ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University

Department of Electrical and Computer Engineering

Lecture 10

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1. Fourier Series Representation of Periodic Continuous-Time Signals
2. Properties of Fourier Series

The Responses of LTI Systems to Complex Exponential Signals

Complex exponentials of the type cz^n and Ae^{st} remain basically invariant under the action of time shifts and derivatives.

The response of an LTI system to a complex exponential input is the same complex exponential with only a change in (complex) amplitude:

Continuous-time LTI system: $e^{st} \to H(s)e^{st}$

Discrete-time LTI system: $z^n \to H(z)z^n$

where the complex amplitude factors $H(s), H(z)$ are functions of the complex variable s or z.

Eigenfunctions of LTI Systems

Input signals like $x[n] = z^n$ and $x(t) = e^{st}$ are called *eigenfunctions* of LTI systems.

Fact: the response of an LTI system to such a signal is the input signal multiplied by a complex constant.

The complex gains are the system's *eigenvalues* corresponding to the eigenfunctions.

Eigenfunctions of CT LTI systems

Let the impulse response of an continuous-time LTI system be h(t). Then the response of the system to $\boldsymbol{e^{st}}$ is:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau := H(s)e^{st}$$

The system's response has the form $y(t) = H(s)e^{st}$

Thus $\boldsymbol{e^{st}}$ is an eigenfunction of an LTI system.

Eigenfunctions of DT LTI systems

Let h[n] be the impulse response of a discrete-time LTI systems. Its response to a complex exponential z^n is:

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

The system's response has the form $y[n] = H(z)z^n$, where

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}.$$

Thus, z^n is an eigenfunction of DT LTI systems.

Harmonically-Related Complex Exponentials

Recall: periodic signals satisfy $x(t) = x(t+T), -\infty < t < +\infty$ for some positive value of T. The smallest such T is called the *fundamental period* of the signal, and its *fundamental frequency* is defined as $\omega_0 = \frac{2\pi}{T}$ (radians/s).

Note: the signal $x(t)$ is entirely determined by its values over one fundamental period T.

Also recall: harmonically-related complex exponentials have frequencies that are multiples of ω_0:

$$\phi_k(t) := e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

The orthogonal property of harmonics

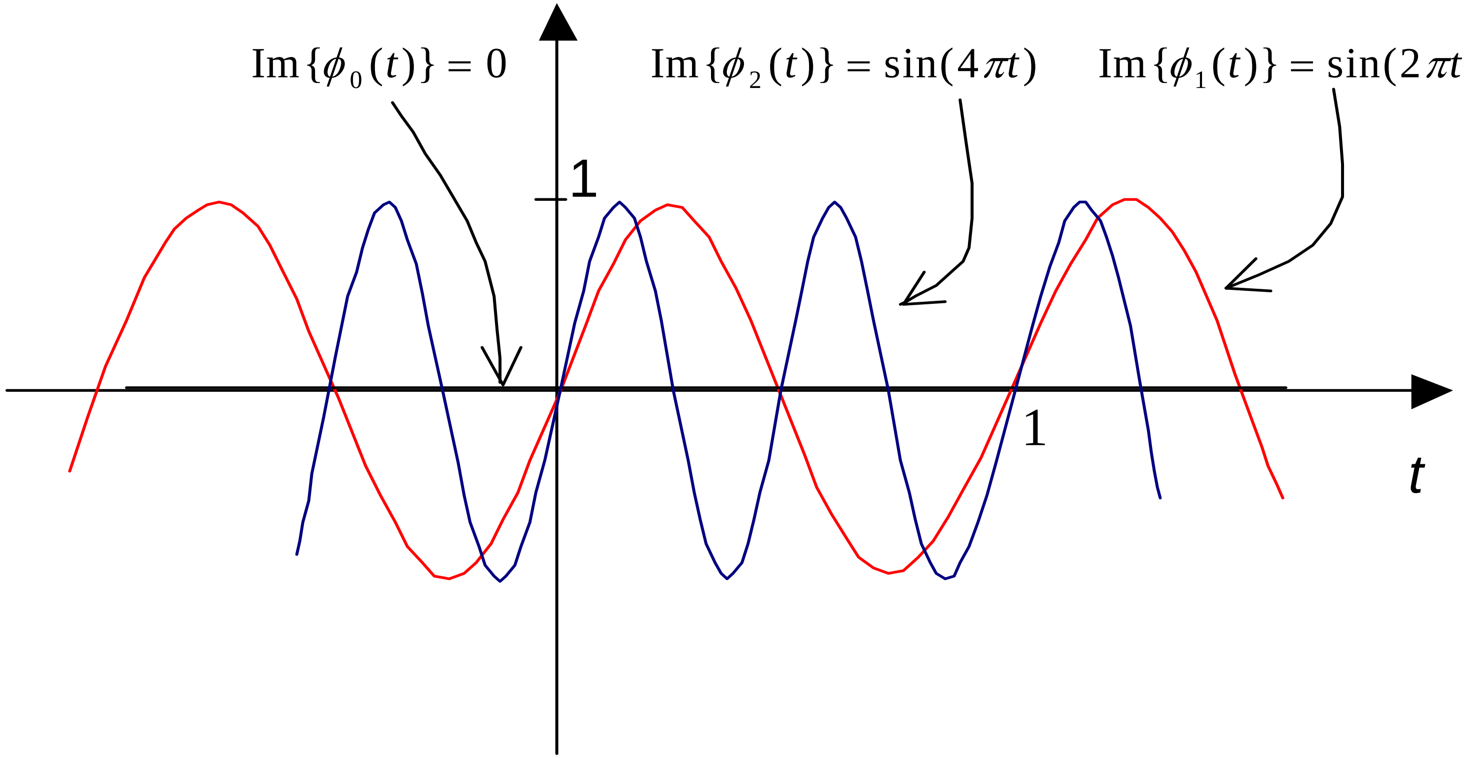
Recall: Harmonically related signals are *orthogonal over the fundamental period:*

$$\int_0^{\frac{2\pi}{\omega_0}} \phi_k(t)\overset{*}{\phi}_m(t)dt = \int_0^{\frac{2\pi}{\omega_0}} e^{jk\omega_0 t}e^{-jm\omega_0 t}dt = \begin{cases} 0, & k \neq m \\ \dfrac{2\pi}{\omega_0} = T, & k = m \end{cases}$$

Example of harmonics

Each one of these periodic signals has a fundamental frequency that is a multiple of ω_0.

Let's have a look at the imaginary part of $\phi_k(t)$ for $k = 0,1,2$ and $T = 1$ second.



A linear combination of harmonics

A linear combination of the complex exponentials $\phi_k(t)$ is also periodic with fundamental period T:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\frac{2\pi}{T})t} .$$

The two terms with $k = \pm 1$ in this series are collectively called the *fundamental components* or the *first harmonic components* of the signal.

The two terms with $k = \pm 2$ are referred to as the *second harmonic components* (with fundamental frequency $2\omega_0$), and more generally the components for $k = \pm N$ are called the *N^{th} harmonic components*.

Example 4.1: Consider the periodic signal with fundamental frequency $\omega_0 = \pi/2$ rad/s made up of the sum of five harmonic components:

$$x(t) = \sum_{k=-5}^{5} a_k e^{jk\frac{\pi}{2}t},$$

$$a_0 = 0,\ \ a_{\pm1} = -0.2026,\ \ a_{\pm2} = 0,$$

$$a_{\pm3} = -0.0225,\ \ a_{\pm4} = 0,\ \ a_{\pm5} = -0.0081$$

Collecting the harmonic components together, we obtain

$$x(t) = -0.2026(e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}) - 0.0225(e^{j\frac{3\pi}{2}t} + e^{-j\frac{3\pi}{2}t}) - 0.0081(e^{j\frac{5\pi}{2}t} + e^{-j\frac{5\pi}{2}t})$$

$$= -0.4052\cos(\frac{\pi}{2}t) - 0.0450\cos(\frac{3\pi}{2}t) - 0.0162\cos(\frac{5\pi}{2}t)$$

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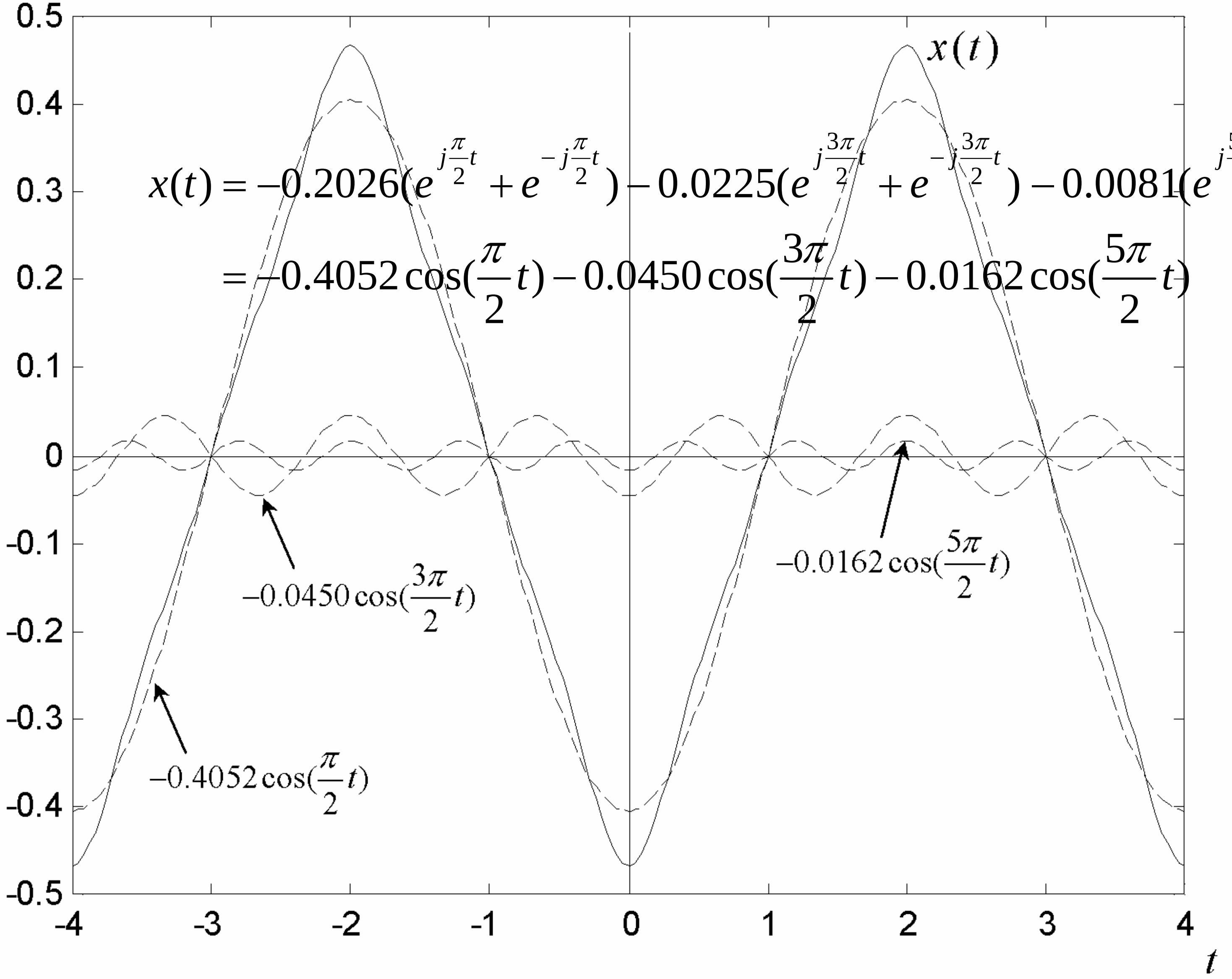


Representing periodic signals using Fourier series

Most engineering periodic signals can be represented using a linear combination of harmonically related complex exponentials:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\frac{2\pi}{T})t} \qquad (4.7)$$

where $$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_0^T x(t)e^{-jk(\frac{2\pi}{T})t}dt \qquad (4.8)$$

$$T = \frac{2\pi}{\omega_0}$$

(4.7) is referred to as the Fourier representation of periodic signals, and as the *synthesis equation*. (4.8) is referred to as the *analysis equation*.

The Coefficients of Fourier Series for a Continuous Time Periodic Signal

The coefficient of the Fourier Series is obtained by considering the orthogonality of harmonics.

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \int_0^T e^{jk\omega_0 t} e^{-jn\omega_0 t} dt = Ta_n$$

Therefore, the n^th coefficient of the Fourier series is:

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_0^T x(t)e^{-jn(\frac{2\pi}{T})t}dt$$

Representing a signal in the time-domain and the frequency domain

The Fourier series (4.7) gives us a ***time-domain representation*** of the signal as a sum of periodic complex exponential signals.

The Fourier series coefficients a_k (4.8) give us a ***frequency-domain representation*** or the ***spectral coefficients*** of the signal. Each of these complex coefficients measures how much the corresponding harmonic component of a given frequency contributes to the signal $x(t)$.

The coefficient a_0 is the dc component of the signal.

Properties of Continuous-Time Fourier Series

The set of spectral coefficients $\{a_k\}_{k=-\infty}^{+\infty}$ determines $x(t)$ completely. The duality between the signal and its spectral representation is denoted as $x(t) \overset{FS}{\leftrightarrow} a_k$. The following properties of the Fourier series are easy to show (do it as an exercise.)

Linearity

The operation of calculating the Fourier series of a periodic signal is linear. For $x(t) \overset{FS}{\leftrightarrow} a_k$, $y(t) \overset{FS}{\leftrightarrow} b_k$, if we form the linear combination $z(t) = Ax(t) + By(t)$, then we have

$$z(t) \overset{FS}{\leftrightarrow} Aa_k + Bb_k.$$

Time Shifting

Time shifting leads to a multiplication by a complex exponential. For

$x(t) \overset{FS}{\leftrightarrow} a_k$,

$$x(t - t_0) \overset{FS}{\leftrightarrow} e^{-jk\omega_0 t_0} a_k .$$

Remark: The magnitudes of the Fourier series coefficients are not changed, only their phases.

Time Reversal

Time reversal leads to a "sequence reversal" of the corresponding sequence of Fourier series coefficients:

$$x(-t) \overset{FS}{\leftrightarrow} a_{-k}.$$

Interesting consequences:

- For $x(t)$ even, the sequence of coefficients is also even $(a_{-k} = a_k)$
- For $x(t)$ odd, the sequence of coefficients is also odd $(a_{-k} = -a_k)$

Time Scaling

Time scaling applied on a periodic signal changes the fundamental frequency of the signal (but it remains periodic "with the same shape".) For example $x(\alpha t)$ has fundamental frequency $\alpha\omega_0$ and fundamental period $\frac{T}{\alpha}$. The Fourier series coefficients do not change:

$$x(\alpha t) \overset{FS}{\leftrightarrow} a_k,$$

but the Fourier series (the synthesis equation) itself *has changed* as the harmonic components are now at the frequencies $\pm\alpha\omega_0, \pm 2\alpha\omega_0, \pm 3\alpha\omega_0, \ldots$

Multiplication of Two Signals

Suppose that $x(t)$ and $y(t)$ are both periodic with period T.

For $x(t) \overset{FS}{\leftrightarrow} a_k$, $y(t) \overset{FS}{\leftrightarrow} b_k$, we have

$$x(t)y(t) \overset{FS}{\leftrightarrow} \sum_{l=-\infty}^{+\infty} a_l b_{k-l} \quad ,$$

i.e., a convolution of the two sequences of spectral coefficients!

Conjugation and Conjugate Symmetry

Taking the conjugate of a periodic signal has the effect of conjugation and time reversal on the spectral coefficients.

$$x^*(t) \overset{FS}{\leftrightarrow} a^*_{-k}$$

Interesting consequences:

- For $x(t)$ real, the sequence of coefficients is *conjugate symmetric* ($a_{-k} = a^*_k$). This implies

  $$|a_{-k}| = |a_k|, \ \ \text{phase}(a_{-k}) = -\text{phase}(a_k), \ \ a_0 \in \mathbb{R},$$

  $$\text{Re}\{a_{-k}\} = \text{Re}\{a_k\}, \ \ \text{Im}\{a_{-k}\} = -\text{Im}\{a_k\}$$

The Fourier series of a real signal

For a *real* signal $x(t)$, we have $a_{-k} = a_k^*$. Let $a_k = A_k e^{j\theta_k}$, $A_k, \theta_k \in \mathbb{R}$.

Then we have a real form of the Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{+\infty} a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{+\infty} 2\,\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

$$= a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(k\omega_0 t + \theta_k)$$

The Fourier series of real even/odd signals

- For $x(t)$ real and even, the sequence of coefficients is also real and even ($a_{-k} = a_k \in \mathrm{R}$)
- For $x(t)$ real and odd, the sequence of coefficients is also real odd ($a_{-k} = -a^*{}_k$ purely imaginary)
- For even-odd decomposition of the signal

$$x(t) = x_e(t) + x_o(t),\ x_e(t) \overset{FS}{\leftrightarrow} \mathrm{Re}\{a_k\},\ \ x_o(t) \overset{FS}{\leftrightarrow} j\,\mathrm{Im}\{a_k\}$$

Graph of the Fourier Series Coefficients: The Line Spectrum

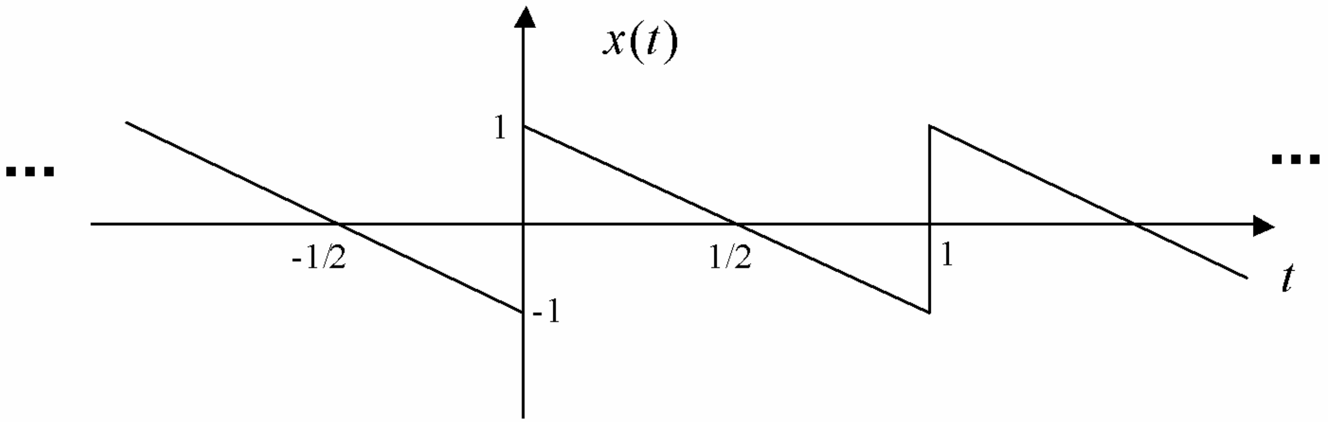
The set of complex Fourier series coefficients $\{a_k\}_{k=-\infty}^{+\infty}$ of a signal can be plotted with separate graphs for their magnitude and phase.

The combination of both plots is called the *line spectrum* of the signal.

Example

Periodic “sawtooth” signal.

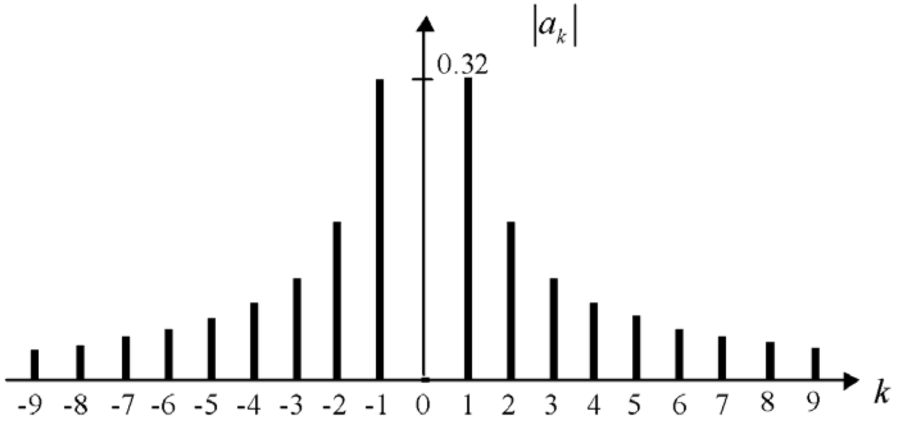
The fundamental period is $T = 1\text{s}$; hence $\omega_0 = 2\pi \text{ rad/s}$. First, the average over one period (the DC value of the signal) is 0, so $a_0 = 0$.



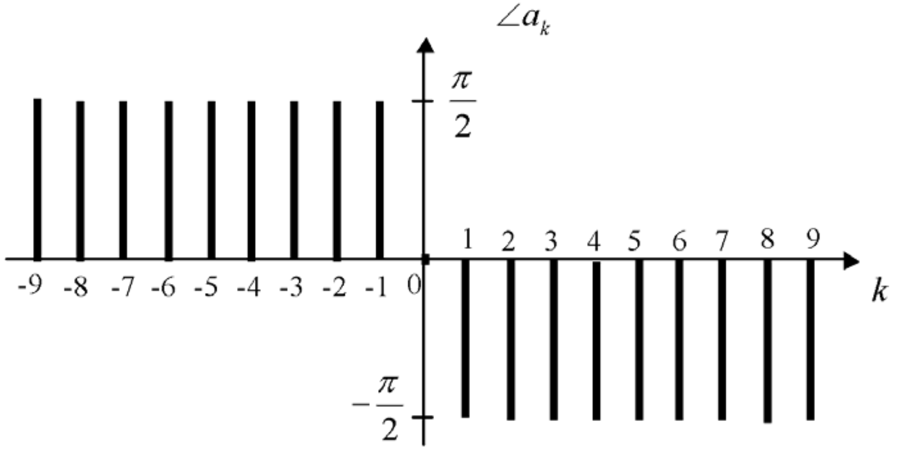
$$
\begin{aligned}
a_k &= \frac{1}{T}\int_0^1 x(t)e^{-jk2\pi t}dt \\
&= \int_0^1 (1-2t)e^{-jk2\pi t}dt \\
&= \frac{-1}{jk2\pi}\Big[(1-2t)e^{-jk2\pi t}\Big]_0^1 - \frac{1}{jk\pi}\underbrace{\int_0^1 e^{-jk2\pi t}dt}_{=0} \ \text{(integration by parts)} \\
&= \frac{1}{jk2\pi} + \frac{1}{jk2\pi} = \frac{1}{jk\pi} \\
&= \frac{-j}{k\pi}
\end{aligned}
$$

Consider the Fourier series coefficients of the sawtooth signal.

Their magnitudes are given by $\left|a_k\right| = \dfrac{1}{\left|k\right|\pi}$, $k \neq 0$, and $\left|a_0\right| = 0$, and their phases are given by $\angle a_k = \begin{cases} -\dfrac{\pi}{2}, & k > 0 \\ \dfrac{\pi}{2}, & k < 0 \end{cases}$ and $\angle a_0 = 0$



$|a_k|$
0.32
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
k



$\angle a_k$
$\frac{\pi}{2}$
1 2 3 4 5 6 7 8 9
-9 -8 -7 -6 -5 -4 -3 -2 -1 0
k
$-\frac{\pi}{2}$

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Lecture 11
September 26, 2008

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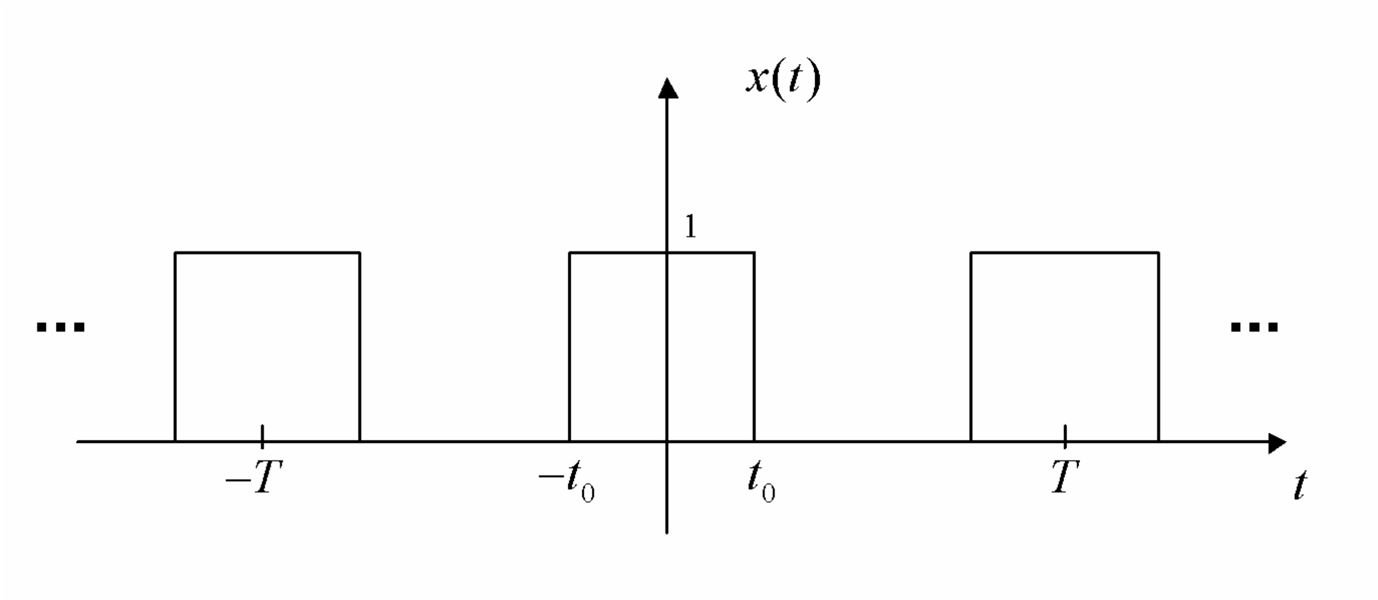
1. Approximating periodic signals using finite sum of exponentials
2. Existence of Fourier series representation
3. Gibbs phenomenon

Periodic even rectangular wave

Consider the following periodic rectangular wave of fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$.



FS coefficients of an even rectangular wave

DC value: $a_0 = \frac{2t_0}{T}$.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-t_0}^{t_0} e^{-jk\omega_0 t}dt$$

$$= -\frac{1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-t_0}^{t_0} = -\frac{1}{jk\omega_0 T}\left(e^{-jk\omega_0 t_0} - e^{jk\omega_0 t_0}\right)$$

$$= \frac{2}{k\omega_0 T}\left(\frac{e^{jk\omega_0 t_0} - e^{-jk\omega_0 t_0}}{2j}\right) = \frac{2\sin(k\omega_0 t_0)}{k\omega_0 T}$$

$$= \frac{\sin\left(\pi k \frac{2t_0}{T}\right)}{\pi k}, \quad k \neq 0$$

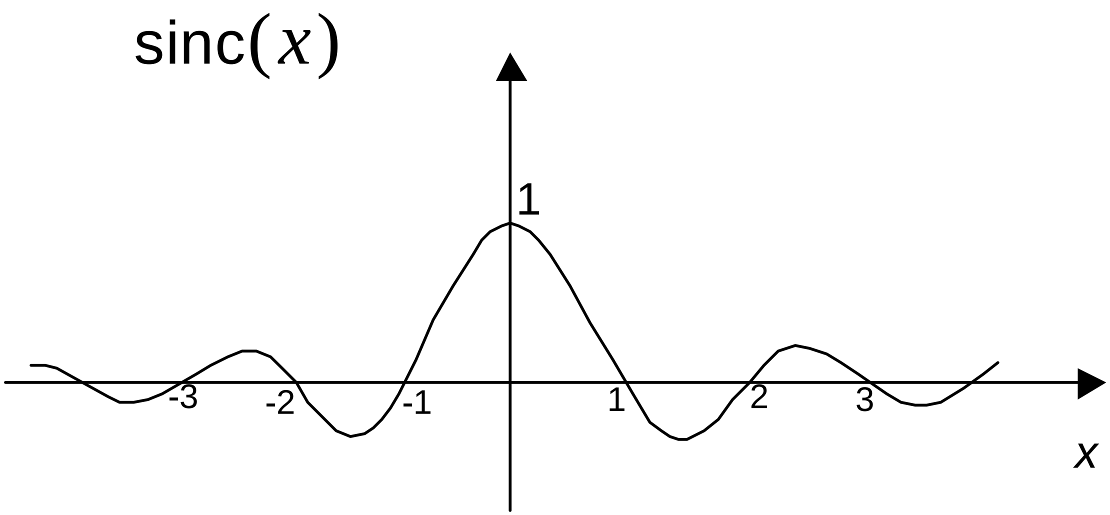
The FS coefficients of even and real rectangular wave signals are also real and even.

The sinc function: *sinc(x)=sin(πx)/πx*

The real continuous "sinc" function is defined as

$$\text{sinc}(x) := \frac{\sin \pi x}{\pi x}.$$

Sinc function is **one** at x=0, and is **zero** at x= ±1, ±2, ±3, …



The duty cycle of a rectangular wave

The duty cycle of the rectangular wave is defined as

$$\eta := \frac{2t_0}{T} .$$

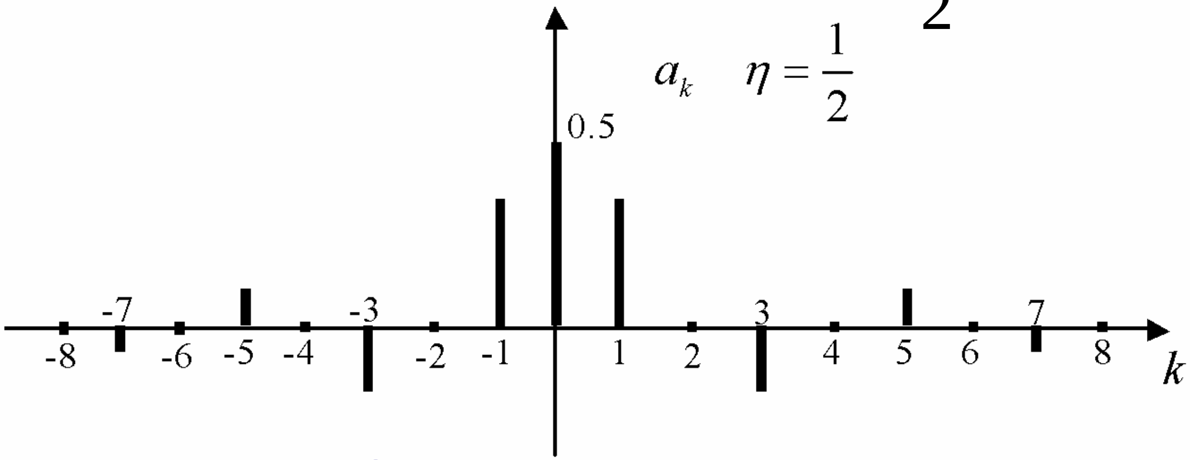
The spectral coefficients of an even rectangular wave are then

$$a_k = \frac{2t_0}{T}\frac{\sin\left(\frac{\pi k 2t_0}{T}\right)}{\frac{\pi k 2t_0}{T}} = \frac{2t_0}{T}\operatorname{sinc}\left(\frac{k2t_0}{T}\right)$$

$$= \eta \operatorname{sinc}(k\eta)$$

For a 50% duty cycle, that is, $\eta = \frac{1}{2}$:

$$a_k = \eta\,\text{sinc}(k\eta) = \frac{1}{2}\text{sinc}\left(\frac{k}{2}\right) = \frac{1}{2}\frac{\sin\left(\frac{k}{2}\pi\right)}{\frac{k}{2}\pi}$$



What does a ***negative frequency*** mean?

