McGill University
Department of Electrical and Computer Engineering

Lecture 34

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Sampling

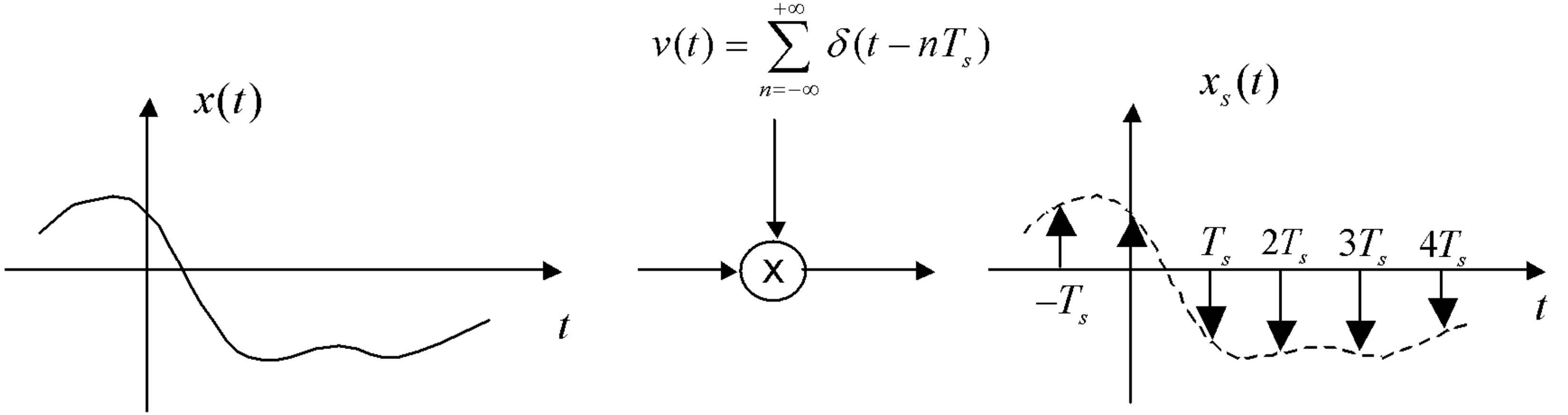
Sampling theorem

Signal Reconstruction

Aliasing and anti-aliasing

Sampling of a continuous signal

The sampling operation can be seen as the multiplication of a continuous-time signal with a periodic impulse train of period T_s .



Recall the property of FT:

Multiplication in the time domain → convolution in the frequency domain, multiplied by $1/2\pi$.

Discrete-time signals as sampled continuous-time signals

Then, the sampled signal in the time-domain can be represented by:

