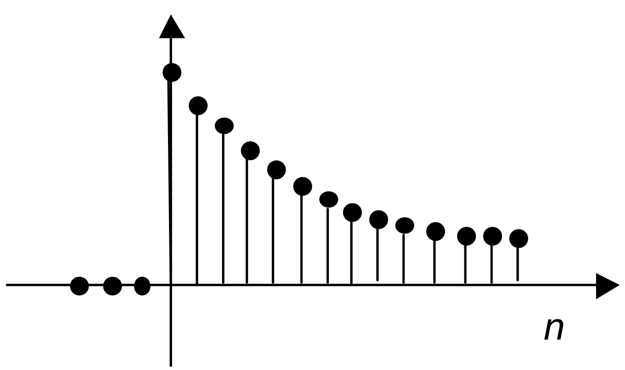
This above upper bound is finite if and only if $|z_m| < 1,$ for all $m = 1, \ldots, N$. Hence the causal LTI difference system is BIBO stable if and only if all the magnitudes of the zeros of its characteristic polynomial are less than 1.

Example: Consider the causal first-order system

$$y[n] - 0.9y[n-1] = x[n].$$

Its characteristic polynomial is p(z)=z-0.9, which has a single zero at z=0.9. Hence this system is stable as |z|=0.9<1. The impulse response h[n] of the system is:

$$h[n] = 0.9^n u[n]$$



Time constant of a 1st-order LTI differential system

The impulse response of an LTI differential system is a linear combination of complex exponentials of the type $Ae^{st}u(t)$ and their derivatives. Consider the stable 1st-order system

$$a_1\dot{y}(t)+a_0y(t)=Kx(t)$$

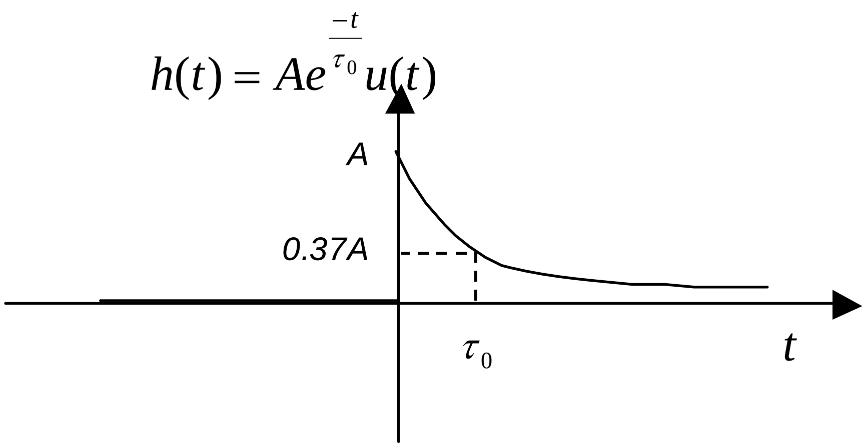
Its impulse response is a single exponential:

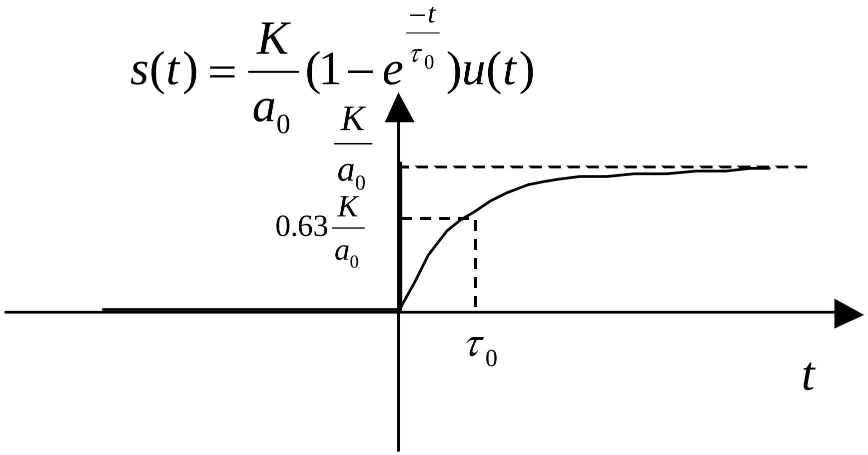
$$h(t)=Ae^{s_1t}u(t)$$

where $s_1=-a_0/a_1$ and $A=K/a_1$. The real number $\omega_n=|s_1|$ is called the *natural frequency* of the first-order system and its inverse $\tau_0=\frac{1}{\omega_n}=\frac{a_1}{a_0}$ is called the *time constant* of the first-order system.

The time constant indicates the decay rate of the impulse response and the rise time of the step response.

At time $t = \tau_0$, the impulse response is $h(\tau_0) = Ae^{-\frac{\tau_0}{\tau_0}} = Ae^{-1} = 0.368A$, so the impulse response has decayed to 36.8% of its value at time $t = 0$.





	Continuous time	Discrete time
Convolution		
Differential/difference Eqs.	$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$	$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$
Homogeneous Eqs.	$\sum_{k=0}^{N} a_k \frac{d^k h_a(t)}{dt^k} = 0$	$\sum_{k=0}^{N} a_k h_a[n-k] = 0$
Solutions to the homogeneous Eq.	Ae^{st}	Cz^n
Characteristic polynomials	$p(s) = a_N s^N + a_{N-1} s^{N-1} + \cdots + a_0$	$p(z) = \sum_{k=0}^{N} a_k z^{N-k}$
Impulse response for the homogeneous Eq.	$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t}$	$h_a[n] = \sum_{k=1}^{N} C_k z_k^n$
Impulse response for general differential/difference Eq.	$h(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} h_a(t)$	$h[n] = \sum_{m=0}^{M} b_m h_a[n-m] = \sum_{m=0}^{M} b_m (\sum_{k=1}^{N} C_k z_k^{n-m})$
Stability	$\mathrm{Re}\{S_k\} < 0$	$\lvert z_k \rvert < 1$