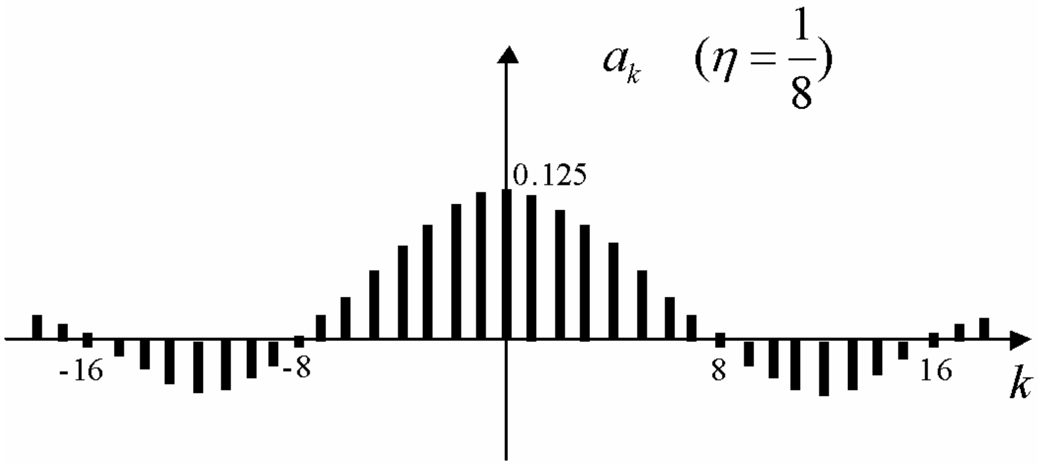
For a 60Hz square wave, the 60 Hz component is:

$$a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} = 2a_1 \cos(\omega_0 t)$$

The power of other harmonic components? --- Assignment 4

The harmonic components of a periodic rectangular wave and its duty cycle

The shorter the duty cycle is, the wider are the lobes of the spectral coefficients, the more coefficients we get in each lobe→ the more harmonic components in a bandwidth.



E.g., a periodic rectangular wave has a fundamental frequency ***f_0=100* Hz**, and a duty cycle **η=0.01**. Then, there are 0^{th}, 1^{st} ,…, 99^{th} harmonic components in the bandwidth 0-100 f_0 Hz.

Approximating a periodic signal using finite sum of harmonic exponentials

Let us approximate a periodic signal with a *finite* sum of exponentials (a truncated version of the infinite sum).

$$x_N(t) := \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

What coefficients can make the "best" approximation?

Examine ***the approximation error:***

$$e_N(t) := x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

The energy of the approximation error in one period

$$E_N = \int_0^T \left(x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t} \right) \left(x^*(t) - \sum_{k=-N}^{+N} a_k{}^* e^{-jk\omega_0 t} \right) dt$$

$$= \int_0^T x(t)x^*(t)dt + \int_0^T \left(\sum_{k=-N}^{+N} a_k e^{jk\omega_0 t} \right) \left(\sum_{n=-N}^{+N} a_n{}^* e^{-jn\omega_0 t} \right) dt$$

$$- \int_0^T x(t) \sum_{k=-N}^{+N} a_k{}^* e^{-jk\omega_0 t} dt - \int_0^T x^*(t) \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t} dt$$

$$= \int_0^T |x(t)|^2 dt + \sum_{k=-N}^{+N} \sum_{n=-N}^{N} a_k a_n{}^* \int_0^T e^{j(k-n)\omega_0 t} dt - \sum_{k=-N}^{+N} a_k{}^* \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$- \sum_{k=-N}^{+N} a_k \int_0^T x^*(t) e^{jk\omega_0 t} dt$$

$$= \int_0^T |x(t)|^2 dt + T \sum_{k=-N}^{+N} a_k a_k{}^* - \sum_{k=-N}^{+N} a_k{}^* \int_0^T x(t) e^{-jk\omega_0 t} dt - \sum_{k=-N}^{+N} a_k \int_0^T x^*(t) e^{jk\omega_0 t} dt$$

The energy of the approximation error in one period

Denote the coefficients in rectangular form:

$$a_k = \alpha_k + j\beta_k.$$

Then the error energy is

$$E_N = \int_0^T |x(t)|^2\,dt + T\sum_{k=-N}^{+N}(\alpha_k{}^2 + \beta_k{}^2) - \sum_{k=-N}^{+N}(\alpha_k - j\beta_k)\int_0^T x(t)e^{-jk\omega_0 t}\,dt$$

$$- \sum_{k=-N}^{+N}(\alpha_k + j\beta_k)\int_0^T x^*(t)e^{jk\omega_0 t}\,dt$$

$$= \int_0^T |x(t)|^2\,dt + T\sum_{k=-N}^{+N}(\alpha_k{}^2 + \beta_k{}^2) - 2\sum_{k=-N}^{+N}\alpha_k \,\mathrm{Re}\left\{\int_0^T x(t)e^{-jk\omega_0 t}\,dt\right\}$$

$$- 2\sum_{k=-N}^{+N}\beta_k \,\mathrm{Im}\left\{\int_0^T x(t)e^{-jk\omega_0 t}\,dt\right\}$$

The optimal coefficients of the finite sum of harmonic exponentials

The coefficients minimizing the error energy can be obtained by taking partial derivatives of the error energy and setting them to zero:

$$\frac{\partial E_N}{\partial \alpha_k} = 2T\sum_{k=-N}^{N}\alpha_k - 2\sum_{k=-N}^{N}\mathrm{Re}\{\int_0^T x(t)e^{-jk\omega_o t}dt\} = 0$$

The coefficient satisfying the above equation is then:

$$\alpha_k = \mathrm{Re}\{\int_0^T x(t)e^{-jk\omega_0 t}dt\}$$

The FS coefficients minimize the approximation error energy

Similarly, minimizing the approximation error energy with respect to β_k yields

$$\beta_k = \mathrm{Im}\{\int_0^T x(t)e^{-jk\omega_0 t}dt\}$$

Thus, the complex coefficients minimizing the approximation error energy is just the FS:

$$a_k = \int_0^T x(t)e^{-jk\omega_0 t}dt$$

The difference between a signal and its FS representation

Now if the signal x(t) has a Fourier series representation, then the approximation error energy is zero as N tends to infinity.

$$\lim_{N\to\infty}\{\int_T |x(t)-x_N(t)|^2\ dt\}=0$$

The Fourier series of a signal converge in the sense that ***the power in the difference between the signal and its Fourier series representation approaches zero***.

Existence of a Fourier Series Representation

What classes of periodic signals have Fourier series representation (i.e., the FS coefficients a_k are finite)?

One that does is the class of periodic signals with ***finite energy*** over one period , i.e., signals satisfy

$$\int_0^T |x(t)|\,dt < \infty$$

Another broad class of signals that have Fourier series representation are signals that satisfy the *Dirichlet conditions*.

Such a signal *equals* its Fourier series representation, except at isolated values of t where $x(t)$ is discontinuous (e.g. finite jumps). At these t values, the Fourier series converges to the average of the values on either side of the discontinuity.

If a signal is continuous very where, then its FS converges and equals the original signal at any value of time t.

Dirichlet Conditions

Condition 1: Over any period, must be absolutely integrable, i.e.,

$$\int_0^T | x(t)dt < \infty$$

Condition 2: In any finite interval of time, *x(t)* must be of bounded variations. This means that must have a finite number of maxima and minima during any single period. A signal ***not satisfying*** this condition is:

$$x(t) = \sin(\frac{2\pi}{t})$$

Condition 3: In any finite interval of time, *x(t)* has a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Note: Engineering signals generally satisfy Dirichlet conditions and the convergence of FS can be guaranteed.

Gibbs Phenomenon: a discontinuous periodic signals is different from its FS

let us compare a discontinuous periodic signal with its truncated Fourier series.

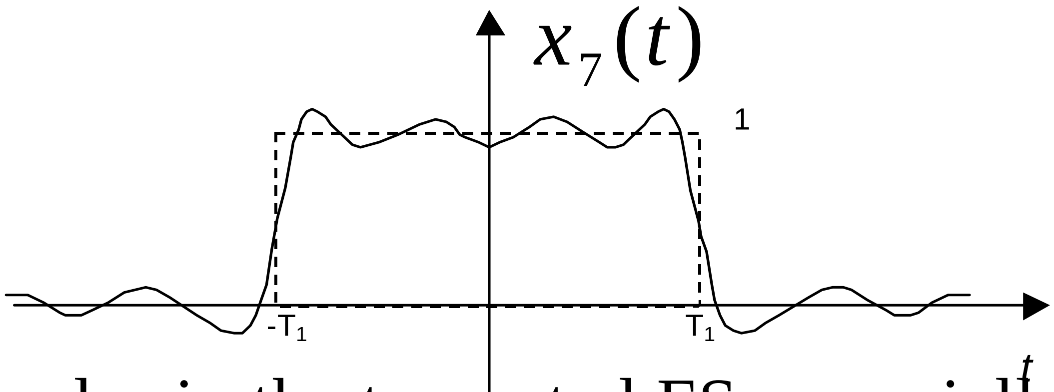
This is easy to do using Matlab:

Compute the spectral coefficients up to $k = \pm 7$ $(N = 7)$, and plot the real approximation to the rectangular wave signal

$$x_N(t) := \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t} = a_0 + 2\sum_{k=1}^{7} a_k \cos(k\omega_0 t).$$

The ripples in the FS representation of a discontinuous signal

The graph over one period looks like this:

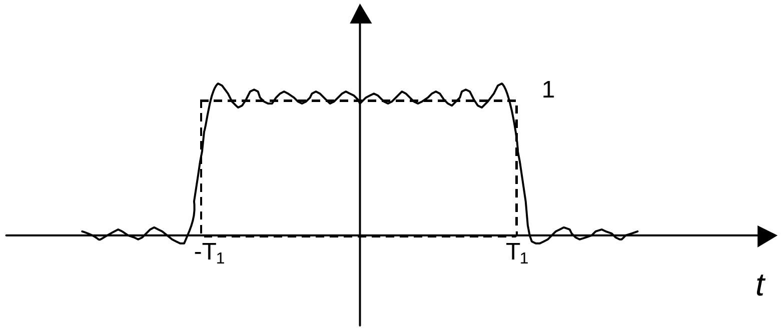


There are ripples in the truncated FS, especially close to the discontinuities in the signal. The *maximum peaks of these ripples don't diminish even if we add more terms in the truncated Fourier series!* This phenomenon is called the *Gibbs phenomenon* after a mathematical physicist who first provided an explanation of this phenomenon.

The energy of the approximation error

For example, for N=19, the approximation gets closer to a square wave but we can still see rather large ripples around the discontinuities.

$x_{19}(t)$



Since the signal x(t) satisfies Dirichlet conditions, $x_N(t)$ should be the average value at either side of the discontinuous points for any large N values.

The energy of the approximation error

As N grows larger, the peak amplitude doesn't diminish and the first overshoot on both sides of the discontinuity remains at 9% of the height of the discontinuity

However, as N approaches infinite, the **energy in these ripples vanishes** because the area of the ripples approaches zero. Also for any fixed time (not at the discontinuity), the approximation tends to the signal value $x_N(t_1) \underset{N\to+\infty}{\longrightarrow} x(t_1)$ (this is called *pointwise convergence*). At the discontinuity for time T_1, the approximation converges to half of the jump.

ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University

Department of Electrical and Computer Engineering

Lecture 12

October 1, 2008

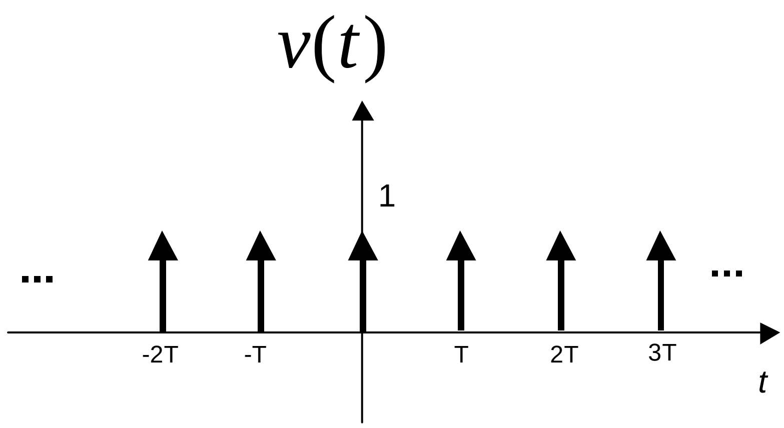
Hui Qun Deng, PhD

1. Impulse Train and its Applications
2. Parseval Theorem
3. Power Spectrum
4. Response of LTI Systems to Periodic Input Signals

Periodic Impulse Train

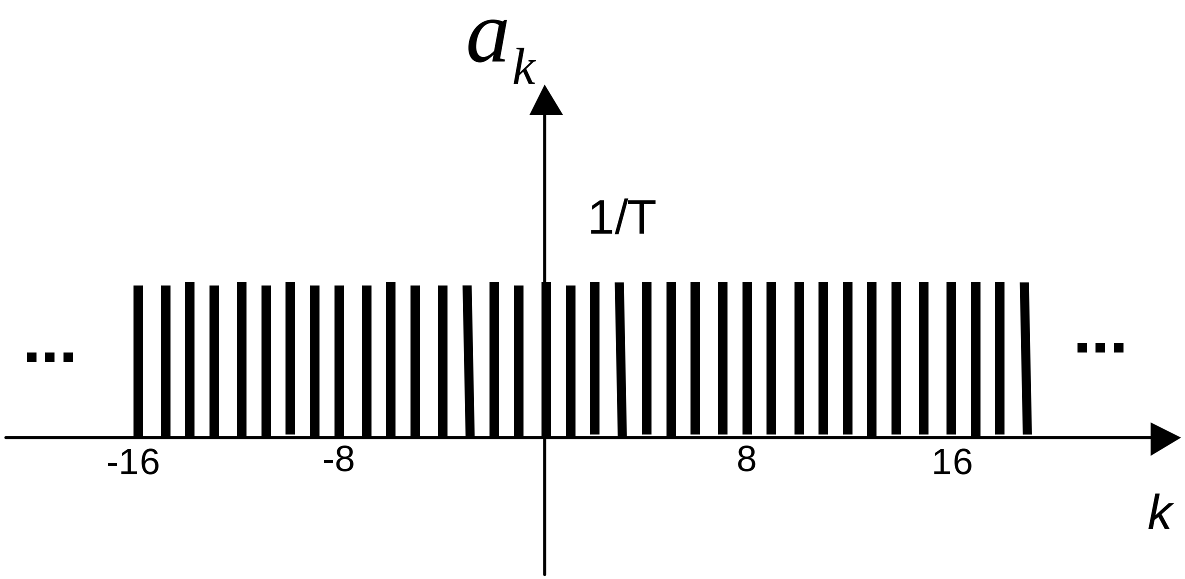
It would be useful to have a Fourier series representation of an *impulse train.*

$$v(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) .$$



Fourier Series of a Periodic of Impulse Train

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$



The spectrum of an impulse train is a real constant sequence.

Periodic signals as convolutions of the impulse train and time-limited signals

A periodic signal x(t) can be described as a convolution of a single period of the signal with a train of impulses. Let

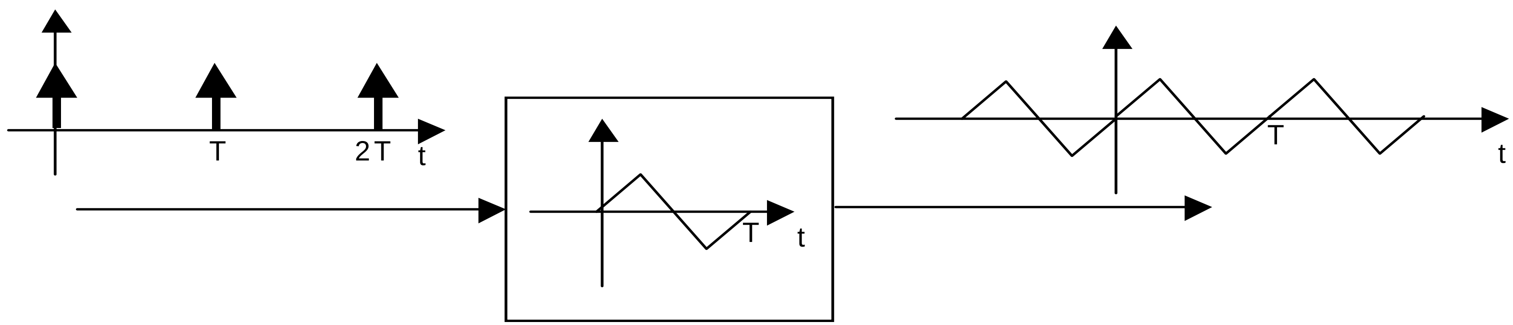
$$x_T(t) = \begin{cases} x(t), & 0 \le t < T \\ 0, & other \end{cases}$$

Then

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT) * x_T(t) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(\tau - nT) x_T(t-\tau) d\tau$$

$$= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(\tau - nT) x_T(t-\tau) d\tau = \sum_{n=-\infty}^{+\infty} x_T(t-nT)$$

$x(t)$

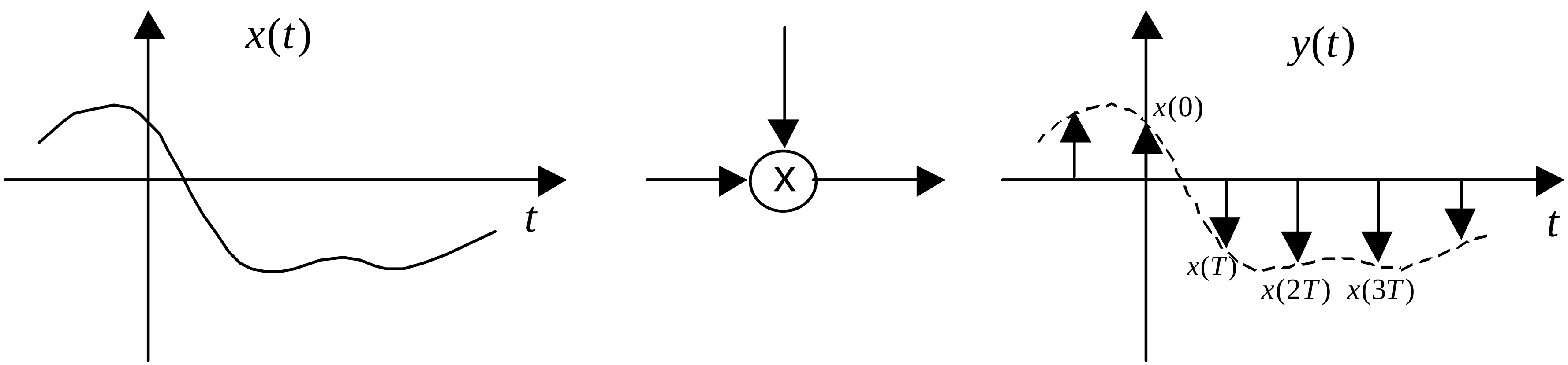


$$h(t) = x_T(t)$$

Application of the impulse train in sampling

Sampling: The operation of periodically sampling a continuous-time signal can also be conveniently represented by a multiplication of an impulse train with the signal (more on this later.)

$$v(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



Parseval Theorem

It can be shown that the **total average power** of a periodic signal *x(t)* is

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

The **average power** in the *k*

$$P_k = \frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2\,dt = \frac{1}{T}\int_T |a_k|^2\,dt = |a_k|^2$$

Parseval Theorem: the total average power of a periodic signal equals the sum of the average powers in all of its harmonic components.

Note: $\boldsymbol{P_k = P_{-k}}$, and the total average power of the k^{th} harmonic component of the signal is $\boldsymbol{2P_k}$.

Example

Compute the total average power of the unit-amplitude square wave with period T and 50% duty cycle.

We have already computed its spectral coefficients:

$$a_k = \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right).$$

According to Paseval's theorem, the total average power is

$$P = \sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k=-\infty}^{\infty} \left|\frac{1}{2}\operatorname{sinc}\frac{k}{2}\right|^2 = \frac{1}{4} + 2\sum_{k=1}^{\infty} \frac{1}{4}\left|\operatorname{sinc}\frac{k}{2}\right|^2$$

$$= \frac{1}{4} + \frac{1}{2}\sum_{k=1,3,5,\ldots} \left|\operatorname{sinc}\frac{k}{2}\right|^2 = \frac{1}{4} + \frac{1}{2}\sum_{k=1,3,5,\ldots} \left|\frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}}\right|^2$$

$$= \frac{1}{4} + 2\sum_{k=1,3,5,\ldots} \frac{1}{k^2\pi^2} = \frac{1}{4} + \frac{2}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \cdots\right] = \frac{1}{4} + \frac{2}{\pi^2}\left[\frac{\pi^2}{8}\right]$$

$$= \frac{1}{2}$$

Now, we computer the total average power of the periodic rectangular wave in the time-domain:

$$P = \frac{1}{T}\int_{T} \left|x(t)\right|^2 dt = \frac{1}{T}\int_{-T/4}^{T/4} 1^2 dt = \frac{1}{T}\left(\frac{T}{4}+\frac{T}{4}\right) = \frac{1}{2}$$

Power Spectrum

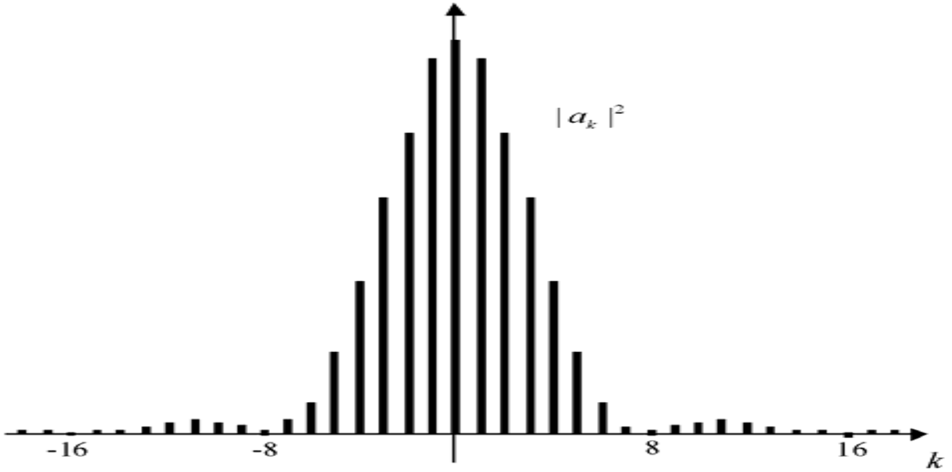
The power spectrum of a signal is the sequence of $|a_k|^2$, i.e., the average powers of harmonic components.

For real periodic signals, the power spectrum is a real even sequence as

$$\left|a_{-k}\right|^2 = \left|a_k^*\right|^2 = \left|a_k\right|^2.$$

Example: Power spectrum of the rectangular wave.

$$\eta = 1/8$$

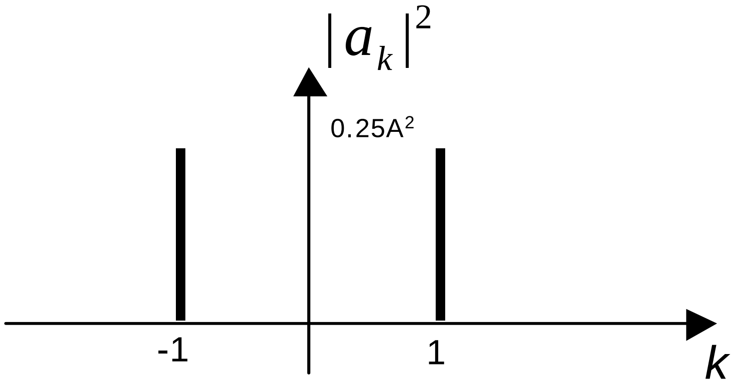


Physical meaning of negative frequency

Example $x(t) = A\sin(\omega_0 t)$.

$$x(t) = A\sin(\omega_0 t) = \frac{A}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}),$$

so $a_1 = -j\frac{A}{2},\ a_{-1} = j\frac{A}{2},\quad a_k = 0,\ k \neq \pm 1$

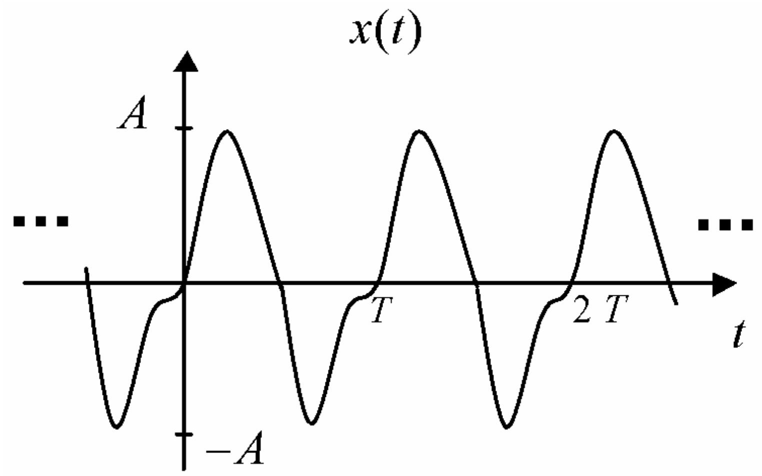


Negative frequency is caused by the use of complex exponentials to represent *sin(kω_0t)* and *cos(kω_0t)* signals. The actual component at $k\omega_0$ is

$$a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}$$

Total Harmonic Distortion

Assume that a signal of a pure sine wave is distorted.

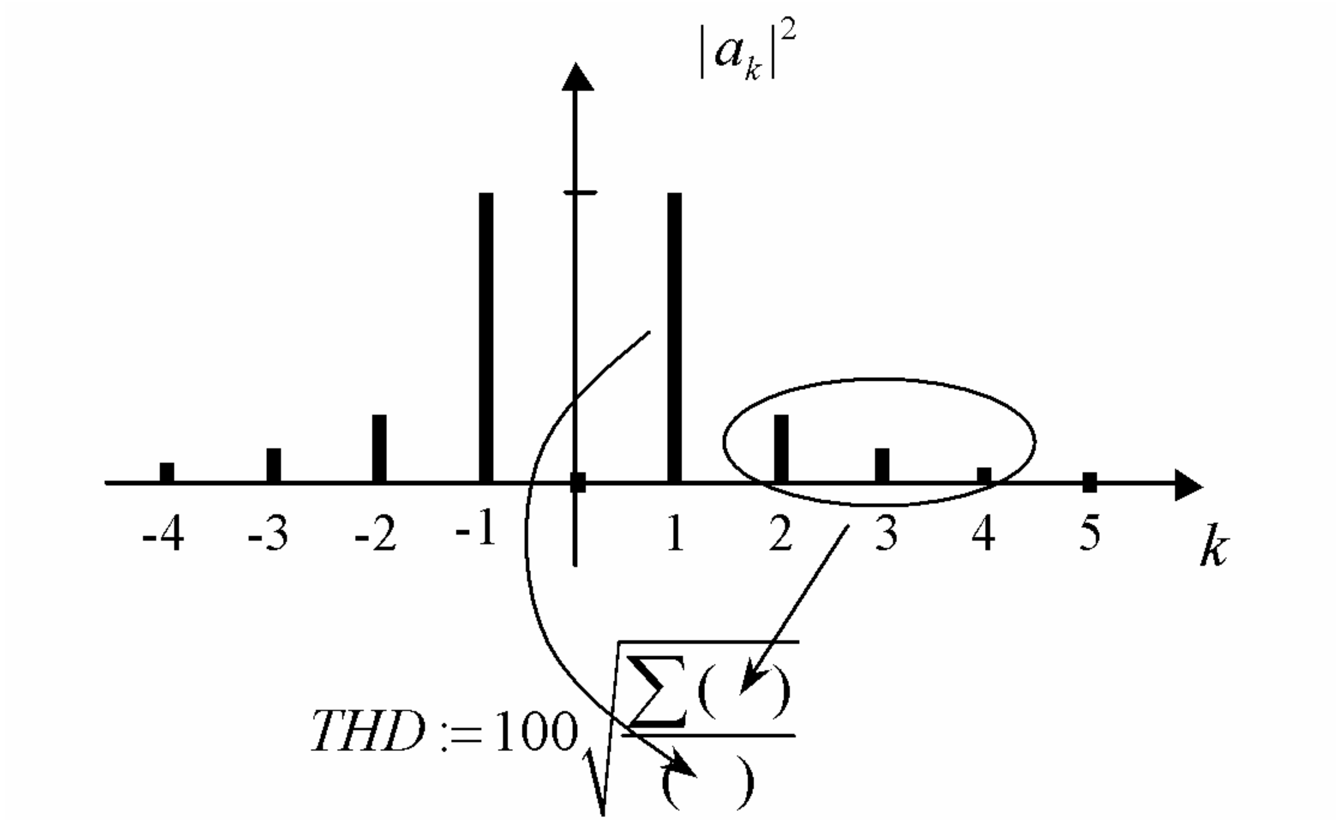


How to measure the degree of distortion?

Total harmonic distortion (THD):

$$THD := 100\sqrt{\frac{\sum_{k=2}^{+\infty}|a_k|^2}{|a_1|^2}}\ \%$$

THD means the RMS (root mean square) of all the harmonics that "should not be there" divided by the RMS of the fundamental component.