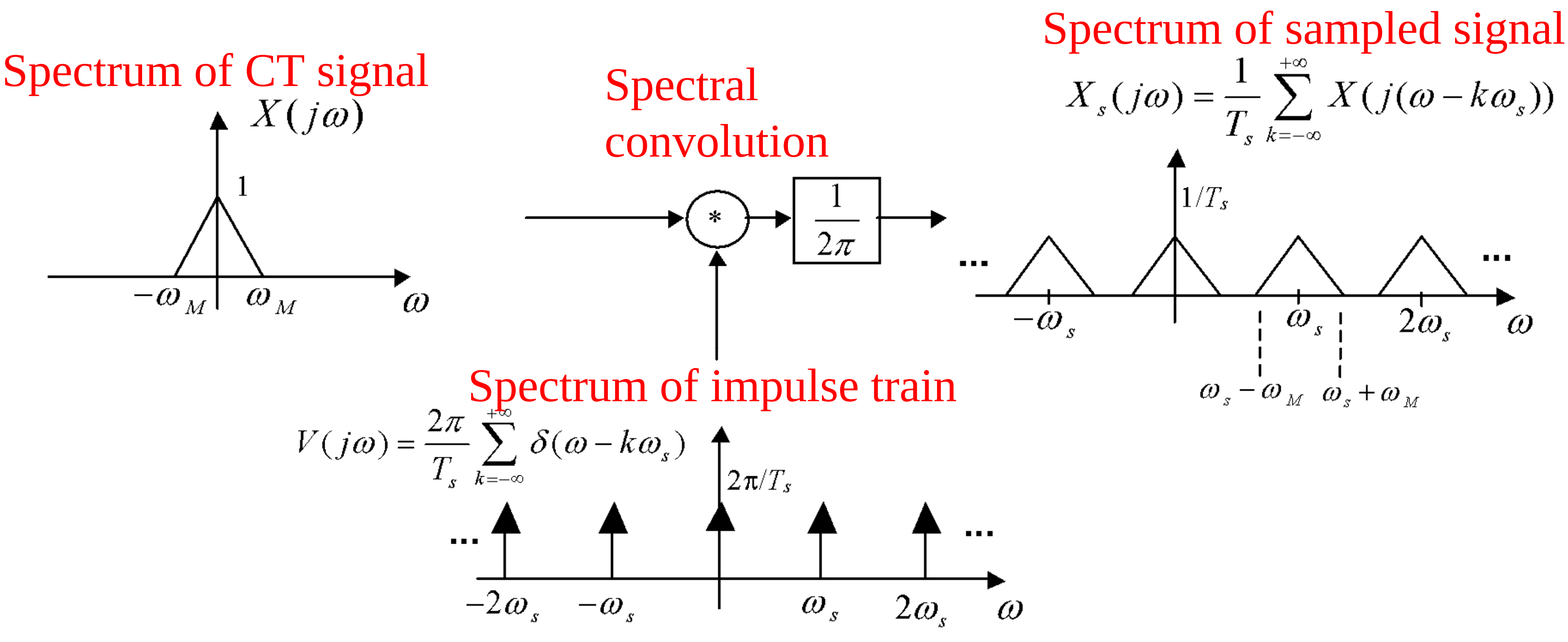
$$x_s(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s)$$

where the impulse at time $t = nT_s$ has a strength (area) equal to the signal sample at that time. T_s is called sampling period.

Recall: sampling property of δ(t):

$x(t)\delta(t-t_0)=x(t_0)\,\delta(t-t_0)$

The spectrum of sampled signals



$$\omega_s = 2\pi/T_s$$

The relationship between the spectrum of $x_s(t)$ and the spectrum of $x(t)$

The spectrum of the sampled signal can be obtained from the multiplication property of FT:

$$X_s(j\omega) = \frac{1}{T_s}\int_{-\infty}^{+\infty} X(j\upsilon)\sum_{k=-\infty}^{+\infty}\delta(\omega-\upsilon-k\omega_s)d\upsilon$$

$$= \frac{1}{T_s}\int_{-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} X(j(\omega-k\omega_s))\delta(\omega-\upsilon-k\omega_s)d\upsilon$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X(j(\omega-k\omega_s))$$

Thus, the spectrum of the sampled signal is a superposition of replicas of the original signal spectrum, shifted by integer multiples of the sampling frequency ω_s and scaled by $1/T_s$.

The Sampling Theorem
(*Nyquist theorem*, or *Shannon theorem*)

Let $x(t)$ be a band-limited signal with $X(j\omega)=0$ for $|\omega|>\omega_M$. Then, x(t) is uniquely determined by its samples $x(nT_s),\ -\infty<n<+\infty$ if

$$\omega_s > 2\omega_M$$

where

$$\omega_s = \frac{2\pi}{T_s}$$

is the sampling frequency, T_s is the sampling period.