$|a_k|^2$
-4
-3
-2
-1
1
2
3
4
5
k
$THD := 100\sqrt{\frac{\sum(\ )}{(\ )}}$

The Frequency Response of an LTI System

The response of an LTI system with impulse response *h(t)* to a complex exponential signal e^{st} is

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \boxed{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau}$$

$$= e^{st} H(s)$$

For $s = j\omega$, the output is $y(t) = H(j\omega) e^{j\omega t}$.

H(s) is called the *transfer function* (or *system function*).

H(jω) is called the system's *frequency response*.

Remarks:

- "steady-state" means the system has been subjected to the same input from time $t = -\infty$.
- Thus there is no transient response from initial conditions in the output signal.
- Also if the system is unstable, then the output would tend to infinity, so we assume that the system is stable.

FS of the response of a LTI system to a periodic signal

A periodic signal can be represented by a Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

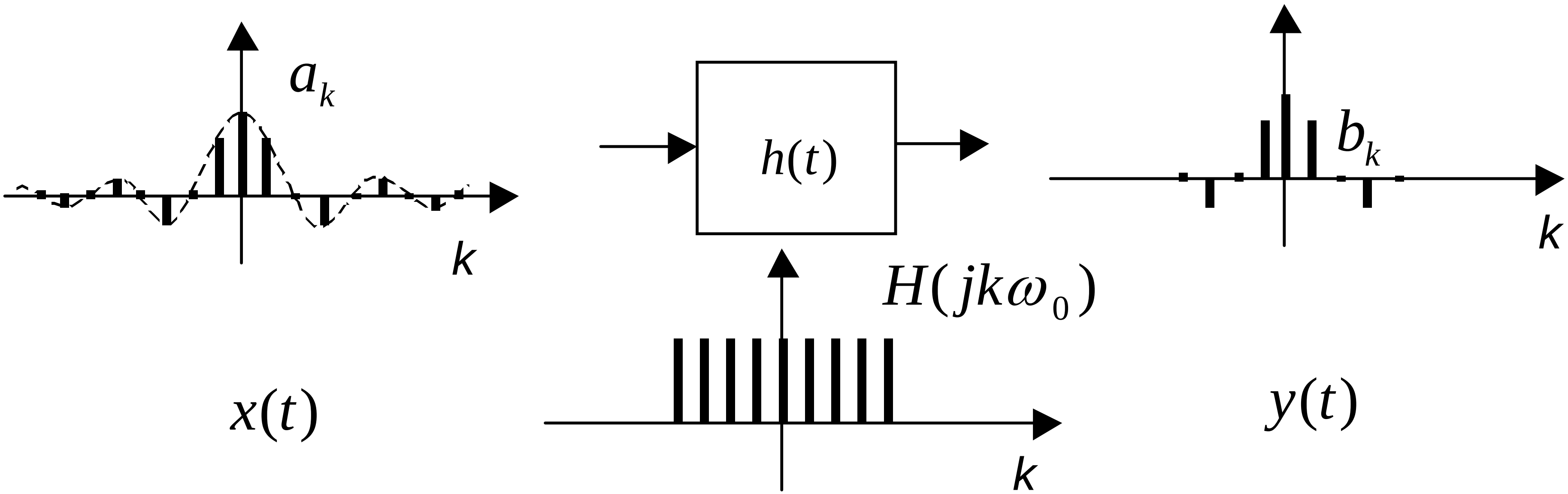
By superposition, the response of an LTI system to x(t) is:

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}.$$

Thus, the Fourier series coefficients of the periodic output y(t) are given by

$$b_k = a_k H(jk\omega_0).$$

Example: A periodic rectangular wave is the input of an LTI system



Filtering

The responses of a system to different components of a signal are different: some components can be amplified while some can be suppressed.

Filtering signals with an LTI system involves the design of a filter with a desirable frequency spectrum $H(jk\omega_0)$ that retains certain frequency components and cuts off others.

Example:

Consider a filter with impulse response $h(t) = e^{-t}u(t)$ (a simple RC circuit with RC=1). The frequency response of this filter is:

$$H(j\omega) = \int_0^{\infty} e^{-\tau}u(\tau)e^{-j\omega\tau}d\tau = \frac{1}{1+j\omega}$$

We can see that as the frequency increase, the magnitude of the frequency response of the filter decreases. In fact, this filter is a low-pass filter (LPF).

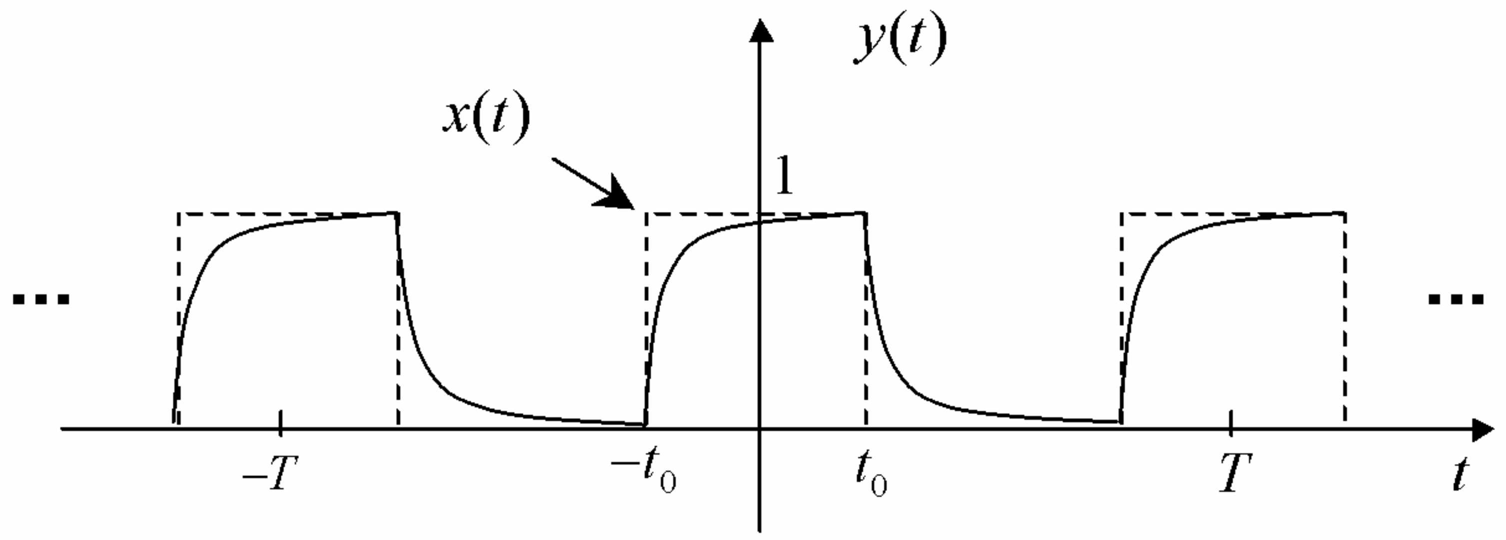
Filtering a rectangular wave

The input signal to the LPF is the rectangular wave, then the output signal will have its Fourier series coefficients b_k given by

$$b_k = a_k H(jk\omega_0) = \frac{\sin(k\omega_0 t_0)}{k\pi(1 + jk\omega_0)}, \quad k \neq 0$$

$$b_0 = a_0 H(0) = \frac{2t_0}{T}$$

The reduced power at high frequencies produces an output signal that's "smoother" than the input signal (remember that discontinuities produce high frequencies).



$y(t)$
$x(t)$
1
...
...
$-T$
$-t_0$
t_0
T
t

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Lecture 13

Hui Qun Deng, PhD

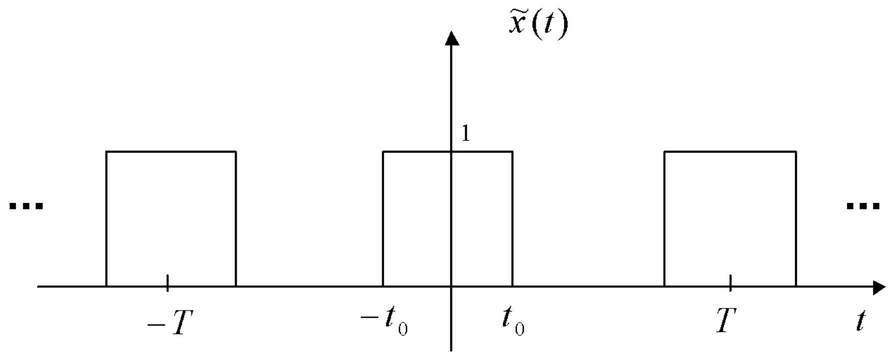
October 3, 2008

Continuous-Time Fourier Transform

1. FT as the limit of FS as period $T \rightarrow$ infinite
2. Properties of FT

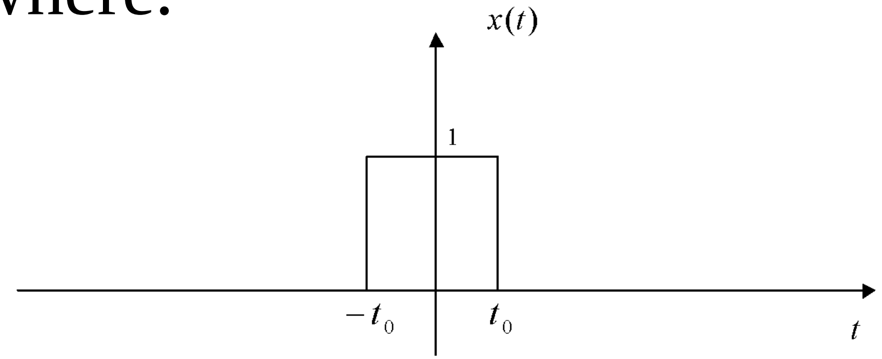
Aperiodic wave as the limit of the periodic wave

The periodic signal:



Now define a signal x(t) equal to the periodic signal over one period and zero elsewhere.

The “single-period” signal:



This aperiodic signal can be thought of being periodic with an infinite fundamental period $T \to +\infty$.

The ***envelop*** of FS coefficients

The FS coefficients a_k of the periodic signal $\tilde{x}(t)$ can be obtained from the "single-period" signal $x(t)$:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt$$

Define:

$$X(j\omega) := \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Then, a_k can be viewed as samples of $X(j\omega)$:

$$a_k = \frac{1}{T}X(jk\omega_0)$$

and $X(j\omega)$ can be viewed as the envelop of the a_k sequence, i.e., the *spectral envelope of the periodic signal*.

Periodic signal in term of its spectral envelop

Now the periodic signal $\tilde{x}(t)$ has the Fourier series representation

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} \ .$$

Or, equivalently, since $\omega_0 = \dfrac{2\pi}{T}$,

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

Aperiodic signal in term of the spectral envelop of its periodic signal

The periodic signal as a function of frequency:

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

As $T \rightarrow +\infty$, we get

- $\omega_0 \rightarrow d\omega$
- $k\omega_0 \rightarrow \omega$
- the summation tends to an integral
- $\tilde{x}(t) \rightarrow x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Transform pair

These two equations are called the *Fourier transform pair*.

Inverse Fourier transform of x(t):

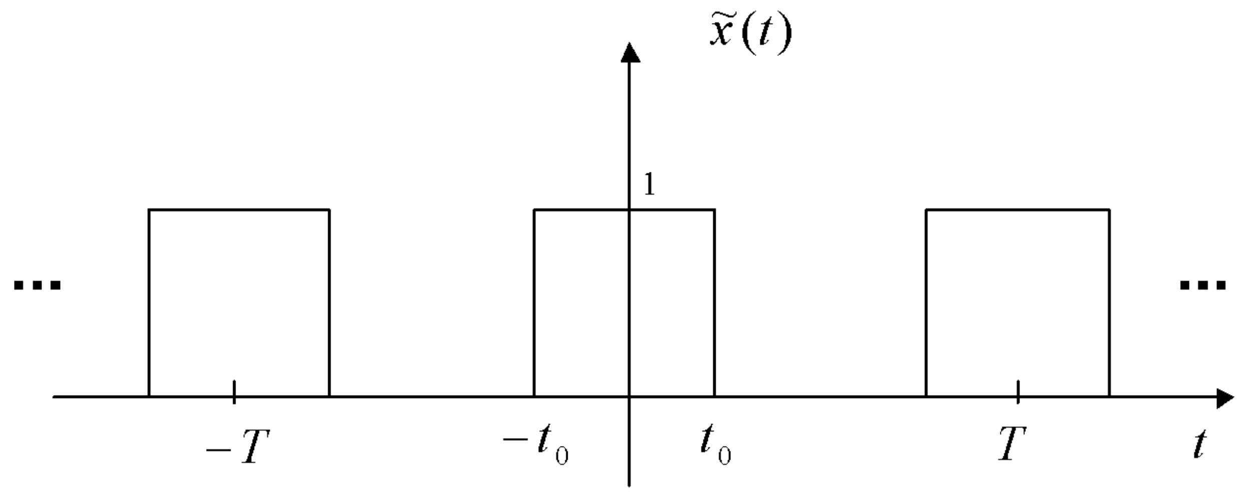
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier transform of x(t)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Example: the Fourier series of a periodic rectangular wave as T→∞

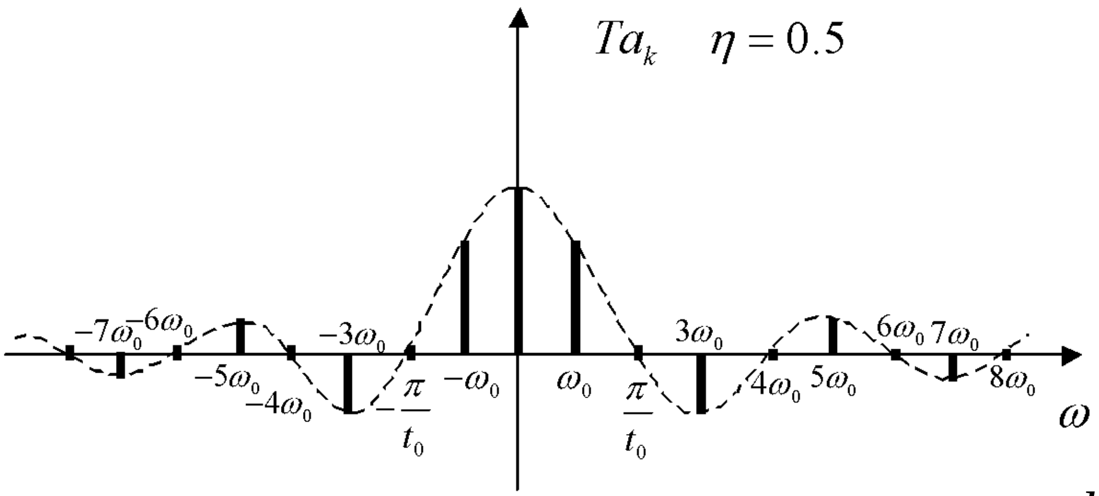
Consider the Fourier series representation of a periodic rectangular signal $\tilde{x}(t)$



The FS of the periodic rectangular wave

Multiplying the spectral coefficients of $\tilde{x}(t)$ by T, and assuming that t_0 is fixed, we get

$$Ta_k = T\eta \text{sinc}(k\eta) = 2t_0 \text{sinc}\left(\frac{k2t_0}{T}\right)$$

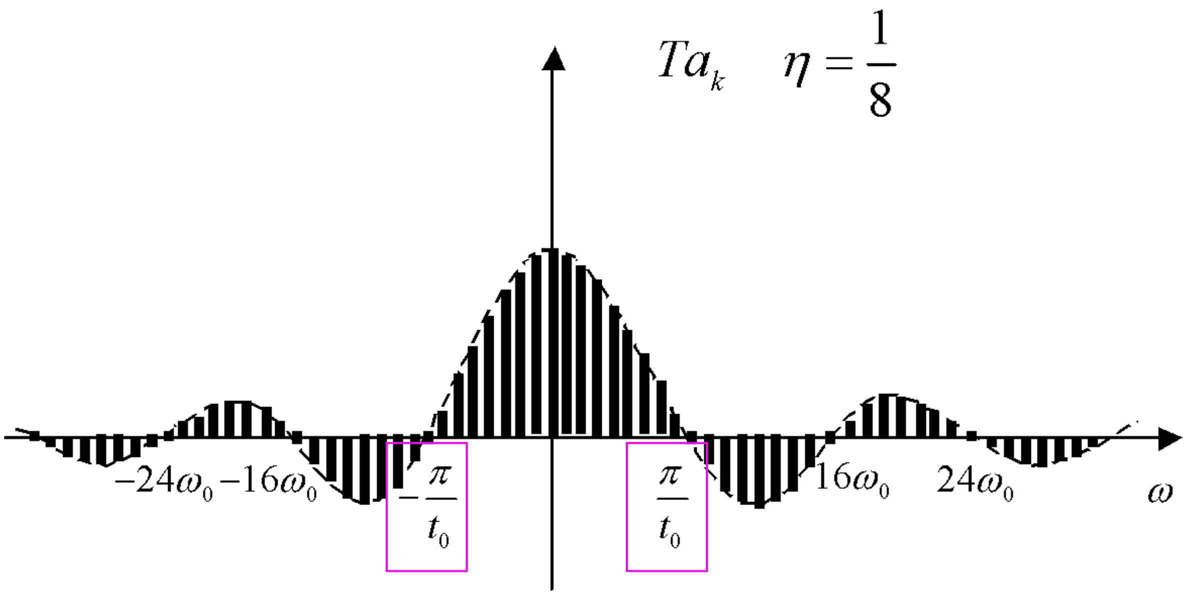


- Note that the spectra is plotted against the frequency variable $k\omega_0$

- The first zeros on each side of the main lobe are at frequencies $\omega = \pm\frac{\pi}{t_0}$

The effect of the period T on the FS of periodic wave

As the period T increases, the frequency interval (ω_0) between adjacent harmonics degreases. As T→ ∞, the harmonics become continuous.



Properties of the Fourier Transform: Linearity

We denote the relationship between a signal and its Fourier transform as $x(t) \overset{\mathfrak{FT}}{\leftrightarrow} X(j\omega)$. Try to derive the following properties as an exercise.

Linearity

The Fourier transform is a linear operation:

$$ax(t) + by(t) \overset{\mathfrak{FT}}{\leftrightarrow} aX(j\omega) + bY(j\omega)$$

Time Shifting

Time Shifting

A time shift results in a phase shift in the Fourier transform:

$$x(t - t_0) \overset{\mathcal{FT}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Time/Frequency Scaling

Scaling the time variable either expands or contracts the Fourier transform.

$$x(\alpha t) \overset{\mathcal{FT}}{\leftrightarrow} \frac{1}{|\alpha|} X(j\omega/\alpha)$$

For $\alpha > 1$, the signal $x(\alpha t)$ is sped up (or compressed in time), so its spectrum (Fourier transform) expands to higher frequencies.

On the other hand, when the signal is slowed down ($\alpha < 1$), the Fourier transform gets compressed to lower frequencies.

Conjugation and Conjugate Symmetry

In general the signal is *complex*, you can prove the FT of its is: $x^*(t) \overset{\mathcal{FT}}{\leftrightarrow} X^*(-j\omega)$

If the signal is *real*, $x^*(t) = x(t)$, then

- the Fourier transform has conjugate symmetry:

$$X^*(j\omega) = X(-j\omega)$$

- $\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$, an even function of ω
- $\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$, an odd function of ω
- $|X(j\omega)| = |X(-j\omega)|$, an even function of ω
- $\angle X(j\omega) = -\angle X(-j\omega)$, an odd function of ω

Even/odd signals and their spectra

If x(t) is real and even, then you can show that

$$X(j\omega) = X(-j\omega) = X^*(j\omega),$$

i.e., the spectrum is even and real.

If x(t) is real and odd, we have

$$X(j\omega) = -X(-j\omega) = -X^*(j\omega),$$

i.e., the spectrum is odd and purely imaginary.

Differentiation in the time domain

Differentiating a signal results in a multiplication of the Fourier transform by $j\omega$.

$$\frac{d}{dt}x(t) \overset{\mathfrak{FT}}{\longleftrightarrow} j\omega X(j\omega)$$

Integration in the time domain

Integrating a signal results in a division of the Fourier transform by $j\omega$. However, to account for the possibility that $x(t)$ has a nonzero average value, i.e., $X(j0)=\int_{-\infty}^{+\infty} x(t)dt \neq 0$, we must add the term $\pi X(0)\delta(\omega)$ to the Fourier transform. That is, a finite energy concentrated at $\omega = 0$ is represented as an impulse at that frequency.

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathfrak{FT}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$$

Convolution in the time domain ←→ multiplication in the frequency domain

The convolution of two signals results in the multiplication of their Fourier transforms in the frequency domain.

$$x(t) * y(t) \overset{\mathfrak{FT}}{\longleftrightarrow} X(j\omega)Y(j\omega)$$

A direct application is the calculation of the response of an LTI system to an arbitrary input signal:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

The output signal in the time-domain is obtained by taking the inverse Fourier transform of its spectrum.

Multiplication in time domain ←→ convolution in the frequency domain

This property is the dual of the convolution property. The multiplication of two signals in the time domain results in the convolution of their spectra.

$$x(t)y(t) \overset{\mathcal{FT}}{\longleftrightarrow} \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

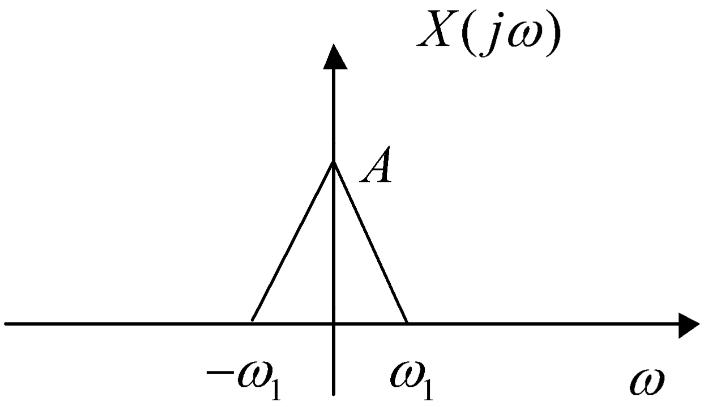
Modulation is based on this property.

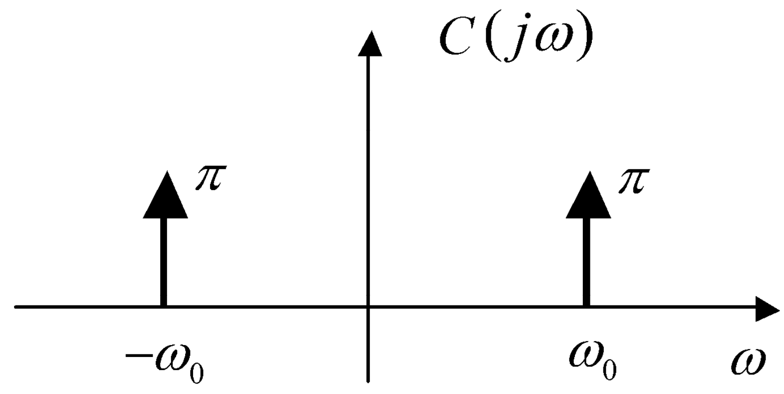
Example: Signal Modulation

Consider the modulation system described by

$$y(t) = \cos(\omega_0 t)x(t) .$$

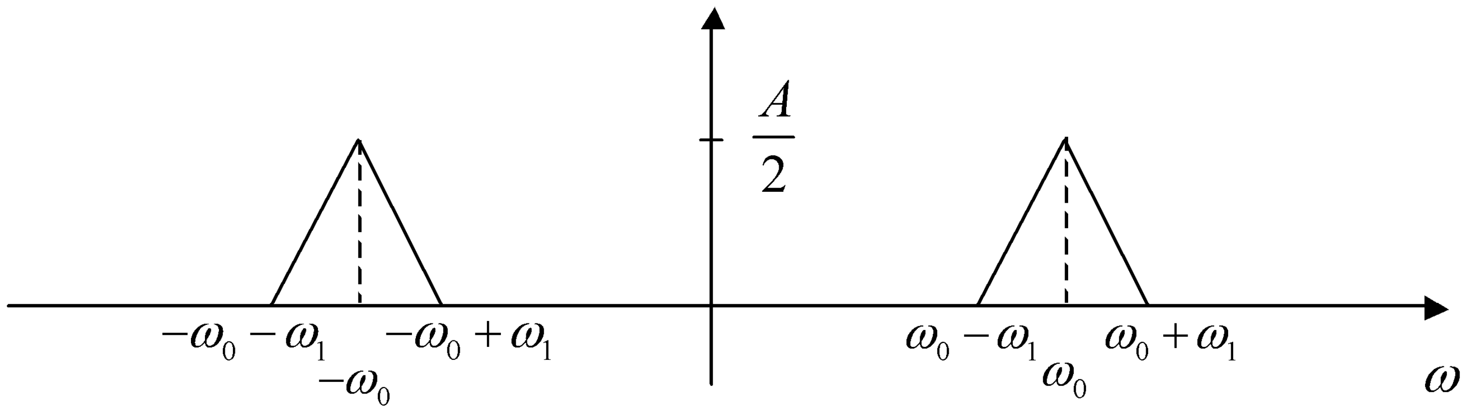
We'll see later that the Fourier transform of $c(t) := \cos(\omega_0 t)$ is composed of two impulses of area π, one at $-\omega_0$ and the other at ω_0. Suppose that the spectrum of $x(t)$ looks like this





Then the Fourier transform of the output signal looks like this

$$Y(j\omega) = \frac{1}{2\pi} C(j\omega) * X(j\omega)$$



Hence multiplication of a signal by a sinusoid shifts its spectrum to another frequency band (it also creates a mirror image in the negative frequencies) for easier transmission over a communication channel. For example, music (bandwidth less than 20kHz) transmitted over typical AM radio has modulation frequencies in the range 500kHz to 1500 kHz.

The energy-density spectrum

The *energy-density spectrum* of an aperiodic signal $x(t)$ is defined as $\left|X(j\omega)\right|^2$. We can find the energy of a signal in a given frequency band by integrating:

$$E_{[\omega_a,\omega_b]} = \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \left|X(j\omega)\right|^2 d\omega$$

Note that for real signals, it is customary to include the negative frequency band as well.

For example, if we wanted to compute the energy contained in a real signal between, say, 5kHz and 10kHz, we would compute

$$E_{[10000\pi,20000\pi]} = \frac{1}{2\pi}\left[\int_{10000\pi}^{20000\pi} \left|X(j\omega)\right|^2 d\omega + \int_{-20000\pi}^{-10000\pi} \left|X(j\omega)\right|^2 d\omega\right]$$

$$= \frac{1}{\pi}\int_{10000\pi}^{20000\pi} \left|X(j\omega)\right|^2 d\omega$$

Parseval's Relation

The total energy in an aperiodic signal is equal to the total energy in its spectrum.

$$\int_{-\infty}^{\infty} |\, x(t)\,|^2 \; dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\, X(\,j\omega)\,|^2 \; d\omega$$

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Lecture 14

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October 6, 2008

1. Parseval's relation
2. Examples of FT
3. Some notes on the midterm 1

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega \qquad (5.21)$$

Note $|X(j\omega)|^2$ is the energy density spectrum of x(t).

Eq. (5.21) means the total energy in the time domain= the total energy in the frequency domain.

Prove:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) \overset{*}{x}(t) \, dt = \int_{-\infty}^{\infty} x(t) [\frac{1}{2\pi} \int_{-\infty}^{\infty} \overset{*}{X}(j\omega) e^{-j\omega t} d\omega] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overset{*}{X}(j\omega) [\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overset{*}{X}(j\omega) X(j\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

The FTs of δ(t) and δ′(t)

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{0} dt = 1$$

$$F\{\frac{d\delta(t)}{dt}\} = j\omega F\{\delta(t)\} = j\omega$$

FT of a rectangular signal

$$f(t) = \begin{cases} E & |t| \le \tau/2 \\ 0 & |t| \ge \tau/2 \end{cases}$$

The FT of the rectangular signal is:

$$F(j\omega) = \int_{-\tau/2}^{\tau/2} E e^{-j\omega t} dt = \frac{2E}{\omega} \sin(\frac{\omega\tau}{2})$$

FT of single-sided e^{at}

$$f(t) = \begin{cases} e^{-at} & t \ge 0 \\ 0 & t < 0 \end{cases} \qquad a > 0$$

The FT of *f(t):*

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \frac{1}{-(a+j\omega)} \int_{0}^{\infty} e^{-(a+j\omega)t} d[-(a+j\omega)t] = \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega}$$

The magnitude spectrum $|F(j\omega)|$ and phase spectrum $\angle F(j\omega)|$ (in class)

Midterm 1 problems and beyond

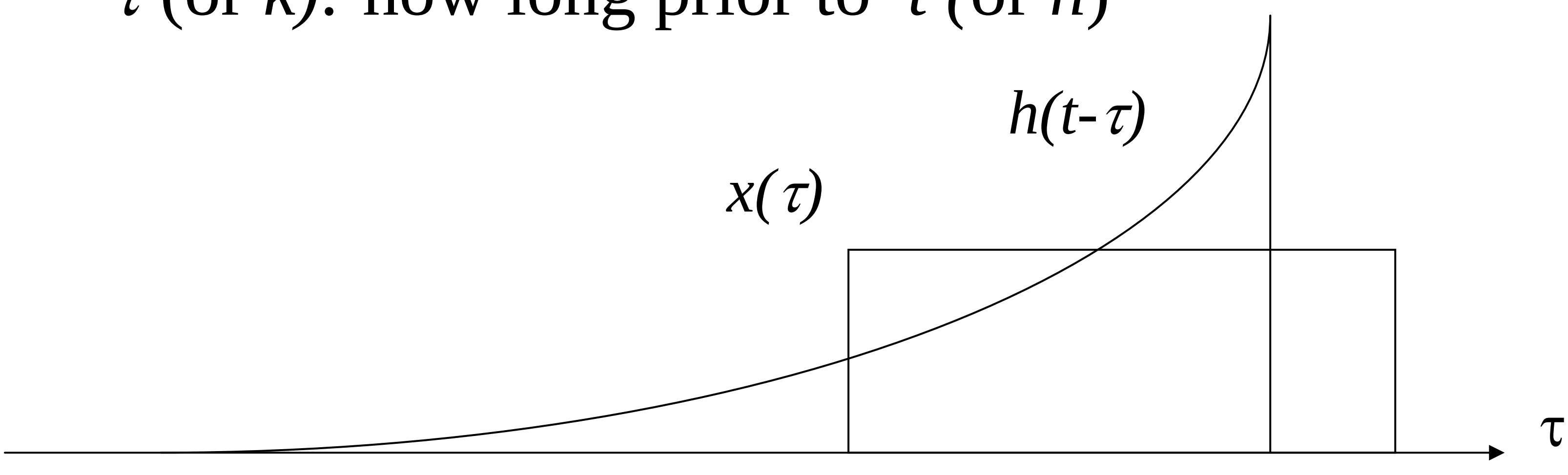
- Properties of signals and systems
- Calculation of convolutions
- Use of unit step function u(t)
- Initial value of ha(0^+)
- Application of superposition property
- System inter connection
- Mathematical skills:
  - Implementation of Eqs.: correct, effective, efficient
  - A language of calculation: expressive, understandable
- Application skills
  - Applications of learned theories/knowledge (mathematics, physics, electricity, computer, signals and systems,…)

Physical meanings of t and τ in the convolution

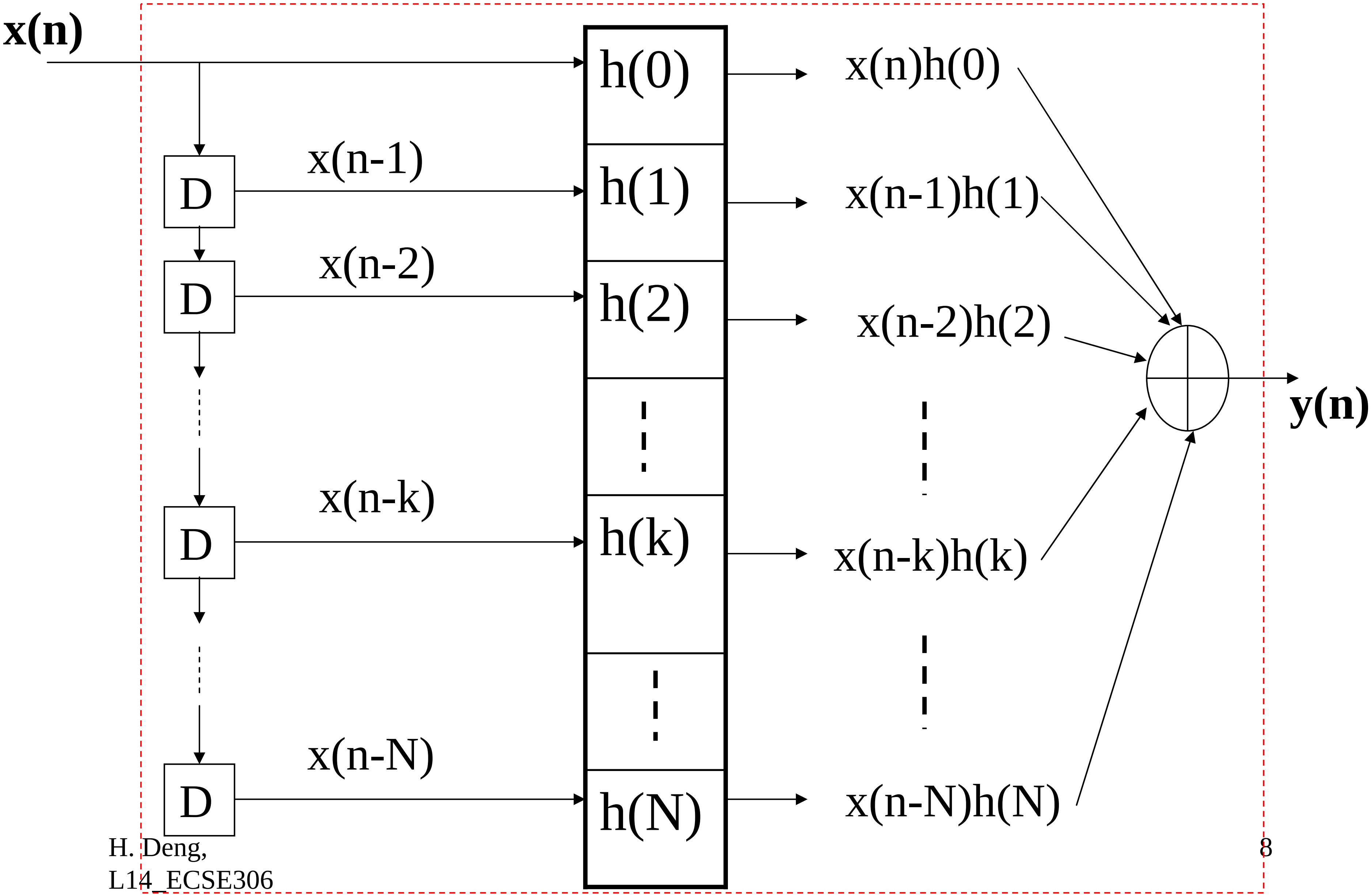
$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$x[n] * h[n] = \sum_{k=\infty}^{\infty} x[k]h[n-k] = \sum_{k=\infty}^{\infty} x[n-k]h[k]$$

- t (or n): the time when the input and output are observed
- τ (or k): how long prior to t (or n)



An implementation of the DT convolution



McGill University
Department of Electrical and Computer Engineering

Lecture 15

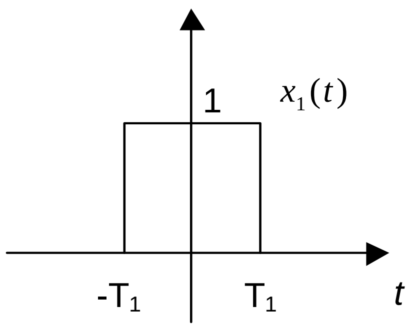
Hui Qun Deng, PhD

October 8, 2008

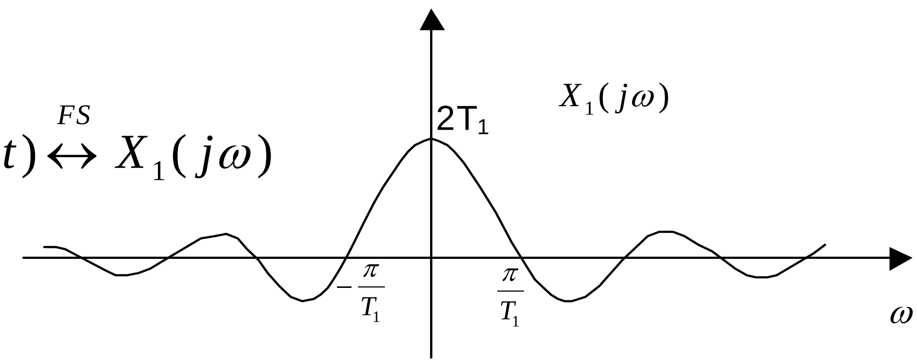
1. The duality of FT pairs
2. Examples of FT
3. Inverse Fourier Transform

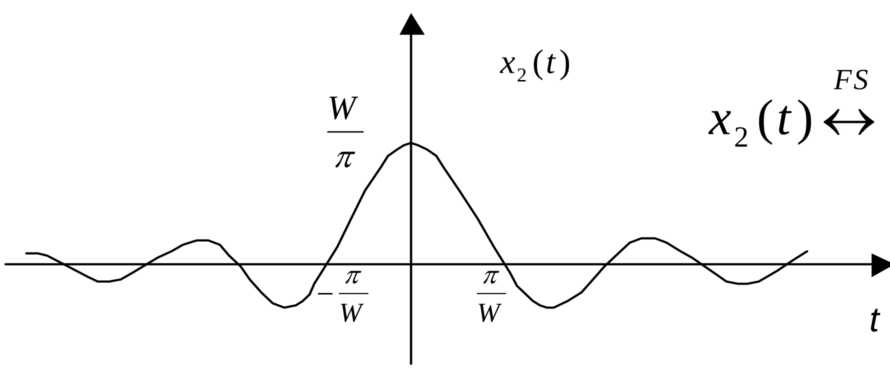
Example of the duality of FT pairs

The Fourier transform and the inverse Fourier transform have a symmetric relationship. This results in a duality between the time domain representation and the frequency domain representation of a signal, e.g., between a rectangular signal and its FT:

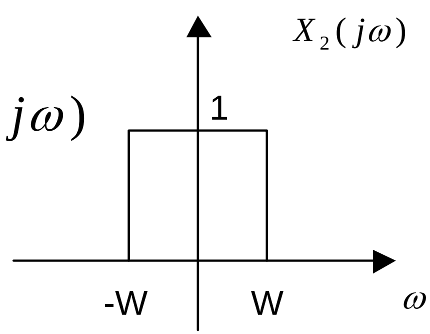


$$x_1(t) \overset{FS}{\leftrightarrow} X_1(j\omega)$$





$$x_2(t) \overset{FS}{\leftrightarrow} X_2(j\omega)$$



Duality of FT pairs

$$x(t) \leftrightarrow X(\omega)$$
$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Denote the FT of *x(t)* as $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

The inverse FT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

Assume a signal *y(t) =X(t)*, what is the FT of y(t)?

$$Y(\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = 2\pi\, x(-\omega)$$

This is because:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{-j\Omega\omega}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)e^{-jt\omega}dt$$

FT of two-sided e^{at}

$$f(t)=\begin{cases} e^{-at} & t\ge 0 \\ e^{at} & t\le 0 \end{cases} \qquad a>0$$

Recall the time/frequency scaling property of FT

$$x(\alpha t)\overset{\mathcal{FT}}{\leftrightarrow}\frac{1}{|\alpha|}X(j\omega/\alpha)$$

Rewrite *f(t)* as:

$$f(t)=f_1(t)+f_1(-t)$$

where

$$f_1(t)=\begin{cases} e^{-at} & t\ge 0 \\ 0 & t<0 \end{cases} \qquad a>0$$

Then the FT of *f(t)* is

$$F(j\omega)=F_1(j\omega)+F_1(-j\omega)=\frac{1}{a+j\omega}+\frac{1}{a-j\omega}=\frac{2a}{\sqrt{a^2+\omega^2}}$$

FT of sgn(t)

$$\text{sgn}(t) := \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

Let $$f(t) = \begin{cases} e^{-at}, & t \geq 0 \\ -e^{at}, & t \leq 0 \end{cases} \qquad a > 0$$

The FT of *f(t)* is:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = -\int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \frac{-2j\omega}{a^2 + \omega^2}$$

Then the FT of sgn(t) is:

$$SGN(j\omega) = \lim_{a->0} F(j\omega) = \lim_{a->0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$

FT of u(t)

$$U(j\omega) = F\{\frac{1}{2} + \frac{1}{2}\text{sgn}(t)\} = F\{\frac{1}{2}\} + \frac{1}{2}F\{\text{sgn}(t)\}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

FT of a constant

Given a signal x(t): $$x(t) = E, \quad -\infty < t < \infty$$

The FT of *x(t)* can be derived according to the duality of FT pairs.

We know the FT of δ(t) is a constant 1:

$$\delta(t) \leftrightarrow 1$$

Then according to the duality of FT pairs, we have

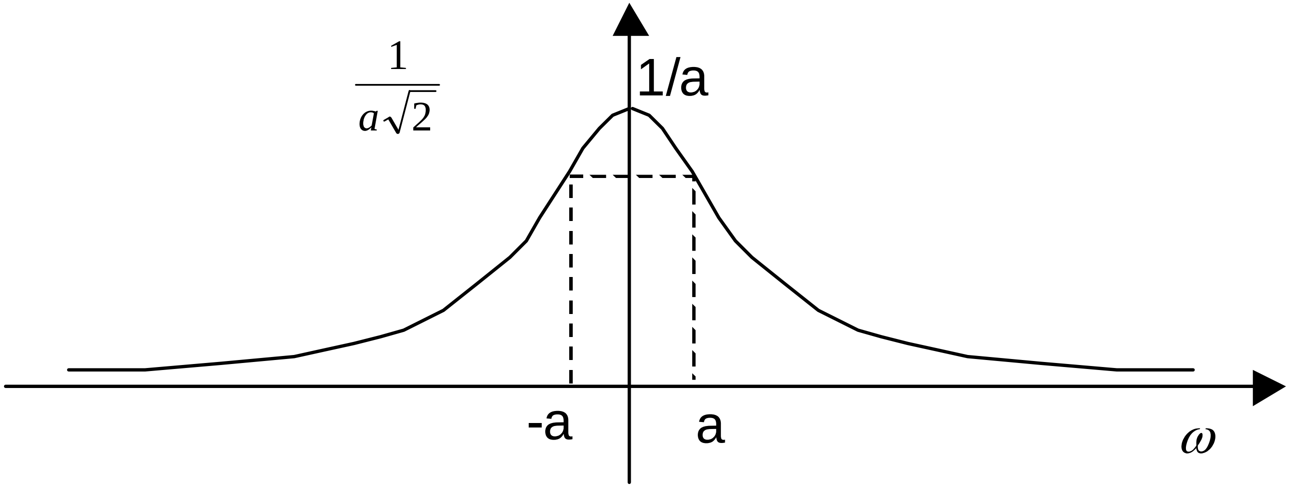
$$E \leftrightarrow 2E\,\pi\delta(\omega)$$ see Appendix D

FT of the Complex Exponential

The Fourier transform of $e^{-at}u(t),\ \ a = \alpha + j\beta, \alpha > 0$ is given by (see Boulet's book):

$$X(j\omega) = \frac{1}{j\omega + a}$$

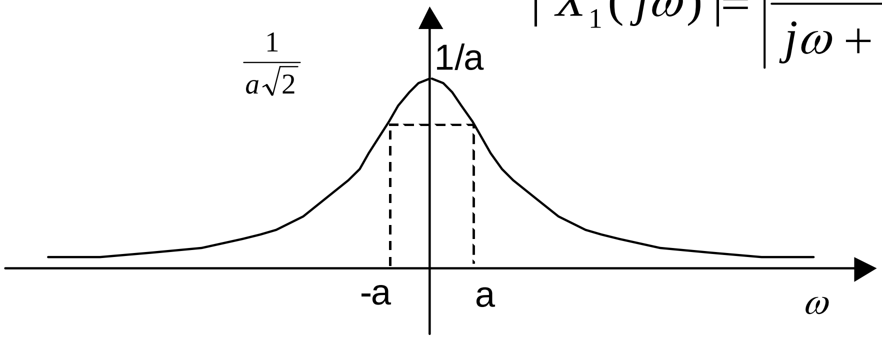
For the case $a > 0$ real, the magnitude is $|X_1(j\omega)| = \left|\frac{1}{j\omega + a}\right|$



Case: *a*>0, real

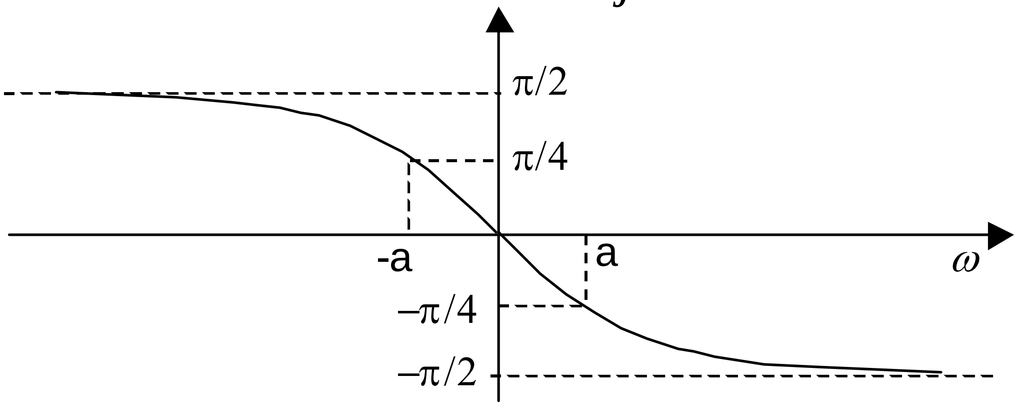
The magnitude spectrum:

$$|X_1(j\omega)| = \left|\frac{1}{j\omega + a}\right|$$



The phase spectrum:

$$\angle X_1(j\omega) = \angle \frac{1}{j\omega + a}$$



Remarks

For the case where $e^{-at}u(t)$ is the impulse response $h(t)$ of a first-order differential LTI system:

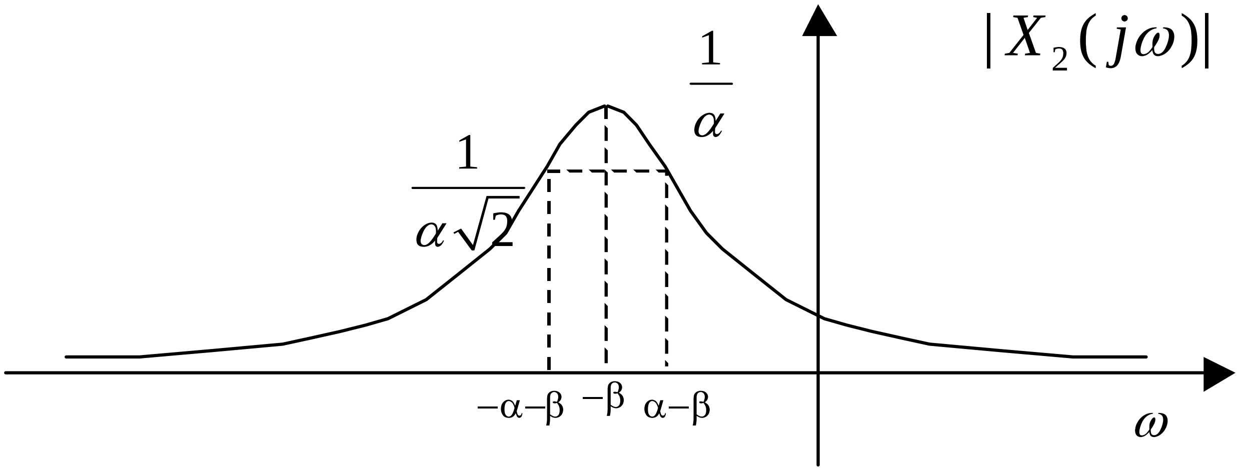
- The system is a lowpass filter with DC gain of $1/a$
- High frequencies in the input signal are attenuated
- The *cutoff frequency* of the filter is $\omega_c = a$, where frequency components of the input signal are attenuated by a factor $1/\sqrt{2}$
- The maximum phase added to the input signal is $-\pi/2$ for $\omega \to +\infty$

Case: a is complex

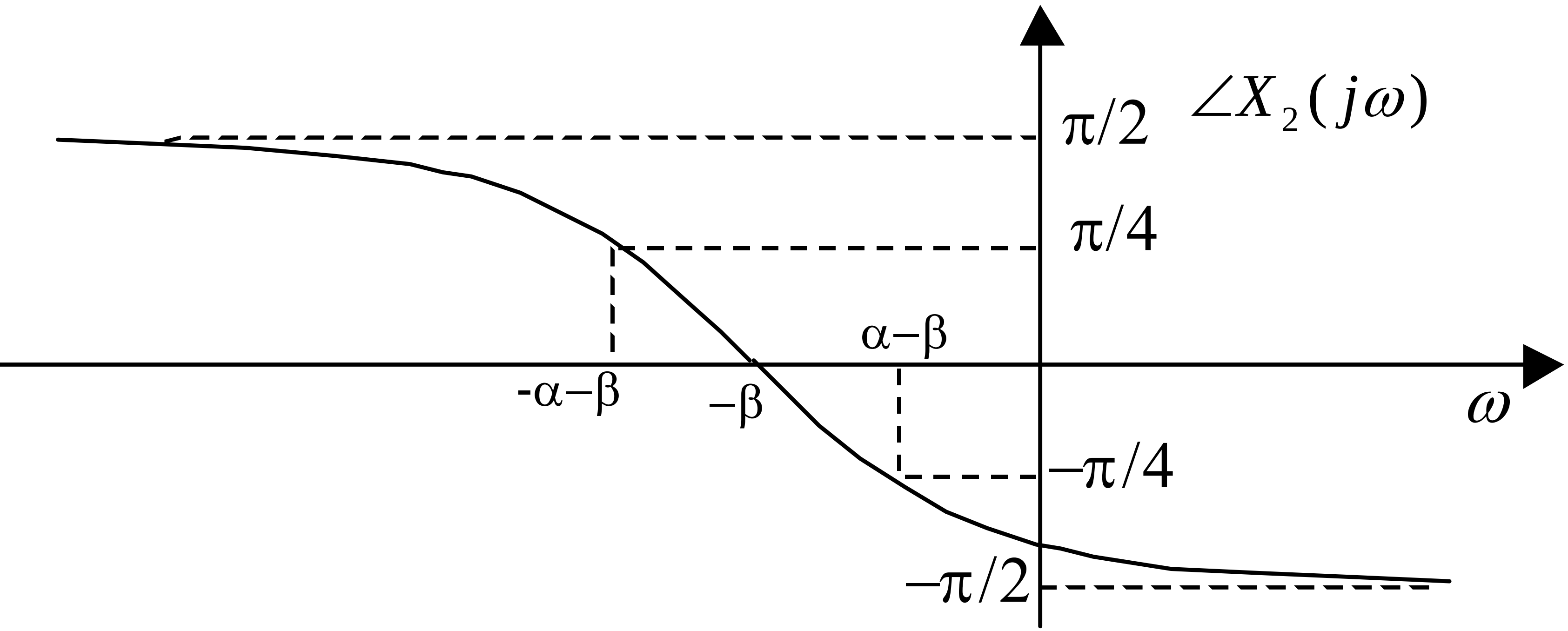
In the case $a = \alpha + j\beta,\ \alpha > 0, \beta > 0$, $X_2(j\omega)$ can be obtained by shifting the magnitude and phase of $X_1(j\omega)$:

$$X_2(j\omega) = X_1(j(\omega + \beta)) = \frac{1}{j(\omega + \beta) + \alpha}.$$

Note that this is a shift to the left in the frequency domain, so the magnitude and phase of $X_2(j\omega)$ are plotted as follows:



Phase:



FT of $e^{-at}sin(\omega_0 t)$

By the linearity and conjugation properties, we can write

$$x_3(t) = e^{-\alpha t}\sin(\omega_0 t)u(t) = \frac{1}{2j}\left(e^{(-\alpha + j\omega_0)t}u(t) - e^{(-\alpha - j\omega_0)t}u(t)\right)$$

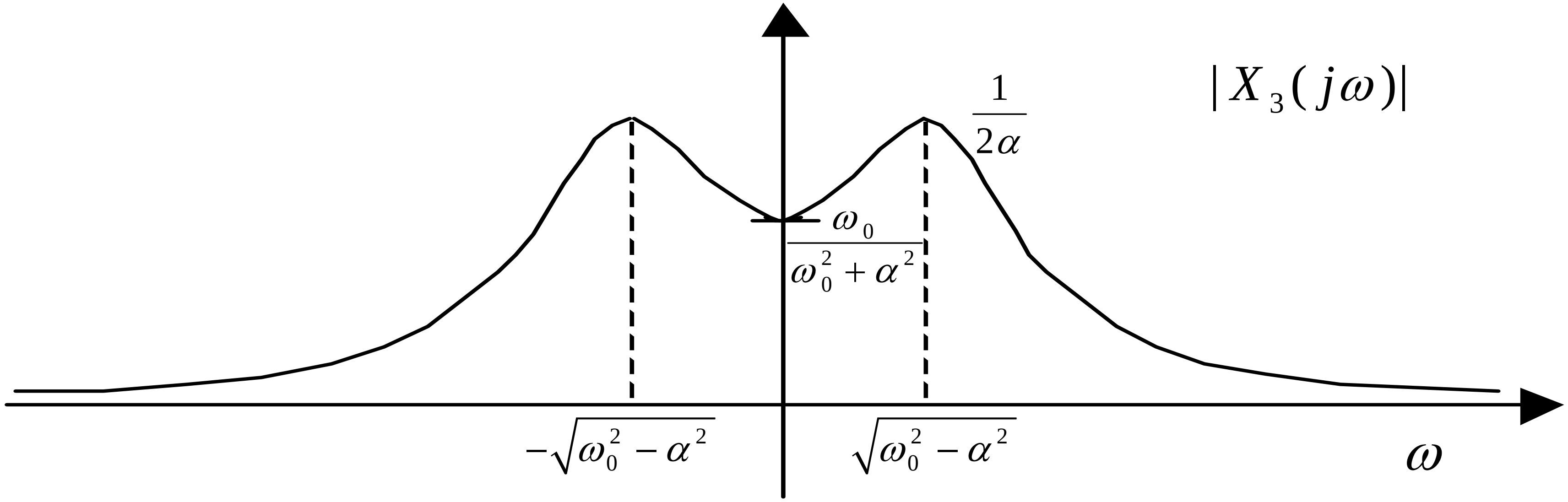
$$\overset{\mathcal{FT}}{\leftrightarrow}$$

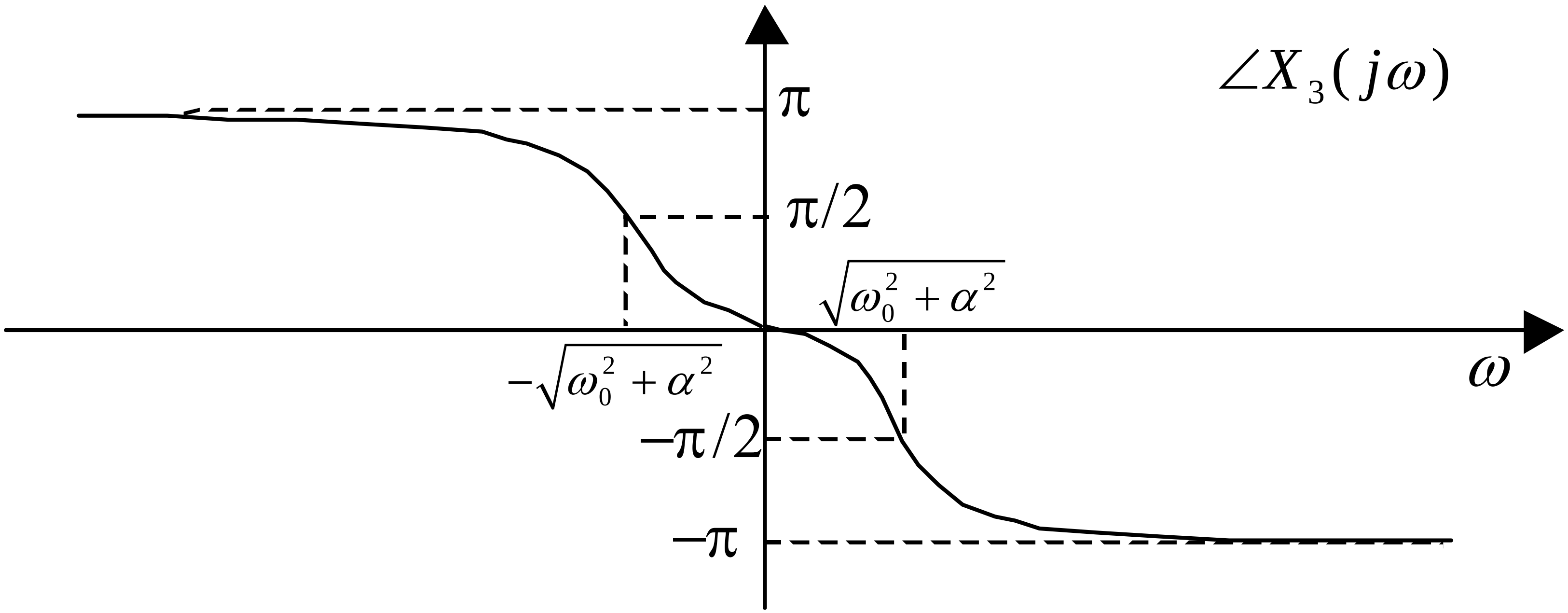
$$X_3(j\omega) = \frac{1}{2j}\left(\frac{1}{j(\omega-\omega_0)+\alpha} - \frac{1}{j(\omega+\omega_0)+\alpha}\right)$$

$$= \frac{1}{2j}\left(\frac{j(\omega+\omega_0)+\alpha-[j(\omega-\omega_0)+\alpha]}{[j(\omega-\omega_0)+\alpha][j(\omega+\omega_0)+\alpha]}\right)$$

$$= \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2}$$

Here the signal was real so we should get an even $|X_3(j\omega)|$ and an odd $\angle X_3(j\omega)$.