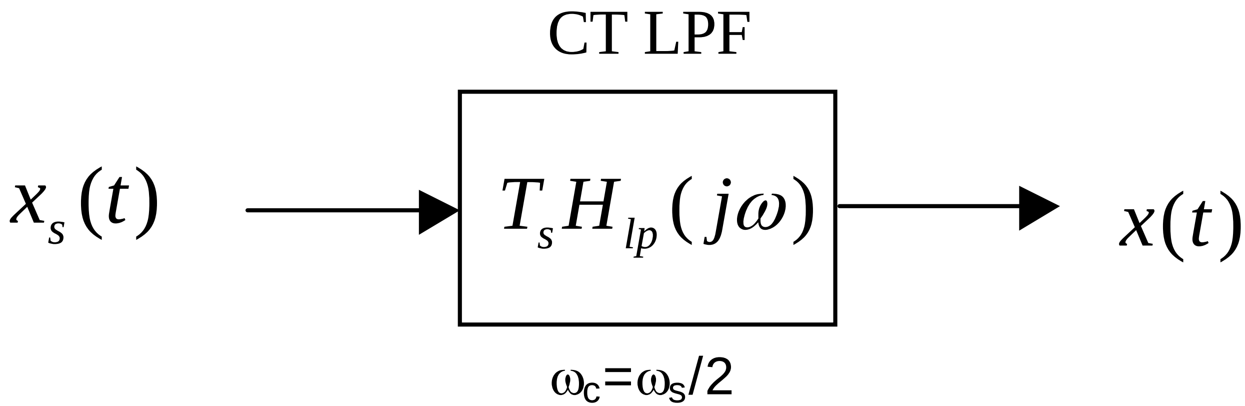
Signal Reconstruction

Assume that we have sampled a band-limited signal *x(t)* at a sampling frequency $\omega_s = \frac{2\pi}{T_s}$ that satisfies the condition of the sampling theorem.

What we have is a discrete-time signal $x[n] = x(nT_s)$.

Ideally, to reconstruct the signal *x(t)*, we need to construct a train of impulses $x_s(t)$ from *x[n]*, and then to filter $x_s(t)$ with an ideal low-pass filter.

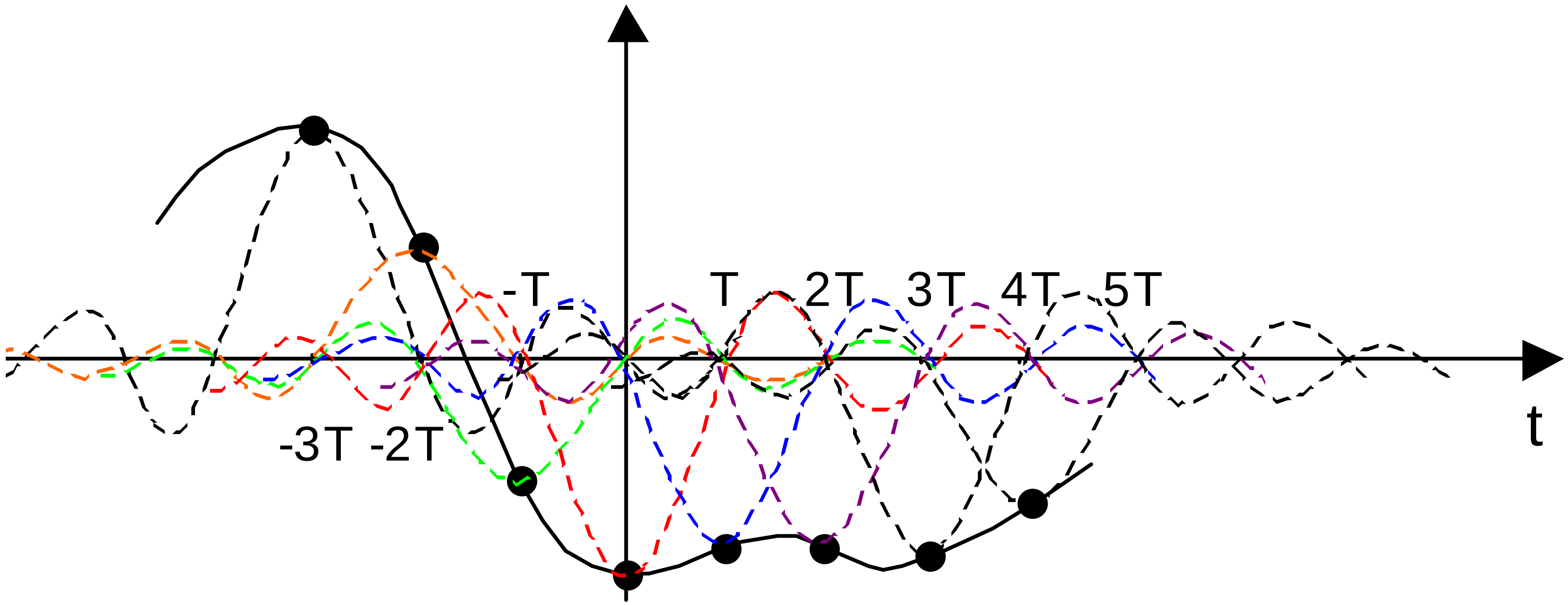
Recover x(t) from its samples



Filtering the sample impulses using LPF with a cutoff frequency $\omega_c = \omega_s/2$ is equivalent to interpolating the samples using time-shifted *sinc* functions with zeros at nT_s, as the impulse response of an LPF is a sinc function:

$$\frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}t) = \frac{1}{T_s}\text{sinc}(\frac{t}{T_s}) = \frac{\sin(t\pi/T_s)}{t\pi}$$