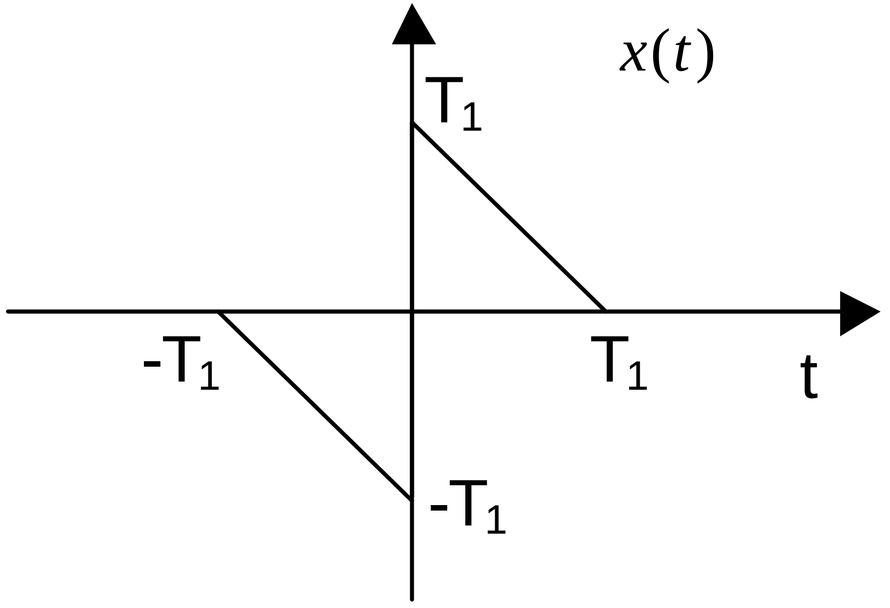




$\angle X_3(j\omega)$
π
π/2
$\sqrt{\omega_0^2+\alpha^2}$
$-\sqrt{\omega_0^2+\alpha^2}$
ω
−π/2
−π

Fourier Transform of a "Sawtooth"

Let's calculate the Fourier Transform of the sawtooth signal $x(t)$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-T_1}^{0} (-T_1 - t)e^{-j\omega t}dt + \int_{0}^{T_1} (T_1 - t)e^{-j\omega t}dt$$

$$= j\left(\frac{2\sin(\omega T_1)}{\omega^2} - \frac{2T_1}{\omega}\right)$$

The Inverse Fourier Transform of an ideal low-pass filter

An ideal lowpass filter with cutoff frequency ω_c as given by its spectrum

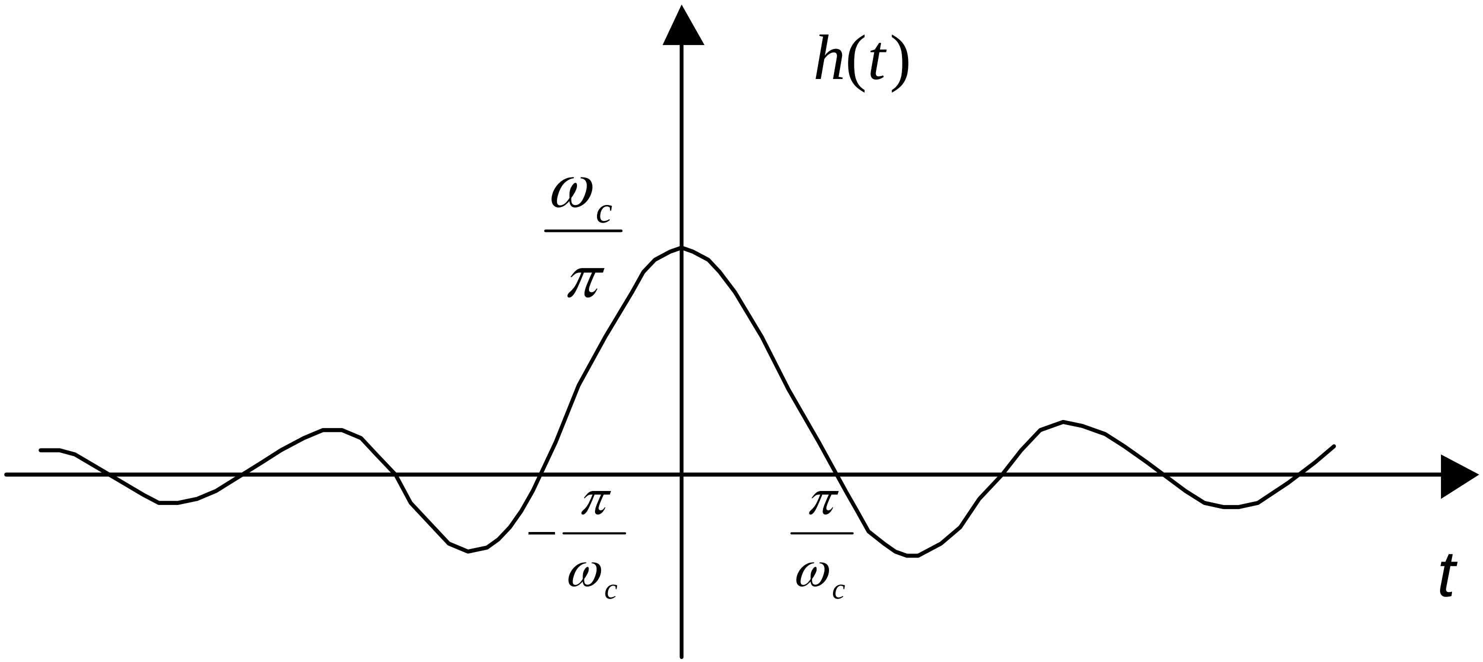
$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}.$$

The corresponding impulse response is calculated:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega t}d\omega = \frac{1}{2\pi(jt)}\left[e^{j\omega t}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{\sin(\omega_c t)}{\pi t} = \frac{\omega_c}{\pi}\frac{\sin(\frac{\omega_c}{\pi}t\pi)}{\frac{\omega_c \pi t}{\pi}} = \frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c}{\pi}t\right)$$

Thus the impulse response of an ideal lowpass filter is a (real) sinc function extending from $t = -\infty$ to $t = +\infty$



Inverse Fourier transforms of rational functions

In the case when the Fourier transform is in the form of a rational function of $j\omega$ (a ratio of two polynomials).

- It is much easier to perform a partial fraction expansion of the Fourier transform, and then to identify each term in this expansion using a table of Fourier transforms and their corresponding time-domain signals.
- This method is usually preferred to obtain the output response of a stable differential LTI system using the convolution property.

Example: obtaining the response of a system via Inverse FT

Consider the response of an LTI system with impulse response $h(t) = e^{-2t}u(t)$, (which meets the Dirichlet conditions) to the input $x(t) = e^{-3t}u(t)$.

Rather than computing their convolution, we will find the response by multiplying the Fourier transforms of the input and the impulse response.

$$X(j\omega) = \frac{1}{j\omega + 3}, \qquad H(j\omega) = \frac{1}{j\omega + 2}$$

Then, $Y(j\omega) = X(j\omega)H(j\omega) = \dfrac{1}{(j\omega + 3)(j\omega + 2)}$.

Inverse Fourier transform Step 1: partial fraction expansion

The *partial fraction expansion* consists of expressing this transform as a sum of simple first-order terms.

$$Y(j\omega) = \frac{1}{(j\omega + 2)(j\omega + 3)} = \frac{A}{(j\omega + 2)} + \frac{B}{(j\omega + 3)}$$

The constants A, B can be determined by substituting values for the frequency ω (e.g., 0) and to solve the resulting system of linear equations.

Another easier technique consists of applying the following procedure.

(1) Equate the transform with its sum of partial fractions, and let $s = j\omega$;

$$\frac{1}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$$

(2) To obtain A, multiply both sides of the equation by $(s+2)$ and evaluate for $s = -a$.

$$\left.\frac{1}{(s+3)}\right|_{s=-2} = A + \left.\frac{(s+2)B}{(s+3)}\right|_{s=-2}$$

$$\Rightarrow \quad A = \frac{1}{-2+3} = 1$$

Applying step (2) for the constant B, we obtain

$$\left.\frac{1}{(s+2)}\right|_{s=-3} = \left.\frac{(s+3)A}{(s+2)}\right|_{s=-3} + B$$

$$\Rightarrow \quad B = \frac{1}{-3+2} = -1$$

Step 2: Inverse Fourier Transform of the partial fraction expansion

Finally, the partial fraction expansion of the Fourier transform of the output is given by

$$Y(j\omega) = \frac{1}{(j\omega + 2)} - \frac{1}{(j\omega + 3)}$$

Using Table D.1 of basic Fourier transform pairs in the textbook, we find that

$$y(t) = e^{-2t}u(t) - e^{-3t}u(t).$$

McGill University
Department of Electrical and Computer Engineering

Lecture 16

October 10, 2008

Hui Qun Deng, PhD

1. Application of Fourier transform in obtaining responses of LTI systems
2. Fourier transform of periodic signals
3. Convergence of FT

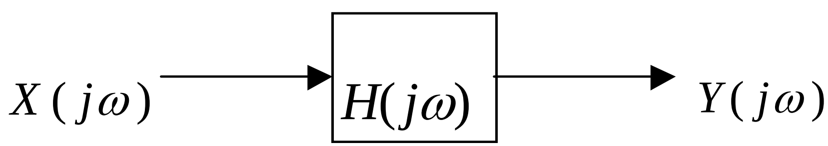
Obtaining the responses of LTI systems using FT

Obtaining the output of a stable LTI system with impulse response *h(t)* and an input *x(t)*:

- Method 1: convolution y(t)=x(t)*h(t)
- Method2 : applying FT and the inverse FT

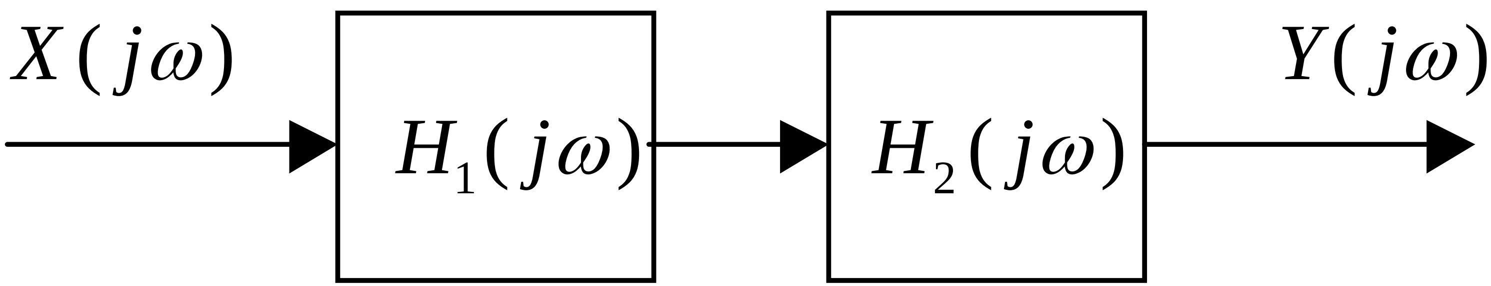
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}d\omega$$



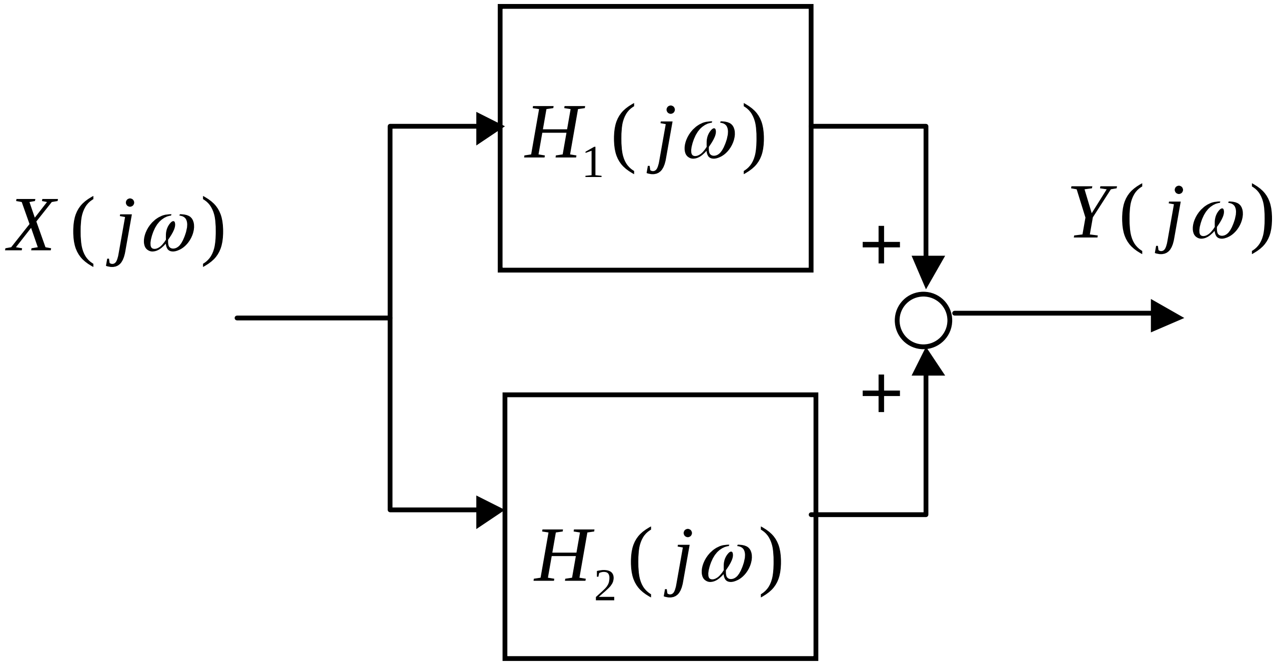
Frequency responses of connected LTI systems

For a cascade of two stable LTI systems with impulse responses $h_1(t),\ h_2(t)$, we have



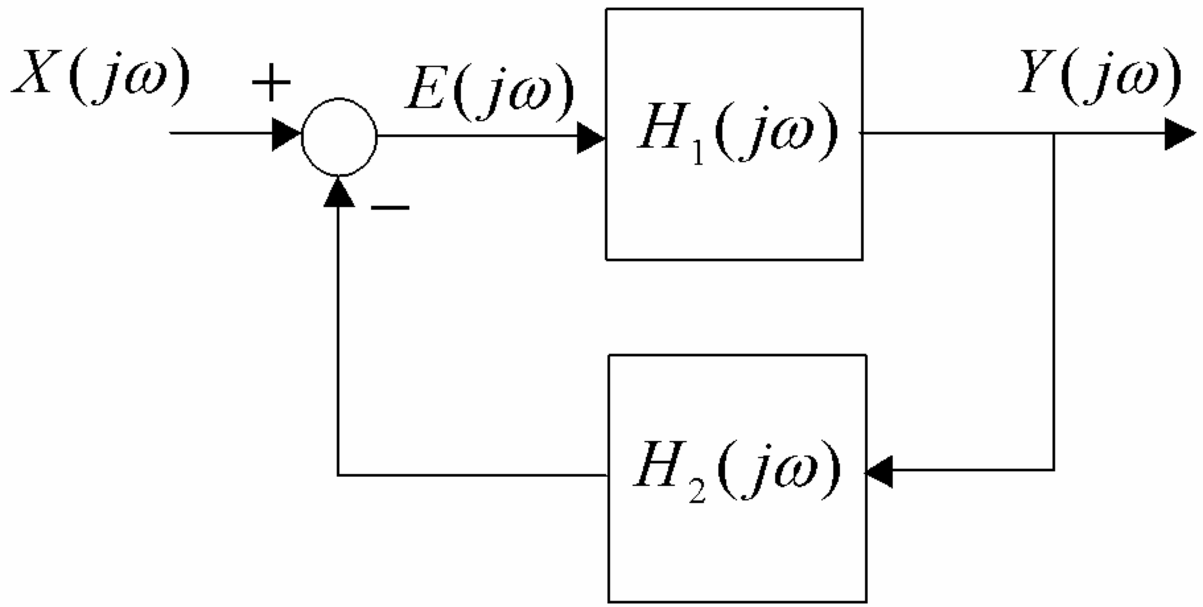
$$Y(j\omega) = H_2(j\omega)H_1(j\omega)X(j\omega)$$

For a parallel connection of two stable LTI systems with impulse responses $h_1(t),\ h_2(t)$, we have



$$Y(j\omega)=[H_1(j\omega)+H_2(j\omega)]X(j\omega)$$

For a feedback interconnection of two stable LTI systems with impulse responses $h_1(t),\ h_2(t)$, we would have



$$Y(j\omega) = \frac{H_1(j\omega)}{1 + H_1(j\omega)H_2(j\omega)} X(j\omega)$$

The frequency response of an LTI Differential System

Consider the stable LTI system defined by an N^{th}-order linear constant-coefficient differential equation initially at rest:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} .$$

Assume that $X(j\omega)$, $Y(j\omega)$ denote the Fourier transforms of the input $x(t)$ and the output $y(t)$ respectively. Taking FT on LHS and RHS of the Eq., and applying the derivative property of FT, we have

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) .$$

The frequency response of a differential system

The frequency response of the system is given by:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Example 1: Obtaining the step response of a differential system

A second-order LTI differential system is defined by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} - x(t)$$

Suppose we want to obtain the step response of the system.

Step 1. Calculating the frequency response of this system by taking FT on the two sides of the Eq.

$$[(j\omega)^2 + 3j\omega + 2]Y(j\omega) = (j\omega - 1)X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega - 1}{(j\omega)^2 + 3j\omega + 2}$$

Step 2: Obtain the FT of the step response

From Table D.1 and Lecture 15, we know the Fourier transform of the step function is:

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

From the convolution property of FT, the FT of the step response is:

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega - 1}{(j\omega)^2 + 3j\omega + 2}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{j\omega - 1}{[(j\omega)^2 + 3j\omega + 2]j\omega} + \frac{1}{2}\pi\delta(\omega)$$

Step 3: represent the FT of the response in terms of partial fractions

Expanding the rational function on the right-hand side into partial fractions.

Let s=jω, we get

$$\frac{s-1}{\left[s^2+3s+2\right]s}=\frac{s-1}{(s+1)(s+2)s}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2},$$

where the coefficients A, B and C are computed as follows:

$$A=\left.\frac{s-1}{(s+1)(s+2)}\right|_{s=0}=-\frac{1}{2} \qquad B=\left.\frac{s-1}{s(s+2)}\right|_{s=-1}=\frac{-2}{-1}=2 \qquad C=\left.\frac{s-1}{s(s+1)}\right|_{s=-2}=-\frac{3}{2}$$

Hence, $$Y(j\omega)=-\frac{1}{2}[\frac{1}{j\omega}+\pi\delta(\omega)]+\frac{2}{j\omega+1}-\frac{3}{2}\frac{1}{j\omega+2}$$

Step 4: Look up FT pair Table and obtain the inverse FT of the partial fractions

$$Y(j\omega) = -\frac{1}{2}[\frac{1}{j\omega} + \pi\delta(\omega)] + \frac{2}{j\omega+1} - \frac{3}{2}\frac{1}{j\omega+2}$$

In the TF pair Table (Appendix D), we find:

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

$$e^{at}u(t) \leftrightarrow \frac{1}{j\omega - a}, \quad \mathrm{Re}(a) < 0$$

Then, the inverse FT of $Y(j\omega)$ is

$$y(t) = [-\frac{1}{2} + 2e^{-t} - \frac{3}{2}e^{-2t}]u(t)$$

Remark: When M>N in differential equation, the frequency response has a numerator polynomial with higher order than the denominator polynomial.

$$\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} y(t) = \sum_{m=0}^{N} b_m \frac{d^m}{dt^m} y(t) \quad \rightarrow \quad H(j\omega) \quad \rightarrow \quad H(s) = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n}$$

Example 2

Consider the LTI differential system initially at rest described by

$$2\frac{dy(t)}{dt}+y(t)=\frac{d^2x(t)}{dt^2}-\frac{dx(t)}{dt}-2x(t).$$

The frequency response of this system is given by

$$H(j\omega)=\frac{(j\omega)^2-j\omega-2}{2(j\omega+0.5)}=\frac{(j\omega-2)(j\omega+1)}{2(j\omega+0.5)}.$$

Let $s = j\omega$ and write $H(s)$ as

$$H(s) = \frac{s^2 - s - 2}{2(s+0.5)} = As + B + \frac{C}{(s+0.5)}.$$

Multiplying both sides by $2(s+0.5)$, we can identify each coefficient.

$$s^2 - s - 2 = 2(s+0.5)(As+B) + 2C = 2As^2 + (A+2B)s + 2C + B$$

$$A = \frac{1}{2},\ B = -\frac{3}{4},\ C = -\frac{5}{8}$$

Thus, $$H(s) = \frac{s^2 - s - 2}{2(s+0.5)} = \frac{1}{2}s - \frac{3}{4} - \frac{5}{8}\frac{1}{(s+0.5)}$$

Expanded frequency response:

$$H(j\omega) = \frac{1}{2} j\omega - \frac{3}{4} - \frac{5}{8} \frac{1}{(j\omega + 0.5)} \; .$$

Finally, the impulse response is

$$h(t) = \frac{1}{2} \frac{d}{dt} \delta(t) - \frac{3}{4} \delta(t) - \frac{5}{8} e^{-0.5t} u(t) \; .$$

Example 3

Stable second-order LTI differential system, whose characteristic polynomial has complex zeros:

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

The frequency response of this system is given by:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{(j\omega)^2 + j\omega + 1} = \frac{1}{(j\omega + \frac{1}{2} - j\frac{\sqrt{3}}{2})(j\omega + \frac{1}{2} + j\frac{\sqrt{3}}{2})}$$

Letting $s = j\omega$, and expanding the right-hand side into partial fractions, we get

$$\frac{1}{s^2+1s+1} = \frac{1}{(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})(s+\frac{1}{2}-j\frac{\sqrt{3}}{2})} = \frac{A}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}} + \frac{B}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}$$

the coefficients are computed as follows:

$$A = \left.\frac{1}{s+\frac{1}{2}-j\frac{\sqrt{3}}{2}}\right|_{s=-\frac{1}{2}-j\frac{\sqrt{3}}{2}} = j\frac{1}{\sqrt{3}}, \quad B = \left.\frac{1}{s+\frac{1}{2}+j\frac{\sqrt{3}}{2}}\right|_{s=-\frac{1}{2}+j\frac{\sqrt{3}}{2}} = -j\frac{1}{\sqrt{3}}$$

Hence,

$$H(j\omega) = \frac{1}{(j\omega + \frac{1}{2} + j\frac{\sqrt{3}}{2})(j\omega + \frac{1}{2} - j\frac{\sqrt{3}}{2})}$$

$$= j\frac{1}{\sqrt{3}}\frac{1}{j\omega + \frac{1}{2} + j\frac{\sqrt{3}}{2}} - j\frac{1}{\sqrt{3}}\frac{1}{j\omega + \frac{1}{2} - j\frac{\sqrt{3}}{2}}$$

Using Table D.1 of Fourier transform pairs in the book, we find the output by inspection

$$h(t) = \left[\frac{j}{\sqrt{3}} e^{-\frac{1}{2}t - j\frac{\sqrt{3}}{2}t} - \frac{j}{\sqrt{3}} e^{-\frac{1}{2}t + j\frac{\sqrt{3}}{2}t} \right] u(t)$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \operatorname{Re}\left\{ e^{-j(\frac{\sqrt{3}}{2}t - \frac{\pi}{2})} \right\} u(t)$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t - \frac{\pi}{2}) u(t)$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t) u(t)$$

Fourier transform of $e^{jk\omega_0 t}$

Consider a signal *x(t)* with Fourier transform that is a single impulse of area 2π at frequency $k\omega_0$:

$$X(j\omega) = 2\pi\delta(\omega - k\omega_0).$$

Taking the inverse Fourier transform yields

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - k\omega_0)e^{jk\omega_0 t}d\omega = e^{jk\omega_0 t}$$

Thus,

$$e^{jk\omega_0 t} \leftrightarrow 2\pi\delta(\omega - k\omega_0) \qquad (5.67, 5.68)$$

Fourier transform of general periodic Signals

A general periodic signal has the time-domain Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad (5.70)$$

Taking Fourier transform of the above LHS and RHS and applying Eqs. (5.67, 5.68), we get:

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)} \qquad (5.69)$$

Therefore, the Fourier transform of a periodic signal is a train of impulses of area $2\pi a_k$, occurring at the frequencies $k\omega_0$, with a_k's being the FS coefficients of the signal.

Example

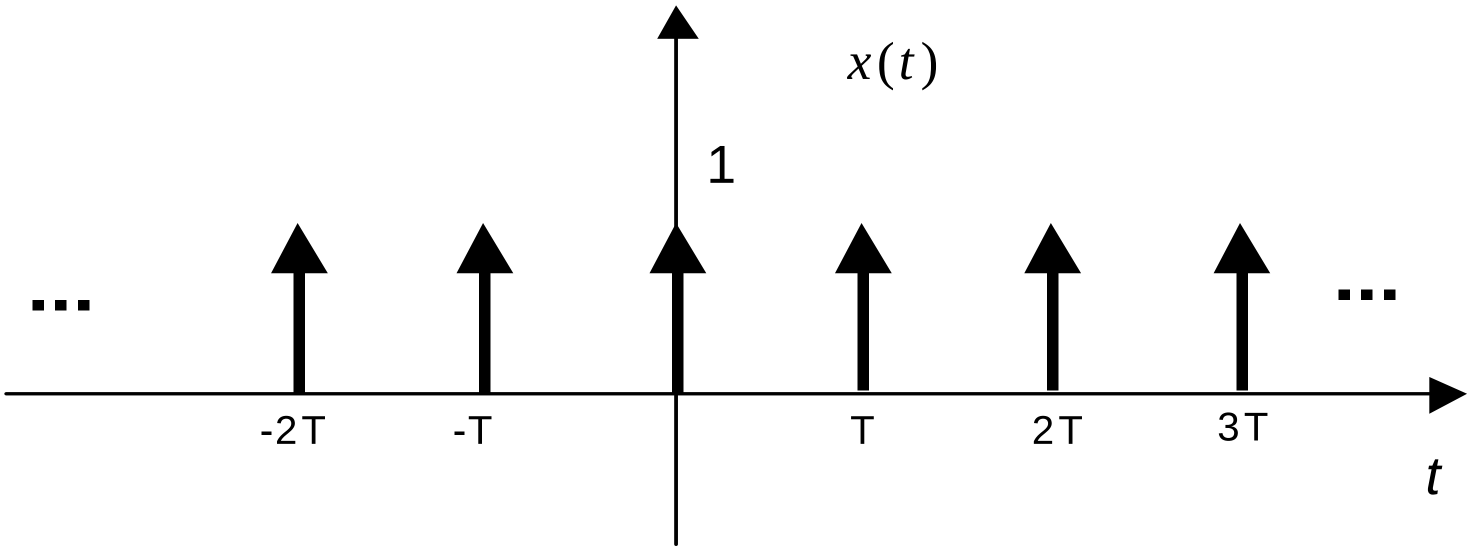
The Fourier transform of a sinusoidal signal of the form $x(t) = A\sin(\omega_0 t)$ is

$$X(j\omega) = jA\pi\delta(\omega+\omega_0) - jA\pi\delta(\omega-\omega_0)$$

FT of impulse train

Consider the impulse train signal

$$x(t) = \sum_{-\infty}^{+\infty} \delta(t - kT) \, .$$



We know Eq. (5.69)

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

We know that the FS coefficients the impulse train are $a_k = 1/T$, hence

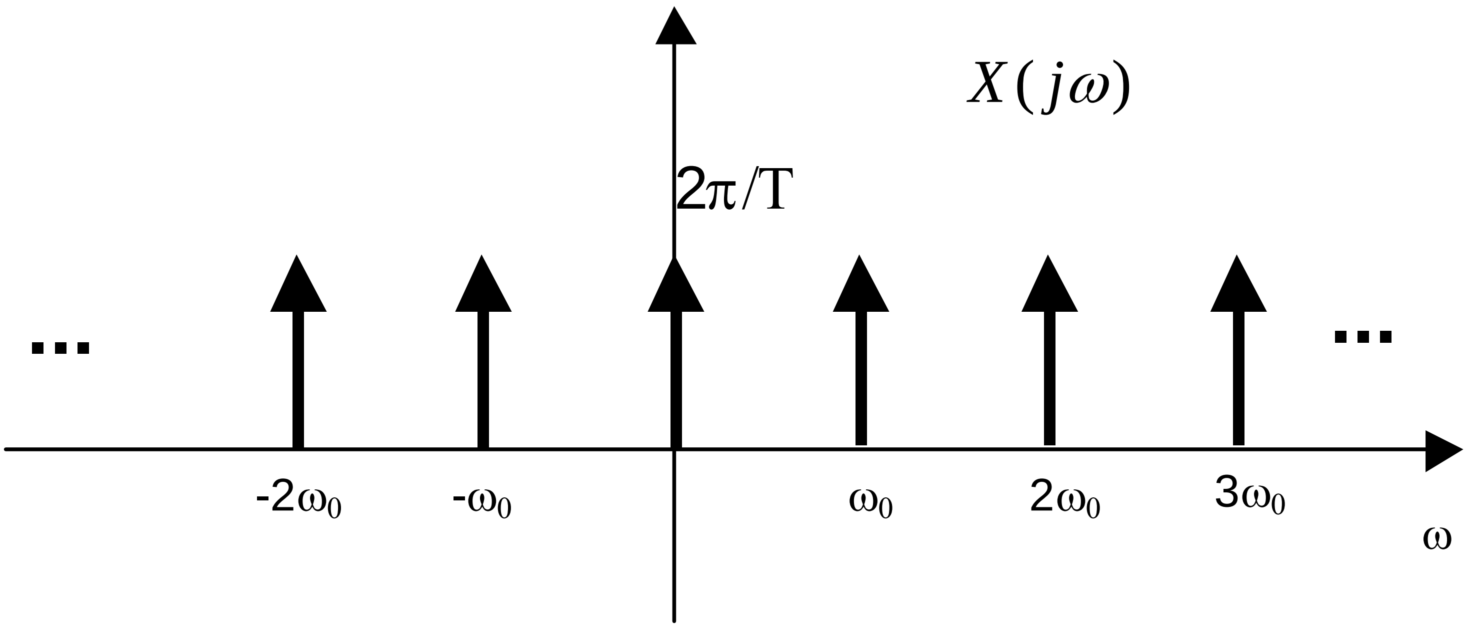
$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

We can also calculate the Fourier transform of the impulse train using the integral formula. This yields

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} e^{-j\omega kT}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} e^{-j\omega kT}$$

This Fourier transform is actually a periodic train of impulses of period $\omega_0 = 2\pi/T$ (note that the period is a frequency here!) in the frequency domain. That is, the above series converges to the impulse train shown below.



Convergence of the Fourier Transform

There are two important classes of signals for which the Fourier transform converges.

1. Signals of finite total energy, i.e. $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$

2. Signals that satisfy the Dirichlet conditions:

(1) $x(t)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$,

(2) $x(t)$ has a finite number of maxima and minima over any finite interval of time,

(3) $x(t)$ has a finite number of discontinuities over any finite interval of time. Furthermore, each of these discontinuities must be finite.

Types of convergence

Signals of finite energy: there is no energy in the error between a signal $x(t)$ and its inverse Fourier transform

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\,.$$

Signals satisfying the Dirichlet conditions: $\tilde{x}(t)$ is equal to $x(t)$ at every time t (pointwise convergence), except at discontinuities where $\tilde{x}(t)$ will take on the average of the values on either side of the discontinuity.

McGill University
Department of Electrical and Computer Engineering

Lecture 17

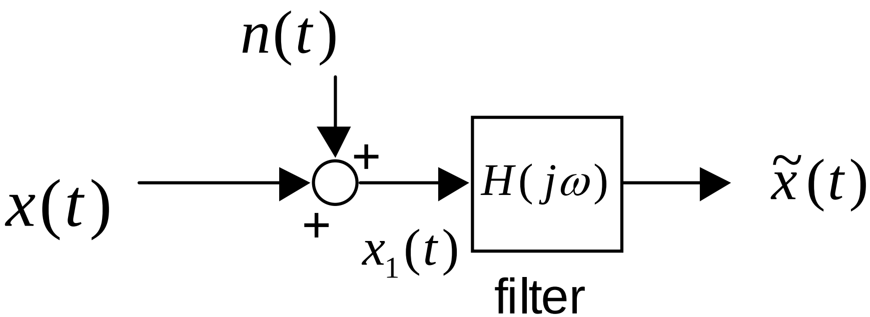
October 15, 2008

Hui Qun Deng, PhD

1. LPF, HPF and BPF
2. Laplace Transform
3. Inverse Laplace Transform

Frequency selective filter

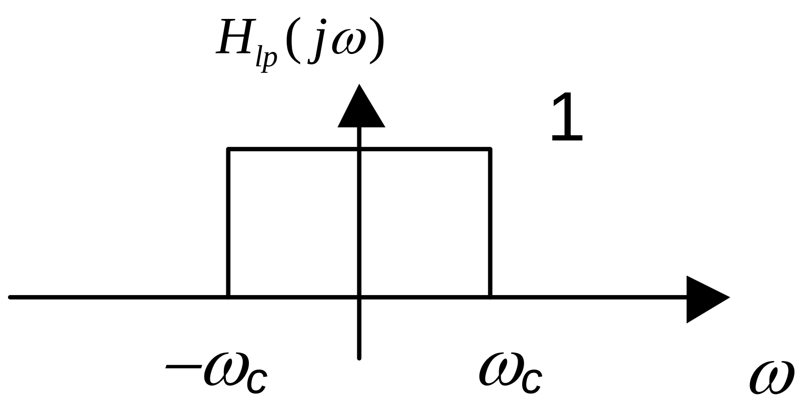
Frequency-selective filters are filters that allow frequency components over a given frequency band (the *passband*), components to pass undistorted, while attenuate components at other frequencies (the *stopband*).



The frequency response of an ideal low-pass filter

An ideal lowpass filter cuts off frequency components higher than a *cutoff frequency* ω_c. The frequency response of this filter is given by:

$$H(j\omega) := \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases}.$$

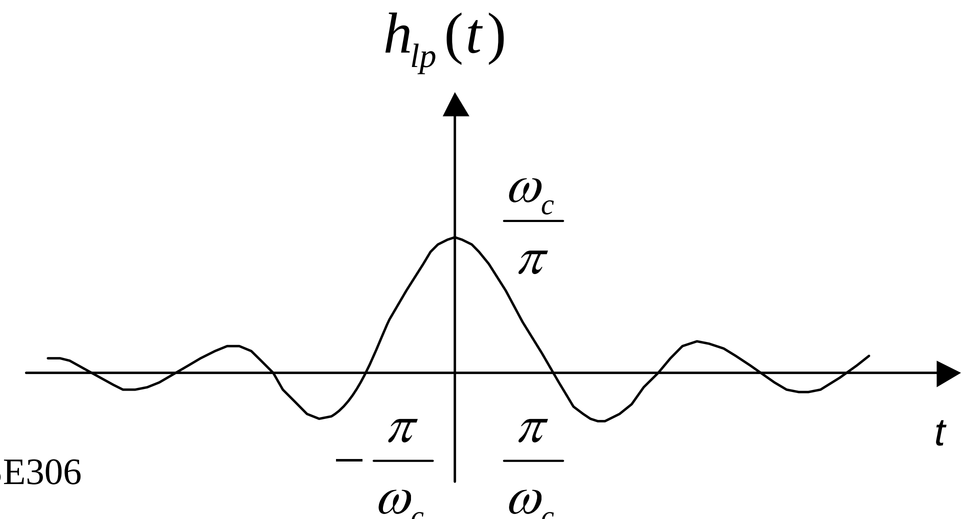


The impulse response of an ideal LP

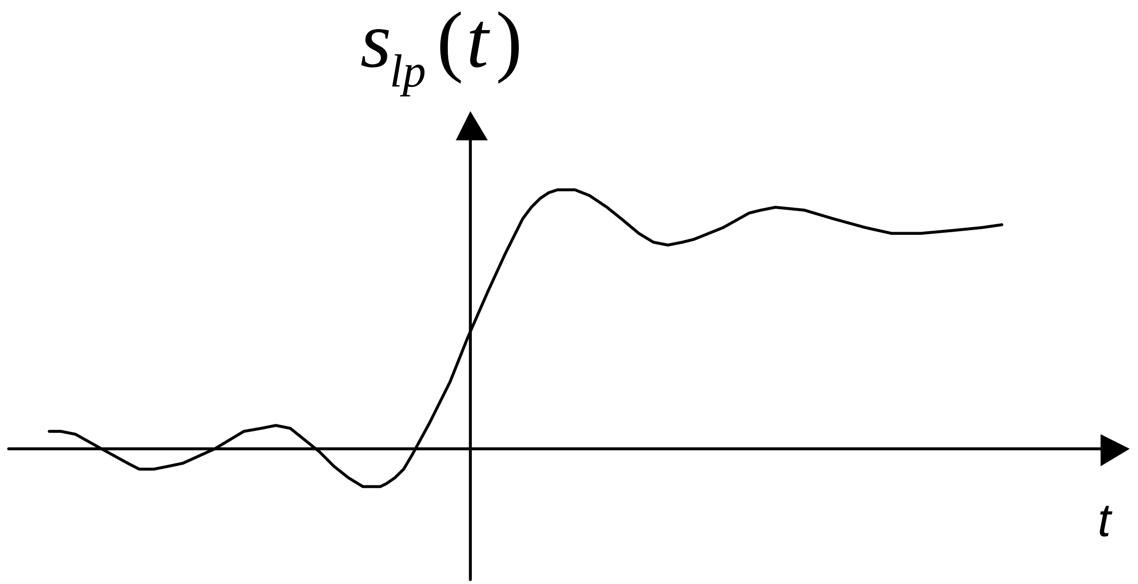
Recall Lecture 12: the frequency response of an LTI system and its impulse response constitute a FT pair: $h(t) \overset{FT}{\leftrightarrow} H(j\omega)$

Thus, the impulse response of the LPF can be derived via inverse FT of the frequency response $H(j\omega)$ of the LPF:

$$h_{lp}(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\,d\omega = \frac{1}{2\pi jt} e^{j\omega t}\Big|_{-\omega_c}^{\omega_c} = \frac{\sin(\omega_c t)}{\pi t} = \frac{\omega_c \operatorname{sinc}(\omega_c t/\pi)}{\pi}$$



Although the above filter is termed "ideal" in reference to its frequency response, it may not be so desirable in the time-domain for some applications because of the ripples in its step response.



Butter-worth LPF

One approximation to the ideal lowpass filter is the *Butterworth filter*. The magnitude of the frequency response of an *N^{th}-order* Butter-worth LPF is given by:

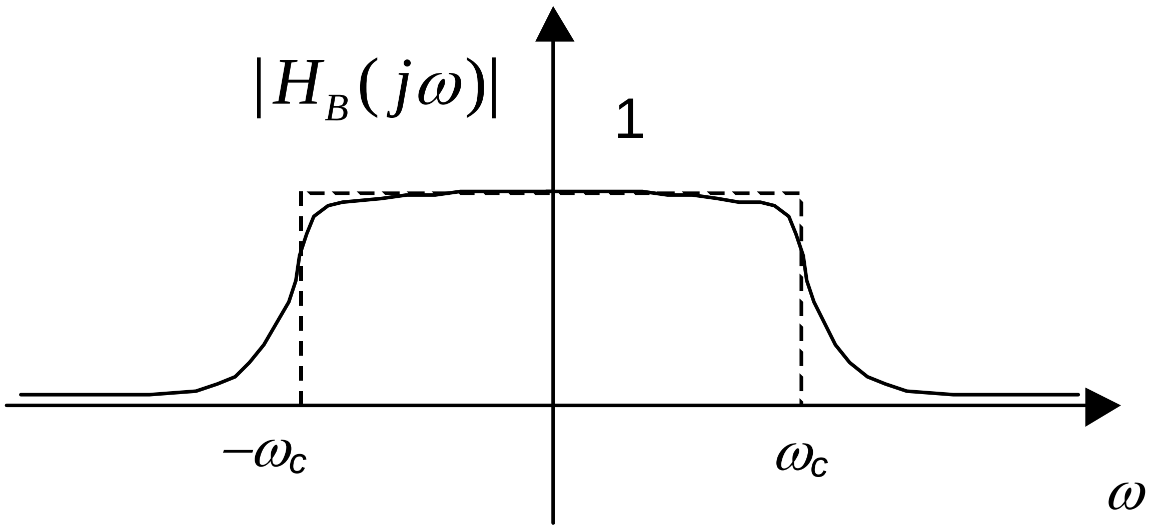
$$|H_B(j\omega)| = \frac{1}{[1+(\frac{\omega}{\omega_c})^{2N}]^{1/2}}$$

The magnitude of the frequency response of a 2nd-order Butterworth filter with cutoff frequency ω_c is given by:

$$H_B(j\omega) = \frac{\omega_c^{\ 2}}{(j\omega)^2 + \omega_c\sqrt{2}\,j\omega + \omega_c^{\ 2}}$$

The transitional band of LPFs

The magnitude of the frequency response of a 2nd-order butter-worth LPF:



Around ω_c is the *transition band,* where the magnitude "rolls off". The higher the order, the narrower the transition band becomes.

Representing the LPF using a differential equation

The second-order Butterworth filter is defined by its characteristic polynomial

$$p(s) = s^2 + \omega_c \sqrt{2} s + \omega_c{}^2$$

Therefore the differential equation relating the input and output signals of this filter must have the form

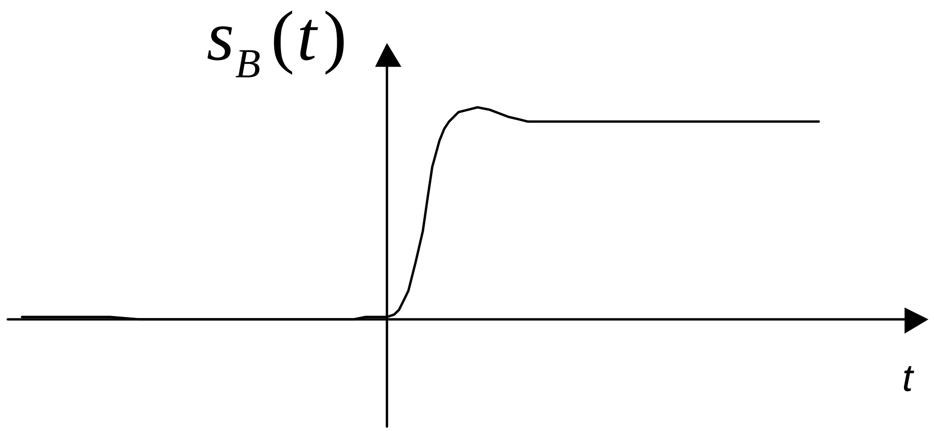
$$\frac{d^2 y(t)}{dt^2} + \omega_c \sqrt{2} \frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t) .$$

The impulse response of a 2nd-order butter-worth filter is given by (Assignment 6.1):

$$h_B(t) = \sqrt{2}\omega_c e^{-\frac{\omega_c}{\sqrt{2}}t} \sin(\frac{\omega_c}{\sqrt{2}}t)u(t).$$

This impulse response does not oscillate much even though it is a damped sinusoid. The decay rate is fast enough to damp out the oscillations.

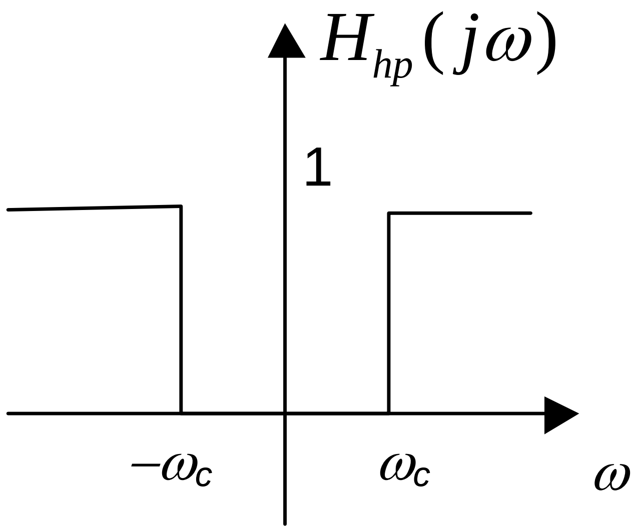
The step response of this second-order Butterworth filter is:



High-Pass Filters

A High-pass filter cuts off frequency components lower than its cutoff frequency ω_c. The frequency response of an ideal HPF is given by:

$$H_{hp} := \begin{cases} 0, & |\omega| \le \omega_c \\ 1, & |\omega| > \omega_c \end{cases}.$$



Obtaining an HPF from an LPF

Notice that the frequency response of an ideal highpass filter can be written as the difference between 1 and the frequency response of an ideal lowpass filter.

$$H_{hp}(j\omega) = 1 - H_{lp}(j\omega)$$

The resulting impulse response is simply

$$h_{hp}(t) = \delta(t) - h_{lp}(t)$$

This suggests one possible but naïve approach to obtaining a realizable highpass filter:

First, design a lowpass filter with cutoff frequency ω_c and desirable characteristics in the transition band and the stopband.

Second, form the frequency response of the highpass filter using above equation

Example

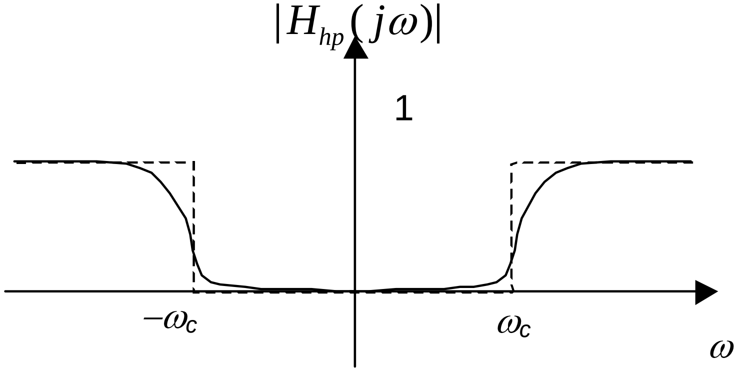
We can design an HPF using the second-order lowpass Butterworth filter of the above example.

$$H_{hp}(j\omega) = 1 - H_B(j\omega)$$

$$= 1 - \frac{\omega_c^2}{(j\omega)^2 + \omega_c\sqrt{2}\,j\omega + \omega_c^2}$$

$$= \frac{(j\omega)^2 + \omega_c\sqrt{2}\,j\omega + \omega_c^2 - \omega_c^2}{(j\omega)^2 + \omega_c\sqrt{2}\,j\omega + \omega_c^2}$$

$$= \frac{(j\omega)^2 + \omega_c\sqrt{2}\,j\omega}{(j\omega)^2 + \omega_c\sqrt{2}\,j\omega + \omega_c^2}$$



The differential Eq. for the HPF

The causal LTI differential equation corresponding to the above $H(j\omega)$ is

$$\frac{d^2 y(t)}{dt^2} + \omega_c\sqrt{2}\frac{dy(t)}{dt} + \omega_c^2 y(t) = \frac{d^2 x(t)}{dt^2} + \omega_c\sqrt{2}\frac{dx(t)}{dt}$$

and the impulse response is given by:

$$h_{hp}(t) = \delta(t) - \sqrt{2}\omega_c e^{-\frac{\omega_c}{\sqrt{2}}t}\sin(\frac{\omega_c}{\sqrt{2}}t)u(t)$$

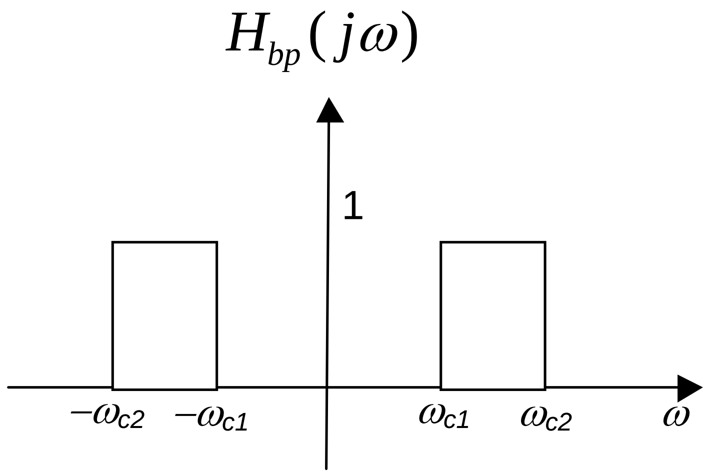
Band-Pass Filters

An ideal bandpass filter cuts off frequencies lower than its first cutoff frequency ω_{c1} and higher than its second cutoff frequency ω_{c2}.

The frequency response of such a filter is given by:

$$H_{bp} := \begin{cases} 1, & |\omega_{c1}| < \omega | < \omega_{c2} \\ 0, & otherwise \end{cases}$$

.



Obtaining a BPF from a LPF and an HPF

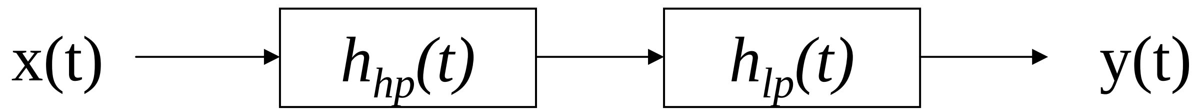
One approach to design bandpass filters:

The frequency response of an ideal bandpass filter can be written as the product of the frequency responses of ideal overlapping lowpass and highpass filters.

$$H_{bp}(j\omega) = H_{hp}(j\omega)H_{lp}(j\omega)$$

The highpass filter should have a cutoff frequency of ω_{c1} and the lowpass filter ω_{c2} .

An implementation of BPF:



The Definition of Bilateral Laplace Transform

The bilateral Laplace transform of x(t) is defined as:

$$X(s) := \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

where s is a complex variable. Notice that the Fourier transform is given by the same equation, only $s = j\omega$ for the Fourier transform.

Let the Laplace variable be written as $s = \sigma + j\omega$.

$$X(\sigma + j\omega) := \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt$$

Then the Laplace transform can be viewed as the Fourier transform of the *exponentially-weighted signal* $x(t)e^{-\sigma t}$.

Motivations for Laplace transform

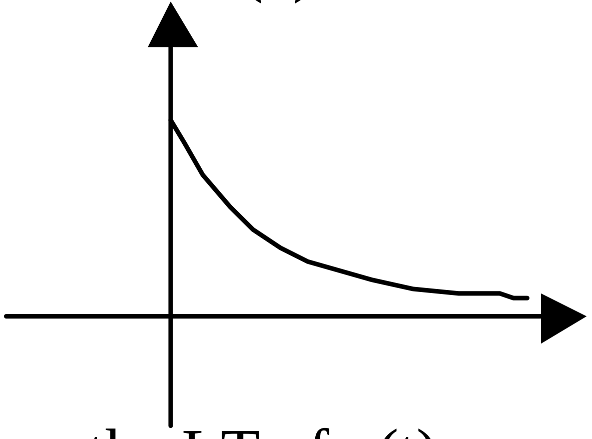
- Laplace transform can analyze unbounded signals or unstable systems.
- The unilateral Laplace transform can be used to analyze differential LTI systems with *nonzero initial conditions*.

In contrast, FT can only analyze bounded signals and zero-initial systems.

Example 1: LT of $e^{-at}u(t)$, $a>0$

Find the Laplace transform of $x(t) = e^{-at}u(t),\ a$ real .

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \frac{1}{s+a}, \qquad \mathrm{Re}\{s\} > -a$$

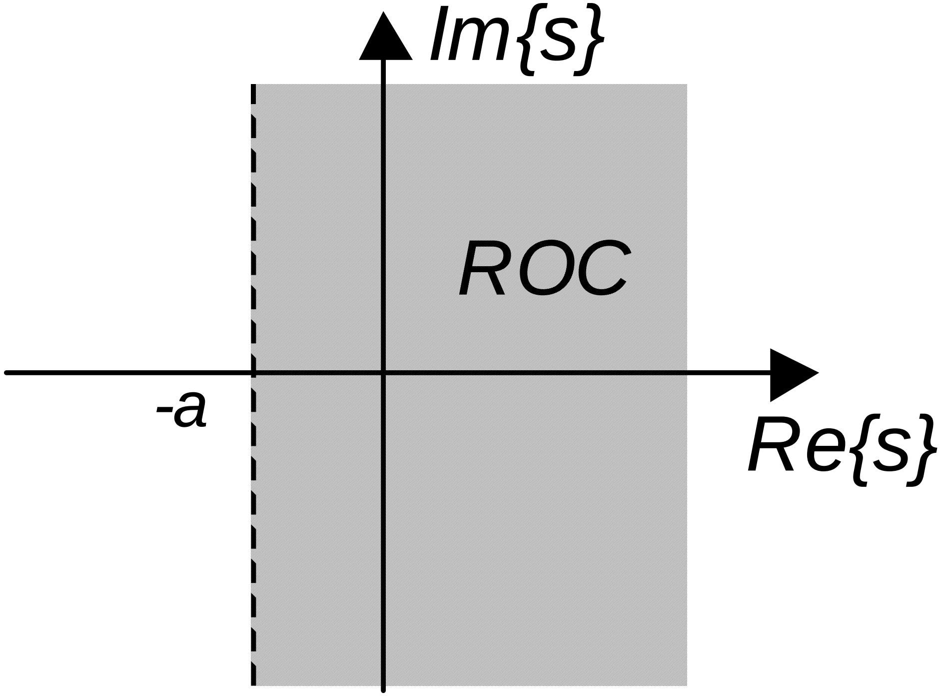


Note: for a=0, we have the LT of u(t).

Region of convergence of the LT

This Laplace transform converges only for values of s in the open half-plane to the right of $s = -a$.

This half plane is the region of convergence (ROC) of the Laplace transform. It is represented as follows:



Compare: ROC of FT and ROC of LT

Consider the signal $x(t) = e^{-at}u(t), \;\; a \in \mathbb{R}$

Its Fourier transform of converges only for $a > 0$ (decaying exponential).

Whereas, its Laplace transform converges for any a (even for growing exponentials!), as long as $\mathrm{Re}\{s\} > -a$.

In other words, the Fourier transform of

$$x(t)e^{-\sigma t} = e^{-(a+\sigma)t}u(t)$$

converges for the region where $\mathrm{Re}\{s\} = \sigma > -a$.

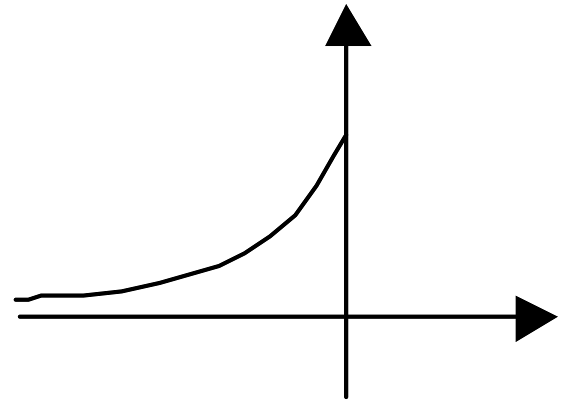
Example 2: LT of $e^{-at}u(-t)$, $a<0$

Find the Laplace transform of $x(t)=e^{-at}u(-t),\ a \text{ real}$.

$$X(s)=\int_{-\infty}^{+\infty} e^{-at}u(-t)e^{-st}dt$$

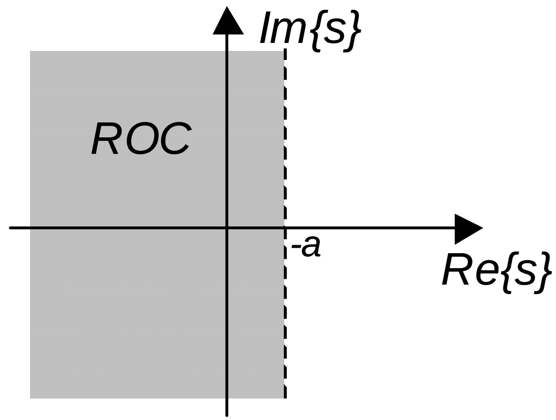
$$=\int_{-\infty}^{0} e^{-(s+a)t}dt$$

$$=-\frac{1}{s+a},\quad \text{Re}\{s\}<-a$$



Region of Convergence of LT

The Laplace transform of the signal in example 2 converges only in the ROC, which is the open half-plane to the left of $s = -a$.



<u>Important note:</u> *The ROC is an integral part of a Laplace transform. It must be specified.*

Without ROC, you can't tell what the corresponding time-domain signal is!

Inverse Laplace Transform

The inverse Laplace transform is in general given by

$$x(t) := \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

This integral is rarely used because we are mostly dealing with linear systems and standard signals whose Laplace transforms are found in tables of Laplace transform pairs.

We will mainly use the partial fraction expansion technique to find the continuous-time signal corresponding to a Laplace transform.

LT pairs

TABLE D.4 Laplace Transform Pairs

Time domain $x(t)$		**Laplace domain** $X(s)$	ROC
$x(t)=\frac{1}{j2\pi}\int_{\alpha-j\infty}^{\alpha+j\infty}X(s)e^{st}ds$	$\{s\in\mathbb{C}:\mathrm{Re}\{s\}=\alpha\}$ $\in ROC$	$X(s)=\int_{-\infty}^{+\infty}x(t)e^{-st}dt$	$s\in\mathrm{ROC}$
$\delta(t)$		1	$\forall s$
$u(t)$		$\frac{1}{s}$	$\mathrm{Re}\{s\}>0$
$tu(t)$		$\frac{1}{s^2}$	$\mathrm{Re}\{s\}>0$
$t^k u(t)$	$k=1,2,3,\ldots$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}\{s\}>0$
$e^{at}u(t)$	$a\in\mathbb{C}$	$\frac{1}{s-a}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{a\}$
$-e^{at}u(-t)$	$a\in\mathbb{C}$	$\frac{1}{s-a}$	$\mathrm{Re}\{s\}<\mathrm{Re}\{a\}$
$e^{-\alpha t}\sin(\omega_0 t)u(t)$	$\alpha,\omega_0\in\mathbb{R}$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-\alpha$
$e^{-\alpha t}\cos(\omega_0 t)u(t)$	$\alpha,\omega_0\in\mathbb{R}$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-\alpha$
$\sin(\omega_0 t)u(t)$	$\omega_0\in\mathbb{R}$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\cos(\omega_0 t)u(t)$	$\omega_0\in\mathbb{R}$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
$\begin{cases}1, & \lvert t\rvert<t_0\\ 0, & \lvert t\rvert>t_0\end{cases}$	$t_0\in\mathbb{R},\ t_0>0$	$\frac{e^{st_0}-e^{-st_0}}{s}$	$x(t)$
$te^{at}u(t)$	$a\in\mathbb{C}$	$\frac{1}{(s-a)^2}$	$\mathrm{Re}\{s\}>\mathrm{Re}\{a\}$

McGill University
Department of Electrical and Computer Engineering

Lecture 18

October 17, 2008

Hui Qun Deng, PhD

1. Regions of convergence of bilateral LT
2. Inverse LT using partial fraction technique
3. Properties of LT

Region of Convergence of the bilateral LT

Bilateral LT:
$$X(s) := \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad (6.1)$$

i.e.,
$$X(\sigma + j\omega) := \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt \qquad (6.2)$$

Convergence of the Laplace integral **depends on the value of σ,** i.e. $Re\{s\}$, for which the Fourier transform of the exponentially-weighted x(t) converge. In the s-plane, the *region of convergence* of *X(s)* is either:

- a left half-plane: $\sigma<\sigma_1$ (if x(t) is left sided)
- a right half-plane: $\sigma>\sigma_2$ (if x(t) is right sided)
- a vertical strip: $\sigma_2<\sigma<\sigma_1$ (if x(t) is two sided, for $-\infty<t<\infty$)
- entire s-plane: if x(t) is finite duration
- nothing

Laplace Transform and Rational Functions

Many LTs are rational functions (ratios of polynomials of *s*):

$$X(s)=P(s)/Q(s)$$

X(s) can be:

- The LT of sum of complex or real *exponential* signals;
- *H(s)*, the LT of *h(t)*, the impulse response of an LTI system.

The roots of *P(s)* are the zeros, and the roots of *Q(s)* are poles of X(s).

For differential LTI systems (see Eq. 3.2), the zeros of the characteristic polynomial are the poles of the *H(s)*.

To find the inverse LT of *X(s)*, we can express *X(s)* in terms of algebraic expressions listed in Table D4, and then find x(t) .

Case 1: rational *X(s)* has no multiple-order poles

Assume:

1. $X(s)$ has no multiple-order poles in its set of poles $\{p_k\}_{k=1}^{m}$.

2. The order of the denominator polynomial is greater than the order of the numerator polynomial.

Then, $X(s)$ can be expanded as a sum of partial fractions:

$$X(s)=\underbrace{\sum_{k=1}^{N}\frac{A_k}{s-p_k}}_{s\in ROC}=\underbrace{\frac{A_1}{s-p_1}}_{s\in ROC_1}+\underbrace{\frac{A_2}{s-p_2}}_{s\in ROC_2}+\cdots+\underbrace{\frac{A_N}{s-p_N}}_{s\in ROC_N}$$

From the *ROC* of $X(s)$, the ROC_i of each individual partial fraction can be found, and then the inverse transform of each of these terms can be determined using Table D4.

The Inverse LT and ROC

$$X(s)=\underbrace{\sum_{k=1}^{N}\frac{A_k}{s-p_k}}_{s\in ROC}=\underbrace{\frac{A_1}{s-p_1}}_{s\in ROC_1}+\underbrace{\frac{A_2}{s-p_2}}_{s\in ROC_2}+\cdots+\underbrace{\frac{A_N}{s-p_N}}_{s\in ROC_N}$$

The ROC of *X(s)* must contain at least the intersection of all the *ROCs* of the partial fractions $\mathrm{ROC}\supseteq\bigcap_{i=1,\ldots,N}\mathrm{ROC}_i$.

- If the *ROC*$_i$ *i=1,…,N*, are open right half-planes, then the ILT of *X(s)* is:

$$x(t)=A_1e^{p_1t}u(t)+A_2e^{p_2t}u(t)+\cdots+A_Ne^{p_Nt}u(t)$$

- If the *ROC*$_i$ *i=1,…,N*, are open left half-planes, then the ILT of *X(s)* is:

$$x(t)=-A_1e^{p_1t}u(-t)-A_2e^{p_2t}u(-t)-\ldots-A_Ne^{p_Nt}u(-t)$$

The coefficients of partial fractions for case 1

$$X(s)=\underbrace{\sum_{k=1}^{N}\frac{A_k}{s-p_k}}_{s\in ROC}=\underbrace{\frac{A_1}{s-p_1}}_{s\in ROC_1}+\underbrace{\frac{A_2}{s-p_2}}_{s\in ROC_2}+\cdots+\underbrace{\frac{A_N}{s-p_N}}_{s\in ROC_N}$$

$$A_k = X(s)(s-p_k)\Big|_{s=p_k}$$

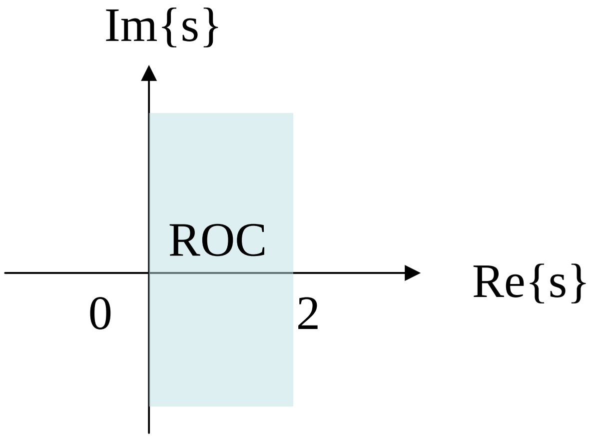
k=1, 2, …, N

Example

Compute the inverse of the following Laplace transform:

$$X(s) = \frac{s+3}{s(s+1)(s-2)}, \quad 0 < \mathrm{Re}\{s\} < 2$$

$$= \underbrace{\frac{A_1}{s+1}}_{s \in ROC_1} + \underbrace{\frac{A_2}{s}}_{s \in ROC_2} + \underbrace{\frac{A_3}{s-2}}_{s \in ROC_3}$$



In order to have $\mathrm{ROC} \supseteq \mathrm{ROC}_1 \cap \mathrm{ROC}_2 \cap \mathrm{ROC}_3$, the only possibility is:

$$X(s) = \underbrace{\frac{A_1}{s+1}}_{\mathrm{Re}\{s\} > -1} + \underbrace{\frac{A_2}{s}}_{\mathrm{Re}\{s\} > 0} + \underbrace{\frac{A_3}{s-2}}_{\mathrm{Re}\{s\} < 2}.$$

The coefficients of partial fractions

For $$X(s) = \frac{s+3}{s(s+1)(s-2)}, \quad 0 < \mathrm{Re}\{s\} < 2$$

$$= \underbrace{\frac{A_1}{s+1}}_{s \in ROC_1} + \underbrace{\frac{A_2}{s}}_{s \in ROC_2} + \underbrace{\frac{A_3}{s-2}}_{s \in ROC_3}$$

The values of A_i are:

$$A_1 = (s+1)\left(\frac{(s+3)}{s(s+1)(s-2)}\right)_{s=-1} = \frac{2}{(-1)(-3)} = \frac{2}{3}$$

$$A_2 = s\left(\frac{(s+3)}{s(s+1)(s-2)}\right)_{s=0} = \frac{3}{(1)(-2)} = -\frac{3}{2}$$

$$A_3 = (s-2)\left(\frac{(s+3)}{s(s+1)(s-2)}\right)_{s=2} = \frac{5}{(2)(3)} = \frac{5}{6}.$$

Hence, the Laplace transform can be expanded as

$$X(s) = \frac{2}{3}\underbrace{\frac{1}{s+1}}_{\text{Re}\{s\}>-1} - \frac{3}{2}\underbrace{\frac{1}{s}}_{\text{Re}\{s\}>0} + \frac{5}{6}\underbrace{\frac{1}{s-2}}_{\text{Re}\{s\}<2}$$

and from Table D.4 of Laplace transform pairs:

$$x(t) = \frac{2}{3}e^{-t}u(t) - \frac{3}{2}u(t) - \frac{5}{6}e^{2t}u(-t).$$

Case 2: *X(s)* has multiple-order poles

For multiple poles in X(s), the partial fraction expansion must contain fractions with all the powers of the multiple poles up to their multiplicity.

To illustrate this, consider *X(s)* with ROC $\text{Re}\{s\} > \text{Re}\{p_N\}$

$$X(s) = \frac{n(s)}{(s-p_1)\cdots(s-p_{m-1})(s-p_m)^r(s-p_{m+1})\cdots(s-p_N)},$$

$$\underbrace{X(s)}_{\text{Re}\{s\}>\text{Re}\{p_{m+1}\}} = \frac{A_1}{\underbrace{s-p_1}_{\text{Re}\{s\}>\text{Re}\{p_1\}}} + \cdots + \frac{A_m}{\underbrace{s-p_m}_{\text{Re}\{s\}>\text{Re}\{p_m\}}} + \frac{A_{m+1}}{\underbrace{(s-p_m)^2}_{\text{Re}\{s\}>\text{Re}\{p_m\}}} + \cdots + \frac{A_{m+r-1}}{\underbrace{(s-p_m)^r}_{\text{Re}\{s\}>\text{Re}\{p_m\}}}$$

$$+ \frac{A_{m+r}}{\underbrace{s-p_{m+1}}_{\text{Re}\{s\}>\text{Re}\{p_{m+1}\}}} + \cdots + \frac{A_N}{\underbrace{s-p_N}_{\text{Re}\{s\}>\text{Re}\{p_N\}}}$$

The coefficients of partial fractions for case 2

If p_m is a pole of multiplicity r, the coefficients $A_m, \ldots, A_{m+r-1}$ are computed as follows:

$$A_{m+r-i} = \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[(s - p_m)^r X(s) \right]_{s=p_m}, \, i = 1, \ldots, r \, ,$$

To compute the coefficient of the term with the highest power of the repeated pole:

$$A_{m+r-1} = \left[(s - p_m)^r X(s) \right]_{s=p_m} .$$

If X(s) has complex conjugate poles

If *X(s)* has a pair of complex conjugate poles, we can include a second-order term $\frac{A\omega_0 + B(s+\alpha)}{(s+\alpha)^2 + \omega_0{}^2}$ in the partial fraction expansion.

The idea is to use the damped or growing sinusoids in the table of Laplace transforms, such as

$$e^{-\alpha t}\sin(\omega_0 t) \overset{L}{\leftrightarrow} \frac{\omega_0}{(s+\alpha)^2 + \omega_0{}^2}, \ \ \mathrm{Re}\{s\} > -\alpha$$

$$e^{-\alpha t}\cos(\omega_0 t) \overset{L}{\leftrightarrow} \frac{s+\alpha}{(s+\alpha)^2 + \omega_0{}^2}, \ \ \mathrm{Re}\{s\} > -\alpha$$

Example

Compute the inverse of the following Laplace transform:

$$X(s) = \frac{2s^2 + 3s - 2}{\left(s^2 + 2s + 4\right)s}, \quad \text{Re}\{s\} > 0 \quad .$$

Note that $s^2 + 2s + 4 = (s+1)^2 + (\sqrt{3})^2$, so that the complex poles are $p_1 = -1 + j\sqrt{3}$ and $p_1 = -1 - j\sqrt{3}$

The transform $X(s)$ can be expanded as follows:

$$X(s)=\frac{2s^2+3s-2}{\left(s^2+2s+4\right)s}=\underbrace{\frac{A\sqrt{3}+B(s+1)}{\left(s+1\right)^2+3}}_{\mathrm{Re}\{s\}>-1}+\underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0}.$$

Coefficient *c* is obtained with the partial fraction technique: c= -1/2.

Now, let *s*=-1 to compute $\frac{-3}{-3}=\frac{1}{\sqrt{3}}A+\frac{1}{2}\Rightarrow A=\frac{\sqrt{3}}{2}$,

Then multiply both sides by *s* and let $s\rightarrow\infty$ and get $B=5/2$.

Then we have the following expansion:

$$X(s) = \underbrace{\frac{\frac{\sqrt{3}}{2}(\sqrt{3}) + \frac{5}{2}(s+1)}{(s+1)^2 + 3}}_{\mathrm{Re}\{s\}>-1} - \underbrace{\frac{1/2}{s}}_{\mathrm{Re}\{s\}>0}$$

$$= \underbrace{\frac{\frac{\sqrt{3}}{2}(\sqrt{3})}{(s+1)^2 + 3}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{\frac{5}{2}(s+1)}{(s+1)^2 + 3}}_{\mathrm{Re}\{s\}>-1} - \underbrace{\frac{1/2}{s}}_{\mathrm{Re}\{s\}>0}$$

Taking the inverse Laplace transform using Table D.4

$$x(t) = \left[\frac{\sqrt{3}}{2}e^{-t}\sin(\sqrt{3}t) + \frac{5}{2}e^{-t}\cos(\sqrt{3}t)\right]u(t) - \frac{1}{2}u(t).$$

Example

Consider the signal $x(t) = e^{-2t}u(t) + e^{t}u(-t)$. Its Laplace transform is given by

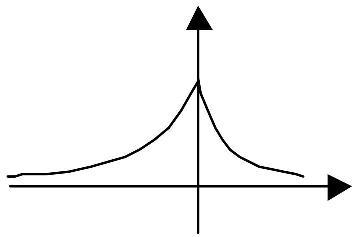
$$X(s) = \int_{-\infty}^{+\infty} e^{-2t}u(t)e^{-st}dt + \int_{-\infty}^{+\infty} e^{t}u(-t)e^{-st}dt$$

$$= \int_{0}^{+\infty} e^{-(s+2)t}dt + \int_{-\infty}^{0} e^{-(s-1)t}dt$$

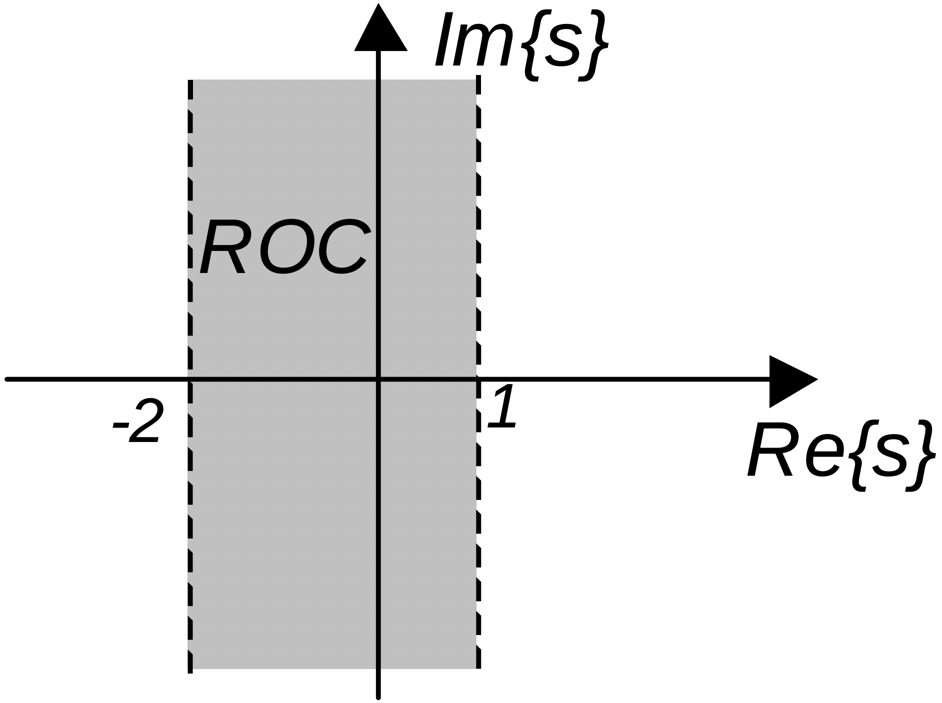
$$= \frac{1}{s+2} - \frac{1}{s-1}, \quad \mathrm{Re}\{s\} > -2 \text{ and } \mathrm{Re}\{s\} < 1$$

$$= \frac{-3}{s^2+s-2}, \quad -2 < \mathrm{Re}\{s\} < 1$$

$x(t)$



The ROC is a vertical strip between the real parts -2 and 1.



Properties of the bilateral LT: Linearity

If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s), \;\; s \in \mathrm{ROC}_1$ and $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s), \;\; s \in \mathrm{ROC}_2$,

then

$$ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\leftrightarrow} aX_1(s) + bX_2(s), \;\; s \in \mathrm{ROC} \supseteq \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

Time-Shifting

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s), \quad s \in \text{ROC}$,

then

$$x(t - t_0) \overset{\mathcal{L}}{\leftrightarrow} e^{-st_0} X(s), \quad s \in \text{ROC}$$

Example:

The impulse response of a zero-order hold is a unit pulse of duration T: $h_0(t) = u(t) - u(t - T)$. Its LT is:

$$H_0(s) = \frac{1}{s} - e^{-sT}\frac{1}{s} = \frac{1 - e^{-sT}}{s}, \quad \forall s \in \mathbb{C}$$

Note: the ROC is the whole complex s-plane. There is no pole at s=0.

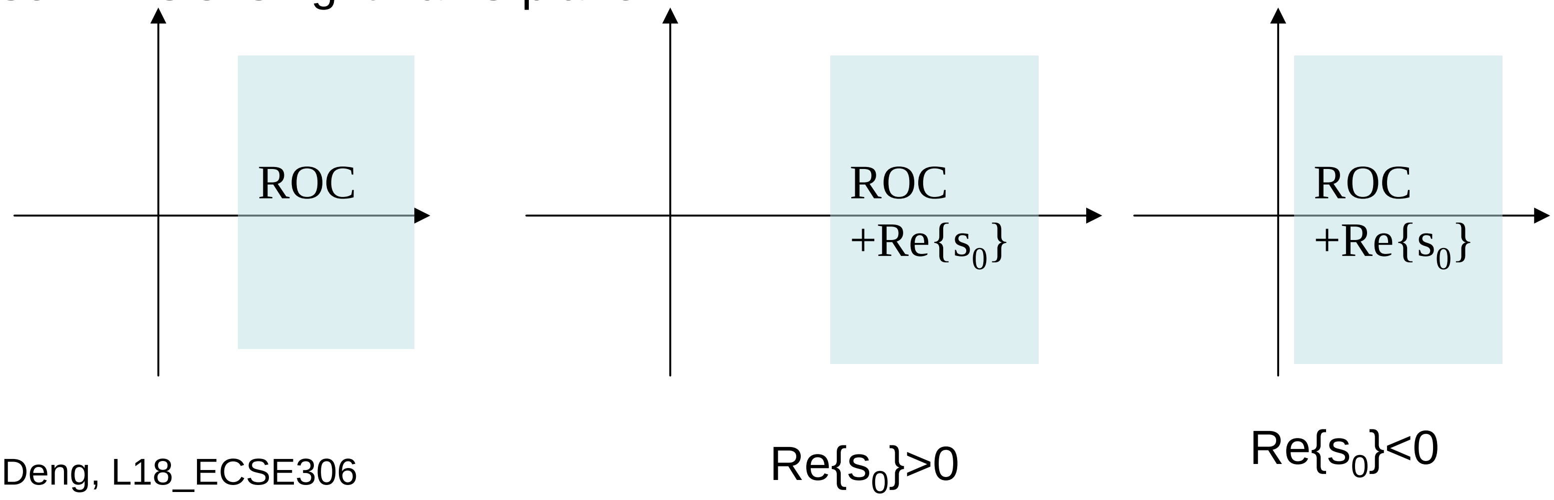
Shifting in the s-Domain

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s), \quad s \in \mathrm{ROC}$, then

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s - s_0), \quad s \in \mathrm{ROC} + \mathrm{Re}\{s_0\}$$

where the new ROC is the original one shifted by $\mathrm{Re}\{s_0\}$, to the right if this number is positive, to the left otherwise.

Case 1: ROC is right half s-plane



Time-Scaling

If $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad s \in \text{ROC}$, then

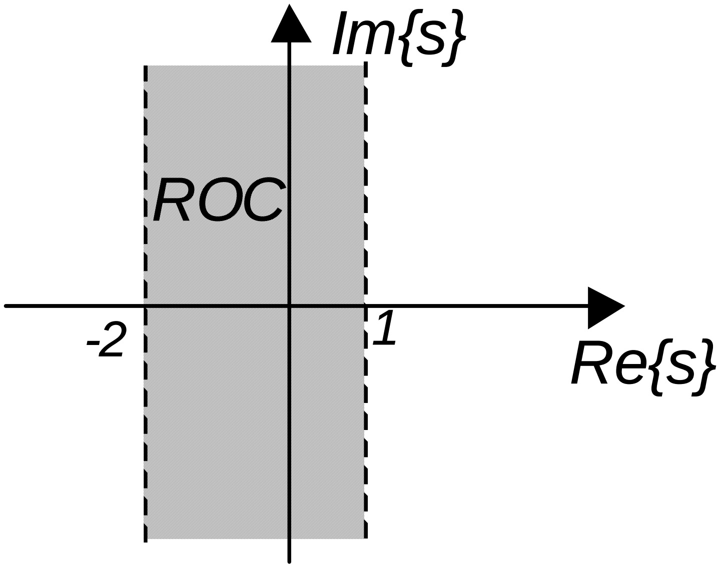
$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X(\tfrac{s}{a}), \quad s \in a\text{ROC}$$

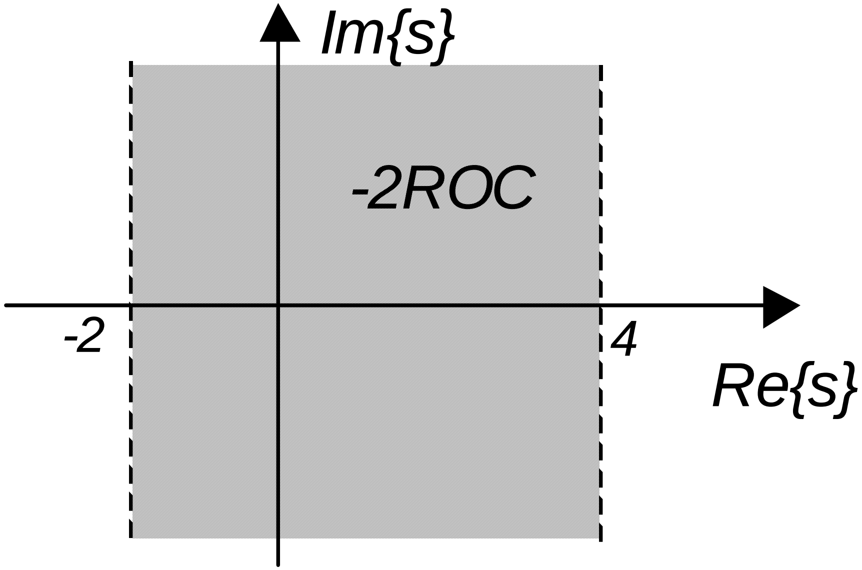
where the aROC is expanded or contracted original ROC.

If a<0, ROC flips around the imaginary axis.

Example:

$$x(-2t) \overset{\mathcal{L}}{\longleftrightarrow} 0.5X(-0.5s), \quad s \in -2\text{ROC}$$





Conjugation

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s), \;\; s \in \text{ROC}$, then

$$x^*(t) \overset{\mathcal{L}}{\leftrightarrow} X^*(s^*), \;\; s \in \text{ROC},$$

Therefore, for $x(t)$ real, $X(s) = X^*(s^*)$.

Important consequence: If *x(t)* is real and if *X(s)* has a pole (or zero) at $s = s_0$, then $X(s)$ has also a pole (or zero) at the complex conjugate point $s = s_0^*$. Thus, *the complex poles or zeros of the Laplace transform of a real signal are conjugate pairs.*

Convolution

If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s),\ \ s \in \mathrm{ROC}_1$ and $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s),\ \ s \in \mathrm{ROC}_2$, then

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s) X_2(s),\ \ s \in \mathrm{ROC} \supseteq \mathrm{ROC}_1 \cap \mathrm{ROC}_2 .$$

Note: the new ROC contains the intersection of the two original ROC's, and may be larger, e.g., when a pole-zero cancellation occurs.

Example: The response of the LTI system with $h(t)=\left[e^{-2t}+e^{-t}\right]u(t)$ to the input $x(t)=-e^{-2t}u(t)+\delta(t)$ is given by the inverse Laplace transform of $Y(s)$:

$$h(t)\overset{\mathcal{L}}{\leftrightarrow}H(s)=\frac{2s+3}{(s+2)(s+1)},\ \ \mathrm{Re}\{s\}>-1$$

$$x(t)\overset{\mathcal{L}}{\leftrightarrow}X(s)=\frac{-1}{s+2}+1=\frac{s+1}{s+2},\ \ \mathrm{Re}\{s\}>-2$$

$$Y(s) = H(s)X(s) = \frac{(2s+3)}{(s+2)(s+1)} \frac{(s+1)}{(s+2)},$$

$$\{s : \mathrm{Re}\{s\} > -2\} \cap \{s : \mathrm{Re}\{s\} > -1\} = \mathrm{Re}\{s\} > -1$$

$$= \frac{(2s+3)}{(s+2)^2}, \ \ \mathrm{Re}\{s\} > -2$$

Expanding this transform into partial fractions, we get

$$Y(s) = \frac{(2s+3)}{(s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2}, \ \ \mathrm{Re}\{s\} > -2 \, .$$

We find the factor B first,

$$\left.\frac{(2s+3)}{1}\right|_{s=-2} = -1 = B \, ,$$

and factor A is given by

$$\left.\frac{2s+3}{s+2}\right|_{s=+\infty} = 2 = A \, .$$

Therefore, using Table D.4 of Laplace transform pairs in the textbook, we obtain

$$y(t) = \left[2e^{-2t} - te^{-2t}\right]u(t)$$

Differentiation in the time domain

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s), \;\; s \in \mathrm{ROC}$, then

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\leftrightarrow} sX(s), \;\; s \in \mathrm{ROC}_1 \supseteq \mathrm{ROC} ,$$

ROC_1 may be larger than the ROC when there is a pole-zero cancellation at $s = 0$.

Differentiation in the Frequency Domain

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s),\ \ s \in \mathrm{ROC}$, then

$$-tx(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{dX(s)}{ds},\ \ s \in \mathrm{ROC} \text{ ,}$$

This property is useful to obtain the Laplace transform of signals of the form $x(t) = te^{-at}u(t)$.

Integration in the Time Domain

If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s), \quad s \in \mathrm{ROC}$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s} X(s), \quad s \in \mathrm{ROC}_1 \supseteq \mathrm{ROC} \cap \left\{ s : \mathrm{Re}\{s\} > 0 \right\}$$

Prove by yourself: view the running integral as u(t)*x(t) and then apply the convolution property of LT.

McGill University
Department of Electrical and Computer Engineering

Lecture 19

October 20, 2008

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1. Stability and causality of LTI systems
2. Unilateral Laplace Transform
3. Initial and final value theorems

The transfer function of an LTI system

The Laplace transform of the output of an LTI system with impulse response *h(t)* is

$$Y(s) = H(s)X(s), \quad \mathrm{ROC_Y} \supseteq \mathrm{ROC_H} \cap \mathrm{ROC_X}$$

The Laplace transform of the impulse response of an LTI system is called the *system function* or *transfer function*.

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

Note:

1. H(s) can also be obtained from the ratio of the LT of the output signal *y(t)* and the LT of the input signal *x(t):*

$$H(s) = \frac{Y(s)}{X(s)}$$

2. The frequency response of the system can be obtained from the H(s): $H(j\omega) = H(s)\big|_{s=j\omega}$.

Causality and ROC

The ROC associated with the transfer function of a causal system is a right half-plane.

Because: for $h(t)e^{-st}$ to be integrable, Re{st}>σ must be satisfied. As h(t) exists only for t>=0 for a causal system, then Re{st}>σ means Re{s}>σ, which defines a right half-plane.

Note:

1. A right half-plane ROC may not imply a causal system. For example, a signal starting at t= -10 also leads to a ROC that is a right half-plane.
2. If the transfer function is a **rational function**, and if the ROC is the right half-plane to the right of the rightmost pole in the s-plane, then the (impulse response of the) system is causal, as given by the partial fraction method of inverse LT.

Examples of ROC and causality

$H(s) = \dfrac{1}{s+1}, \text{Re}\{s\} > -1$ corresponds to a causal system e^{-t} u(t).

$H_1(s) = \dfrac{e^{s}}{s+1}, \text{Re}\{s\} > -1$ is noncausal $e^{-(t+1)}$ u(t+1).

$H_2(s) = \dfrac{e^{-s}}{s+1}, \text{Re}\{s\} > -1$ is causal $e^{-(t-1)}$ u(t-1).

Stability and ROC

Recall: The sufficient and necessary condition for a continuous-time LTI system to be BIBO stable: its impulse response is absolutely integrable, which means its Fourier transform exists.

This condition means the following:

An LTI system is stable if and only if the ***ROC*** *of its transfer function H(s)* ***contains the jω-axis***.

Example of ROC and stability

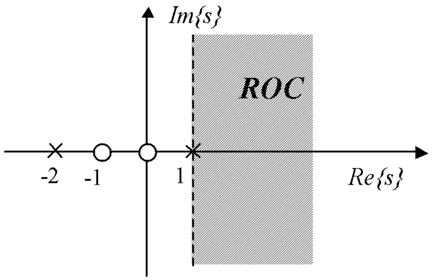
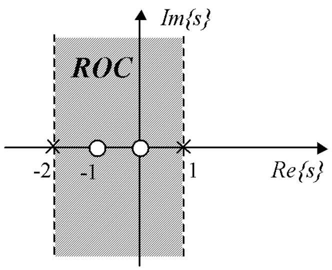
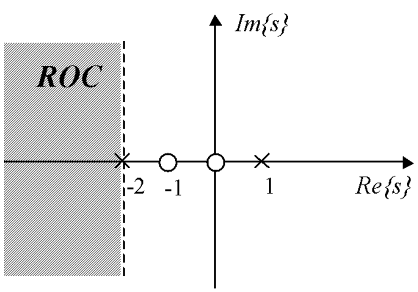
Consider an LTI system with a proper rational transfer function:

$$H(s) = \frac{s(s+1)}{(s+2)(s-1)}.$$

The transfer function can be expanded as

$$H(s) = \frac{s(s+1)}{(s+2)(s-1)} = 1 + \frac{2}{3}\frac{1}{s-1} - \frac{2}{3}\frac{1}{s+2}.$$

3 possible ROCs could be associated with this transfer function. Only one ROC leads to a stable system.

ROC	$h(t)$	**Causal**	**Stable**
	$h(t)=\left[\frac{2}{3}e^{t}-\frac{2}{3}e^{-2t}\right]u(t)+\delta(t)$	YES. ROC is a right half-plane	NO.
	$h(t)=-\frac{2}{3}e^{t}u(-t)-\frac{2}{3}e^{-2t}u(t)$ $+\delta(t)$	NO.	YES. *The* $j\omega$ *-axis lies in the ROC*
	$h(t)=\left[-\frac{2}{3}e^{t}+\frac{2}{3}e^{-2t}\right]u(-t)$ $+\delta(t)$	NO.	NO.

Unilateral Laplace transform

The *one-sided* or *unilateral* Laplace transform of *x(t)* is defined as follows:

$$\mathcal{X}(s) := \int_{0^-}^{\infty} x(t)e^{-st}dt$$

This transform considers only signals from ***t>=0⁻***.

The notation for the unilateral Laplace transform:

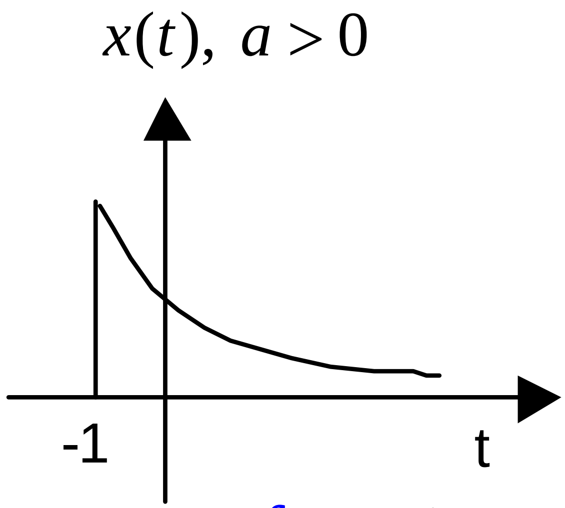
$$x(t) \overset{\mathcal{UL}}{\leftrightarrow} \mathcal{X}(s) = \mathcal{UL}\{x(t)\}$$

Note

- Two signals that are different for $t<0$ but equal for $t\geq0$ have the same unilateral Laplace transform.
- The unilateral Laplace transform of *x(t)* is identical to the (two-sided) Laplace transform of x(t)u(t).
- The ROC of a unilateral Laplace transform is always an open RHP, or the entire *s*-plane.

Compare unilateral LT and bilateral LT

Consider the signal $x(t) = e^{-a(t+1)}u(t+1)$



Its bilateral Laplace transform is

$$X(s) = \frac{e^{s}}{s+a}, \quad \text{Re}\{s\} > -a$$

In contrast, its unilateral Laplace transform is

$$\mathcal{X}(s) = \frac{e^{-a}}{s+a}, \quad \text{Re}\{s\} > -a \, .$$

Properties of the unilateral Laplace Transform

Linearity

The unilateral Laplace transform is linear. If

$$x_1(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_1(s),\ \ s \in \mathrm{ROC}_1 \ \text{and} \ x_2(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_2(s),\ \ s \in \mathrm{ROC}_2 ,$$

then

$$ax_1(t) + bx_2(t) \overset{\mathcal{UL}}{\longleftrightarrow} a\mathcal{X}_1(s) + b\mathcal{X}_2(s),\ \ s \in \mathrm{ROC} \supseteq \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

Time Delay

x(t) is causal, if *x(t)*=0 for *t*<0.

For causal *x(t)*, if $x(t) \overset{\mathcal{UL}}{\leftrightarrow} \mathcal{X}(s), \ s \in \text{ROC}$, then for a time delay of $t_0>0$

$$x(t-t_0) \overset{\mathcal{UL}}{\leftrightarrow} e^{-st_0}\mathcal{X}(s), \ s \in \text{ROC}, t_0 > 0$$

Note:

In the following cases, the resulting and the original unilateral transforms can’t have the above relationship.

1. *x(t)* is nonzero at negative times: a time delay can make a "previously unknown" part of the signal "appear" at positive times.

2. Part of *x(t)* is shifted to negative times for time advance.

Shifting in the s-domain

If $x(t) \overset{\mathcal{UL}}{\leftrightarrow} \mathcal{X}(s), \; s \in \mathrm{ROC}$, then

$$e^{s_0 t} x(t) \overset{\mathcal{UL}}{\leftrightarrow} \mathcal{X}(s - s_0), \; s \in \mathrm{ROC} + \mathrm{Re}\{s_0\},$$

where the new ROC is the original one shifted by $\mathrm{Re}\{s_0\}$, to the right if this number is positive, to the left otherwise.

Time-Scaling

If $x(t) \overset{\mathcal{UL}}{\leftrightarrow} \mathcal{X}(s),\ s \in \mathrm{ROC}$, then for $\alpha > 0$

$$x(\alpha t) \overset{\mathrm{UL}}{\leftrightarrow} \frac{1}{\alpha} \mathrm{X}\left(\tfrac{s}{\alpha}\right),\ s \in \alpha \mathrm{ROC} \qquad (6.64)$$

Conjugation

If $x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s), \ s \in \text{ROC}$, then $x^*(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}^*(s^*), \ s \in \text{ROC}$.

Note:

1. If *x(t)* is real, then $\mathcal{X}(s) = \mathcal{X}^*(s^*)$
2. If *x(t)* is real, then the complex poles and zeros of its unilateral Laplace transform are conjugate pairs.

Proof : Let s_1 be a complex zero of *X(s)*.

$$\because X(s_1) = 0, \quad and \because \overset{*}{X}(\overset{*}{s_1}) = X(s_1)$$

$$\therefore \overset{*}{X}(\overset{*}{s_1}) = 0, \qquad \therefore X(\overset{*}{s_1}) = 0$$

Similar proof holds for complex poles.

Convolution Property

Assume that $x_1(t) = x_2(t) = 0 \text{ for } t < 0$. If

$x_1(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_1(s),\ s \in \mathrm{ROC}_1$ and $x_2(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_2(s),\ s \in \mathrm{ROC}_2$, then

$$x_1(t) * x_2(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_1(s)\mathcal{X}_2(s),\ s \in \mathrm{ROC} \supseteq \mathrm{ROC}_1 \cap \mathrm{ROC}_2$$

This is an extremely useful property for *causal* LTI system analysis with signals that are zero for negative times.

Differentiation in the Time Domain

If $x(t) \overset{\mathcal{UL}}{\leftrightarrow} \mathcal{X}(s),\ s \in \text{ROC}$, then

$$\frac{dx(t)}{dt} \overset{\mathcal{UL}}{\leftrightarrow} s\mathcal{X}(s) - x(0^-),\ s \in \text{ROC}_1 \supseteq \text{ROC}$$

Proof:

$$\int_{0^-}^{\infty} x'(t)e^{-st}dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}dt = s\mathcal{X}(s) - x(0^-)$$

Note:

1. This property is different from that of bilateral LT in Eq. (6.57).
2. $x(0^-)$ =0, if $x(t)$=0 for t<0.
3. $x(0^-) \neq 0$, if $x(t)$ extends to negative times.
4. $x(0^-)$ can be used to set a non-zero initial condition for the output of a causal differential system.

Example

Calculate the output of the following homogeneous causal LTI differential system with initial condition $y(0^-)$.

$$\tau_0 \frac{dy(t)}{dt} + y(t) = 0$$

Let's take the unilateral Laplace transform on both sides:

$$\tau_0 \left[s\mathcal{Y}(s) - y(0^-) \right] + \mathcal{Y}(s) = 0 \, .$$

Solving for $\mathcal{Y}(s)$, we obtain

$$\mathcal{Y}(s) = \frac{y(0^-)}{s + \frac{1}{\tau_0}}, \ \ \mathrm{Re}\{s\} > -\frac{1}{\tau_0}$$

which corresponds to the time-domain output signal (Table D.4)

$$y(t) = y(0^-)e^{-\frac{t}{\tau_0}}u(t) .$$

ULT of Nth-order derivative in the time domain

$$\frac{d^n x(t)}{dt^n} \overset{\mathcal{UL}}{\leftrightarrow} s^n \mathcal{X}(s) - s^{n-1}x(0^-) - \ldots - s\frac{d^{n-2}x(0^-)}{dt^{n-2}} - \frac{d^{n-1}x(0^-)}{dt^{n-1}},$$

$$s \in \mathrm{ROC}_1 \supseteq \mathrm{ROC}$$

This can be derived by successive applications of the differentiation property.

Differentiation in the Frequency Domain

$$-tx(t) \overset{\mathcal{UL}}{\longleftrightarrow} \frac{d\mathcal{X}(s)}{ds}, \;\; s \in \text{ROC}$$

Integration in the Time Domain

$$\int_0^t x(\tau)d\tau \overset{u\mathcal{L}}{\leftrightarrow} \frac{1}{s}\mathcal{X}(s), \quad s \in ROC_1 \supseteq ROC \cap \{s : \mathrm{Re}\{s\} > 0\}$$

This can be proved by using u(t)*x(t) and the convolution property.

The Initial and Final Value Theorems

The *initial-value theorem:*

$$x(0^+) = \lim_{s \to +\infty} s\mathcal{X}(s)$$

The *final-value theorem*:

$$\lim_{s \to +\infty} x(t) = \lim_{s \to 0} s\mathcal{X}(s) \;.$$

The above are the properties of unilateral Laplace transform, as they are applicable only to signals that are 0 for t<0.

Proofs of initial and final value theorems

According to the property of differentiation in the time domain:

$$s\mathcal{X}(s) - x(0^-) = \int_{0^-}^{\infty} x'(t)e^{-st}dt = \int_{0^-}^{0^+} x'(t)e^{-st}dt + \int_{0^+}^{\infty} x'(t)e^{-st}dt$$

$$= x(0^+) - x(0^-) + \int_{0^+}^{\infty} x'(t)e^{-st}dt$$

$$\therefore s\mathcal{X}(s) = x(0^+) + \int_{0^+}^{\infty} x'(t)e^{-st}dt$$

$$\because \lim_{s\to\infty}\{\int_{0^+}^{\infty} x'(t)e^{-st}dt\} = \int_{0^+}^{\infty} x'(t)[\lim_{s\to\infty} e^{-st}]dt = 0$$

$$\therefore \lim_{s\to\infty}\{s\mathcal{X}(s)\} = x(0^+) \qquad \textit{initial value} \quad \textit{theorem}$$

$$\because \lim_{s\to 0}\{\int_{0^+}^{\infty} x'(t)e^{-st}dt\} = \lim_{t\to\infty} x(t) - x(0^+)$$

$$\therefore \lim_{s\to 0}\{s\mathcal{X}(s)\} = \lim_{t\to\infty} x(t) \qquad \textit{Final value} \quad \textit{theorem}$$

Example

Find the initial value $x(0^+)$ of the signal whose unilateral Laplace transform is

$$\mathcal{X}(s) = \frac{10}{s-3}, \;\; \text{Re}\{s\} > 3 \; .$$

Answer:

$$x(0^+) = \lim_{s \to +\infty} s\mathcal{X}(s) = \lim_{s \to +\infty} s \frac{10}{s-3} = 10$$

McGill University
Department of Electrical and Computer Engineering

Lecture 20

October 22, 2008

Hui Qun Deng, PhD

Analysis of LTI Differential Systems Using Laplace Transform

- Transfer functions of differential systems
- Causality and stability of differential systems
- The response of differential system with non-zero initial conditions

Transform Transfer Function of an LTI Differential System

Recall: the transfer function of an LTI system is the Laplace transform of its impulse response.

Given a *differential* LTI system $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$,

we apply the differentiation and linearity properties of the Laplace transform to the LHS and RHS of the above Eq.:

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

and obtain the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Poles and zeros of $H(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

- The poles of H(s) are the N zeros of the characteristic polynomial, i.e., the denominator polynomial.
- The zeros of H(s) are the M roots of numerator polynomial
- If M>N, then $\lim_{s\to\infty} H(s) = \infty$ and the transfer function is sometimes said to have *M-N* poles at ∞.
- If *M*<*N*, then $\lim_{s\to\infty} H(s) = 0$ and the transfer function is sometimes said to have $N - M$ zeros at ∞.
- If *M*=*N*?

ROC of *H(s)* and h(t)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} .$$

- If the ROC is unknown, then there may be many different impulse responses h(t) for the differential equation.
- If the differential system is causal, i.e., *h(t)* is causal, then the ROC is the right half-plane to the right of the rightmost pole in the s-plane.

The causality and the ROC of an LTI differential system

For a rational LT, its inverse LT is causal if and only if its ROC is a right half-plane (recall inverse LT using partial fractions).

Differential LTI systems have rational transfer functions. Thus, an *LTI differential system is causal if and only if the ROC of its transfer function is an open right half-plane located to the right of the rightmost pole.*

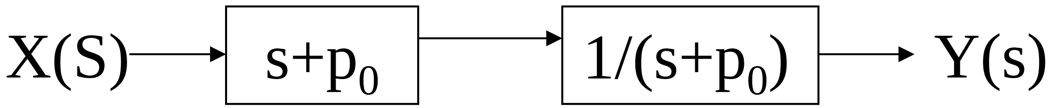
The stability of and LTI differential system

- Recall that an LTI system (including a *differential* LTI system) is stable if and only if the ROC of its transfer function includes the $j\omega$-axis.

Thus, a *causal LTI differential system is stable if and only if all the poles of its transfer function lie in the left side of the $j\omega$-axis.*

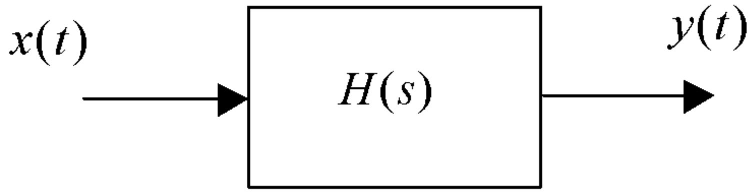
If a zero cancels out a pole

For the case where a zero of the input cancels out an unstable pole (call it p_0, $Re(p_0)>0$) in the transfer function, the corresponding differential LTI system is still considered to be unstable.



- The reason is that any nonzero initial condition would cause the output to either grow unbounded (if $\mathrm{Re}\{p_0\}>0$), oscillate forever (if p_0 is pure imaginary), or settle down to a nonzero value (if $p_0=0$).

Example 1: System Identification



Given that the input of a differential LTI system is $x(t) = 2e^{-t}u(t)$, and that the output was measured to be $y(t) = e^{-2t}\sin(2t)u(t) - te^{-t}u(t)$.

Find the transfer function *H(s)* of the system and its ROC, and determine whether the system is causal and stable,

This is a *system identification* problem, studied here in its simplest, noise-free form.

The LTs of input and output signals

$$X(s) = \frac{2}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

$$Y(s) = \underbrace{\frac{2}{(s+2)^2 + 2^2}}_{\mathrm{Re}\{s\}>-2} - \underbrace{\frac{1}{(s+1)^2}}_{\mathrm{Re}\{s\}>-1}$$

$$= \frac{(2s^2 + 4s + 2) - (s^2 + 4s + 8)}{(s^2 + 4s + 8)(s+1)^2}, \quad \mathrm{Re}\{s\} > -1$$

$$= \frac{s^2 - 6}{(s^2 + 4s + 8)(s+1)^2}, \quad \mathrm{Re}\{s\} > -1$$

The transfer function of the LTI system

Then, the transfer function is simply

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{s^2 - 6}{(s^2 + 4s + 8)(s+1)^2}}{\dfrac{2}{(s+1)}} = \frac{(s-\sqrt{6})(s+\sqrt{6})}{2(s^2+4s+8)(s+1)}$$

To determine the ROC, first note that the ROC of $Y(s)$ should contain the intersection of the ROCs of $H(s)$ and $X(s)$.

The ROC of H(s)

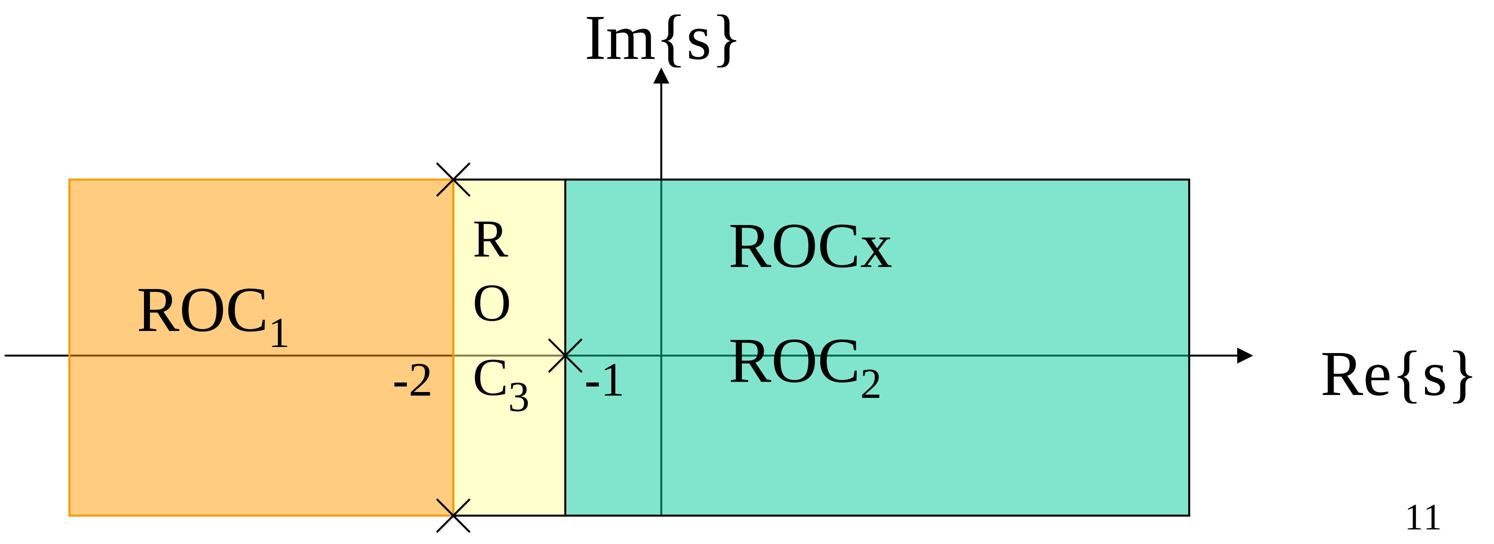
There are 3 possible ROCs for H(s):

ROC_1: an open left half-plane to the left of Re(s)=-2;

ROC_2: an open right half-plane to the right of Re(s)=-1;

ROC_3: a vertical strip between Re(s)= -2 and Re(s)=-1.

Since the ROCs of X(s) and Y(s) are right half-planes, the only possible ROC for H(s) is ROC_2. The system is stable and causal.



Example 2

Suppose we know that the input of a differential LTI system is

$$x(t) = e^{-3t}u(t)$$,

and the output is

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

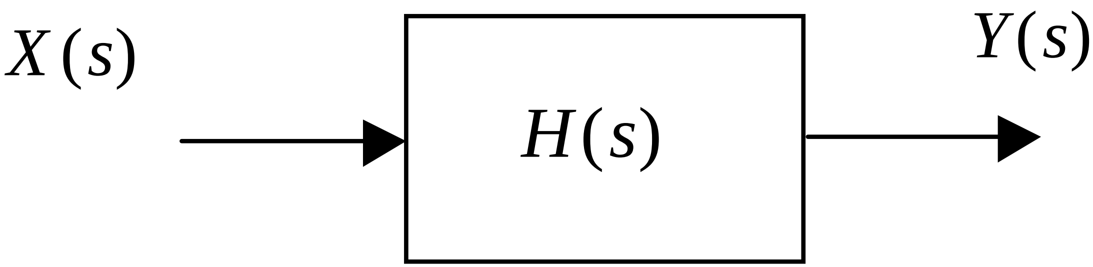
We can deduce the transfer function as follows. First take the Laplace transforms of the input and output signals:

$$X(s) = \frac{1}{s+3}, \quad \mathrm{Re}\{s\} > -3$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1.$$

Then, the transfer function is simply

$$H(s)=\frac{Y(s)}{X(s)}=\frac{\frac{1}{(s+1)(s+2)}}{\frac{1}{s+3}}=\frac{s+3}{(s+1)(s+2)}=\frac{s+3}{s^2+3s+2}$$



To determine the ROC, first note that the ROC of *Y(s)* should contain the intersection of the ROC's of *H(s)* and *X(s)*.

There are three possible ROC's for $H(s)$:

(a) an open left half-plane to the left of $s = -2$,

(b) a strip between $s = -2$ and $s = -1$, and

(c) an open right half-plane to the right of $s = -1$.

But since the ROC of Y(S) is an open right half-planes, the only possible choice is (c).

Hence, the ROC of *H(s)* is Re{s}>-1, and it follows that the LTI system is causal and stable.

Remark:

It is customary to refer to the set $\{s : \mathrm{Re}\{s\} \geq 0\}$ as *the right half-plane* (or to $\{s : \mathrm{Re}\{s\} > 0\}$ as *the open right half-plane*)

and to

$\{s : \mathrm{Re}\{s\} \leq 0\}$ as *the left half-plane* (or to $\{s : \mathrm{Re}\{s\} < 0\}$ as *the open left half-plane*.)

In Boulet's book, the "right half-plane" may contain $\{s: \sigma_0 < \mathrm{Re}\{s\} < 0\}$, and the "left half-plane" may contain $\{s: 0 < \mathrm{Re}\{s\} < \sigma_0\}$.

Analysis of LTI Differential Systems With Non-zero Initial Conditions Using the Unilateral Laplace Transform

Recall the differentiation property of the unilateral Laplace transform:

If $x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s),\ s \in \text{ROC}$, then

$$\frac{dx(t)}{dt} \overset{\mathcal{UL}}{\longleftrightarrow} s\mathcal{X}(s) - x(0^-),\ s \in \text{ROC}_1 \supseteq \text{ROC}.$$

$$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n \mathcal{X}(s) - s^{n-1} x(0^-) - \ldots - s\frac{d^{n-2} x(0^-)}{dt^{n-2}} - \frac{d^{n-1} x(0^-)}{dt^{n-1}}$$

Example 3

Consider the system described by

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+3x(t)$$

and with initial conditions $\frac{dy(0^-)}{dt}=2, \quad y(0^-)=1$. Suppose that this system is subjected to the input signal

$$x(t)=e^{-5t}u(t)\ .$$

What is the output of the system?

Take the unilateral Laplace transform

$$\left[s^2\mathcal{Y}(s) - sy(0^-) - \frac{dy(0^-)}{dt} \right] + 3\left[s\mathcal{Y}(s) - y(0^-) \right] + 2\mathcal{Y}(s)$$

$$= s\mathcal{X}(s) - x(0^-) + 3\mathcal{X}(s)$$

Note that $x(0^-) = 0$, $\mathcal{X}(s) = \dfrac{1}{s+5}, \quad \operatorname{Re}\{s\} > -5$, then

$$\mathcal{Y}(s) = \frac{s+3}{\left(s^2+3s+2\right)\left(s+5\right)} + \frac{s+5}{s^2+3s+2}, \quad \operatorname{Re}\{s\} > -1$$

$$= \frac{s^2+11s+28}{\left(s^2+3s+2\right)\left(s+5\right)}, \quad \operatorname{Re}\{s\} > -1$$

$$= \frac{\frac{9}{2}}{s+1} - \frac{\frac{10}{3}}{s+2} - \frac{\frac{1}{6}}{s+5}$$

Taking the inverse Laplace transform of each term, we obtain

$$y(t) = [\frac{9}{2}e^{-t} - \frac{10}{3}e^{-2t} - \frac{1}{6}e^{-5t}]u(t)$$

ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University
Department of Electrical and Computer Engineering

Lecture 21

October 24, 2008

Hui Qun Deng, PhD

Applications of LT

1. Zero-input response
2. Zero-state response
3. Transient response
4. Steady-state response

Example

Consider the system described by

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+3x(t)$$

and with initial conditions $\frac{dy(0^-)}{dt}=2, \quad y(0^-)=1$.

If the input signal is $x(t)=e^{-5t}u(t)$, then the Unilateral LT of the output of the system is (see last lecture):

$$\mathcal{Y}(s)=\underbrace{\frac{(s+3)\mathcal{X}(s)}{s^2+3s+2}}_{\text{zero-state resp.}}+\underbrace{\frac{sy(0^-)+3y(0^-)+\frac{dy(0^-)}{dt}}{s^2+3s+2}}_{\text{zero-input resp.}}$$

Zero-Input Response and Zero-State Response

The response of a causal LTI differential system with *non-zero initial conditions (i.e., $y^{(n)}(t=0^-)\neq 0$)* and a non-zero input can be viewed as the sum (superposition) of a *zero-state response* and a *zero-input response*.

For the above example,

$$\mathcal{Y}(s)=\underbrace{\frac{(s+3)\mathcal{X}(s)}{s^2+3s+2}}_{\text{zero-state resp.}}+\underbrace{\frac{sy(0^-)+3y(0^-)+\dfrac{dy(0^-)}{dt}}{s^2+3s+2}}_{\text{zero-input resp.}} \leftarrow\rightarrow y(t)=y_{zs(t)}+y_{zi}(t)$$

$y_{zs}(t)$ is obtained under *zero-initial conditions* $y^{(n)}(t=0^-)=0$ using methods in Ch3.

$$y_{zs}(t)=\sum_{i=1}^{N}B_i e^{a_i t}+Dx(t),\quad B_i = function\ \ of\ \ \frac{d^n y(t=0^+)}{dt^n}, n=0,\ldots N-1$$

$$y_{zi}(t)=\sum_{i=1}^{N}C_i e^{a_i t},\quad C_i = function\ \ of\ \ \frac{d^n y(t=0^-)}{dt^n}, n=0,\ldots N-1$$

a_i is a root of the characteristic polynomial of Eq. (3.31).

Transient and Steady-State Responses of LTI Differential Systems

Recall CH3: The complete solution of an LTI differential Eq. is

$y(t)$=homogeneous solution $y_{tr}(t)$+ particular solution $y_p(t)$

$$y_{tr}(t) = \sum_{i=1}^{N} A_i e^{a_i t}, \quad A_i = function \quad of \ y^{(n)}(0^+), n = 0,..N-1$$

$$y_p = Kx(t), \quad K \in Complex$$

The homogeneous response (natural response) of a causal, stable LTI differential system is called *transient response.* The particular solution corresponding to a constant or periodic input is called *steady-state response* (or forced response).

A stable system is said to be in steady-state if the transient component of the output has practically disappeared.

Relationship among transient, steady-state, zero-state, zero-input responses

Total response = transient response + steady-state response

Total response = Zero-input response + Zero-state response

|| ||

Transient response 1 Transient response 2

+ Steady-state response

Transient response = Zero-input response + Zero-state response

- Steady-state response

Transient and Steady-State Analysis Using the Laplace Transform

For a causal, stable LTI system, a partial fraction expansion of the transfer function according poles allows us to determine

- Transient response (the terms with the system poles)
- Steady-state response (the terms with the input poles)

Example 1: identify transient and steady-state responses

For example, consider the step response

$$s(t) = u(t) - e^{-5t}u(t)\ .$$

The transient part of this response is the term $e^{-5t}u(t)$, and the steady-state part is $u(t)$.

Example 2:

Assume that a causal LTI differential system is subjected to the input signal $x(t) = \sin(\omega_0 t)u(t)$, and the resulting output is

$$y(t) = 2\sin(\omega_0 t - \phi)u(t) + e^{-2t}\cos(2t + \theta)u(t).$$

Then, the *transient response* of the system to the input is:

$$e^{-2t}\cos(2t + \theta)u(t)$$

and the *steady-state response is*

$$2\sin(\omega_0 t - \phi)u(t)$$

Example 3: the transient and steady-state responses in the step response

Consider the step response

$$Y(s) = \frac{s+3}{\left(s^2+3s+2\right)s}, \quad \mathrm{Re}\{s\} > 0$$

$$= \underbrace{\frac{A}{s+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{B}{s+2}}_{\mathrm{Re}\{s\}>-2} + \underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0}$$

The steady-state response corresponds to the last term $\frac{C}{s}$, which in the time-domain is *Cu(t)*.

The other two terms correspond to the transient response $Ae^{-t}u(t) + Be^{-2t}u(t)$.

Transfer functions and steady-state responses

The transfer functions and frequency responses of LTI systems are obtained with zero-initial conditions, and thus give us the steady-state response.

Step response: We can apply the final value theorem to determine the steady-state component of a step response. In general, this component is a step function $Au(t)$. The "gain" A is given by:

$$A = \lim_{s \to 0} sH(s)\frac{1}{s} = H(0)$$

Response to a periodic exponential

For $x(t) = Ae^{j\omega_0 t}$, the steady-state response is

$$y_{ss}(t) = H(j\omega_0)Ae^{j\omega_0 t} = \left|H(j\omega_0)\right| Ae^{j\left(\omega_0 t + \angle H(j\omega_0)\right)}$$

Steady response to a sinusoidal input

If the input signal is pure sinusoidal $x(t) = A\sin(\omega_0 t)$, then the steady-state response of the system can be obtained from the frequency response *H(jω)* of the system:

$$y_{ss}(t) = \frac{A}{2j}\left(H(j\omega_0)e^{j\omega_0 t} - H(-j\omega_0)e^{-j\omega_0 t}\right)$$

$$= \frac{A}{2j}\left(\left|H(j\omega_0)\right|e^{j(\omega_0 t + \angle H(j\omega_0))} - \left|H(-j\omega_0)\right|e^{-j(\omega_0 t - \angle H(-j\omega_0))}\right)$$

$$= \frac{A}{2j}\left(\left|H(j\omega_0)\right|e^{j(\omega_0 t + \angle H(j\omega_0))} - \left|H(j\omega_0)\right|e^{-j(\omega_0 t + \angle H(j\omega_0))}\right)$$

$$= \left|H(j\omega_0)\right| A\sin\left(\omega_0 t + \angle H(j\omega_0)\right)$$

Important application:

Steady-state analysis of circuits at a fixed frequency, e.g., 60 Hz.

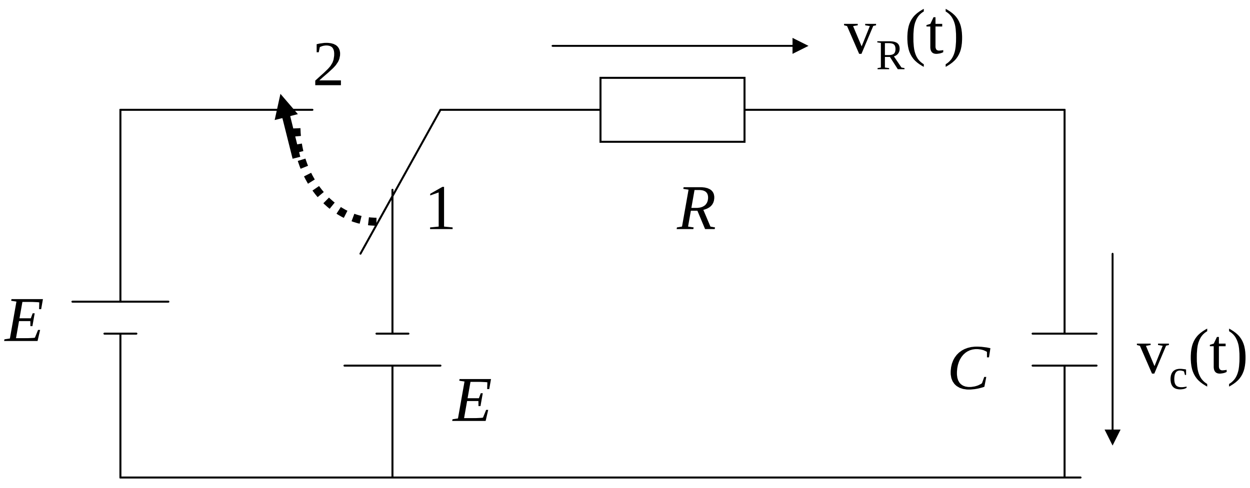
For example, if a circuit is described by its impedance Z(jω), then its steady-state response to a 60 Hz sinusoidal current is characterized by the multiplication of the complex amplitude of the current and the complex number $Z(j2\pi 60)$.

Response to periodic signals

Again, the frequency response of the system gives us the steady state response to a periodic signal admitting a Fourier series representation. For $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$, the steady-state response is

$$y_{ss}(t) = \sum_{k=-\infty}^{+\infty} H(jk\omega_0) a_k e^{jk\omega_0 t} .$$

Example 4: application of unilateral LT in Circuit Analysis



When t<0, the switch is connected at 1, and the state of the system is steady. At t=0, the switch is turned on 2.

Derive $v_c(t)$ and $v_R(t)$.

Step 1: Write the equation about $v_c(t)$

$$RC\frac{dv_c(t)}{dt}+v_c=Eu(t),\quad t\geq 0$$

Step 2: Identify the initial condition at $t=0^-$: $v_c(t=0^-)=-E$

Step 3: Apply unilateral LT on the above differential Eq.

$$RC[sV_c(s)-v_c(0^-)]+V_c(s)=\frac{E}{s}$$

$$V_c(s)=\frac{E(\frac{1}{RC}-s)}{s(s+\frac{1}{RC})}$$

Step 4 and 5: Do partial fraction expansion and inverse LT

Step 4: Partial fraction expansion

$$V_c(s) = \frac{E\left(\frac{1}{RC} - s\right)}{s\left(s + \frac{1}{RC}\right)}$$

$$V_c(s) = E\left(\frac{1}{s} - \frac{2}{s + \frac{1}{RC}}\right)$$

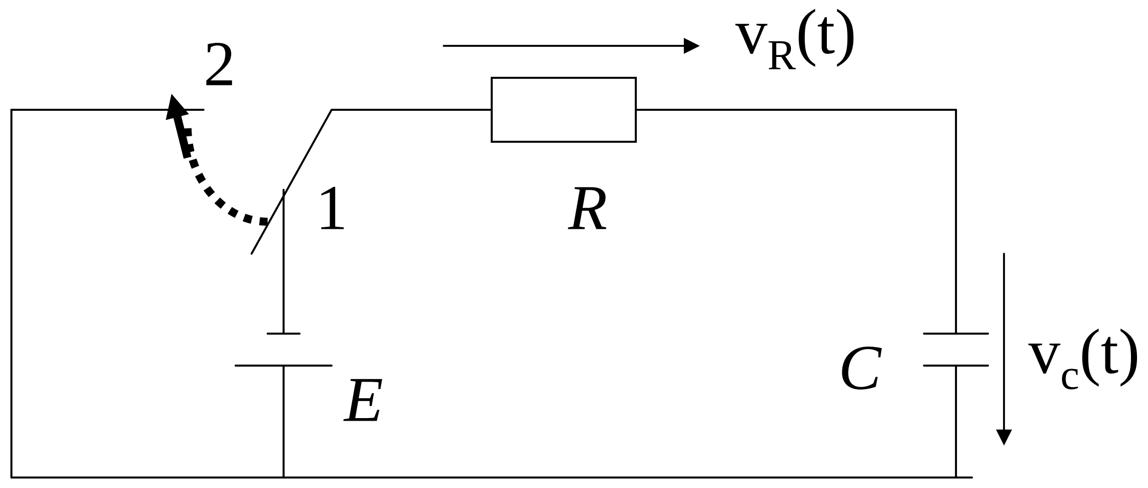
Step 5: Inverse LT

$$v_c(t) = E\left(1 - 2e^{-\frac{1}{RC}t}\right)u(t)$$

And

$$v_R(t) = Eu(t) - v_c(t) = 2Ee^{-\frac{1}{RC}t}u(t)$$

Example 5 (in class)



When t<0, the switch is connected at 1, and the state of the system is steady. At t=0, the switch is turned on 2.

Derive $v_c(t)$ and $v_R(t)$.