Perfect signal interpolation using sinc functions in the time domain



$$x(t) = \sum_{n=-\infty}^{\infty} x[nT]\mathrm{sinc}(\frac{t-nT}{T})$$

x(t) is a superposition of weighted and shifted sinc functions.

For example, the signal at t=T/3 is

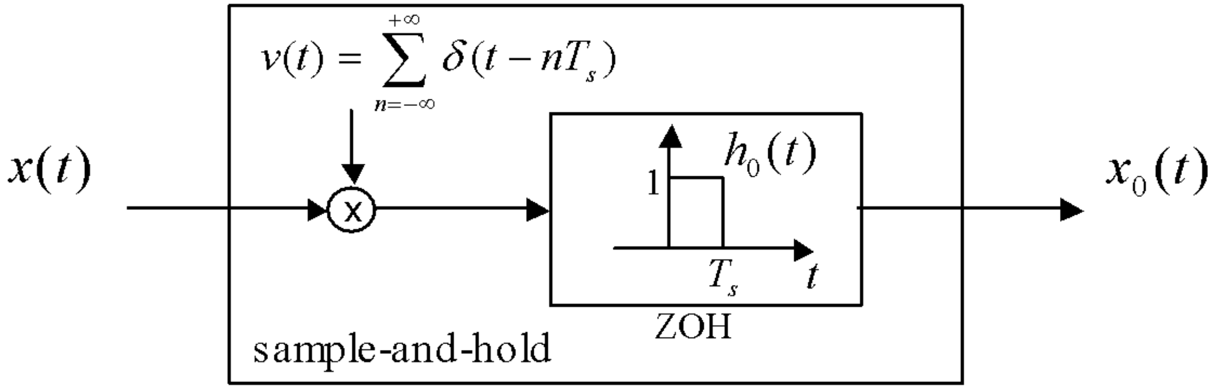
$$x(t = T/3) = \sum_{n=-\infty}^{\infty} x[nT]\mathrm{sinc}(\frac{T/3-nT}{T})$$

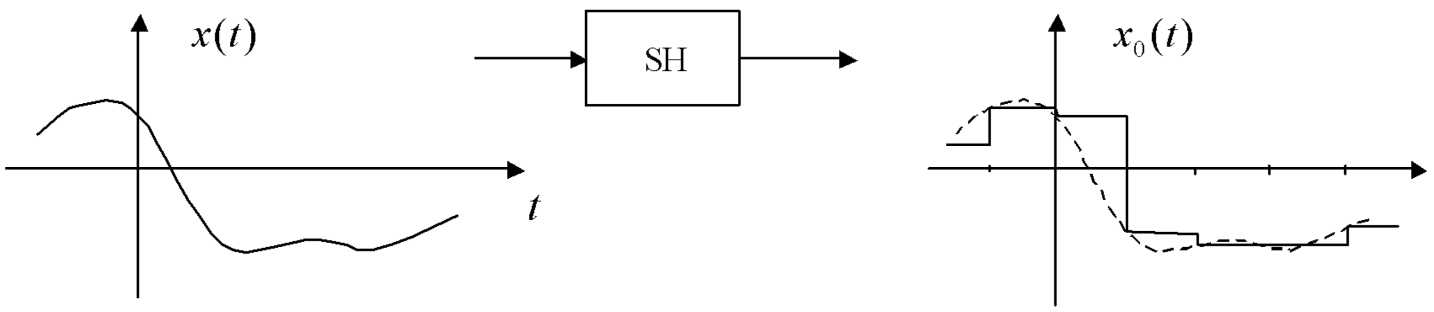
Obtaining sample impulses from $x(t)$ and then filtering them using an ideal LPF is clearly unfeasible, at least an ideal LPF can’t work in real time.

However, there is a number of ways to implement the sampling and reconstruction of the CT signal $x(t)$.

Sampling *x(t)* using a Sample-and-Hold

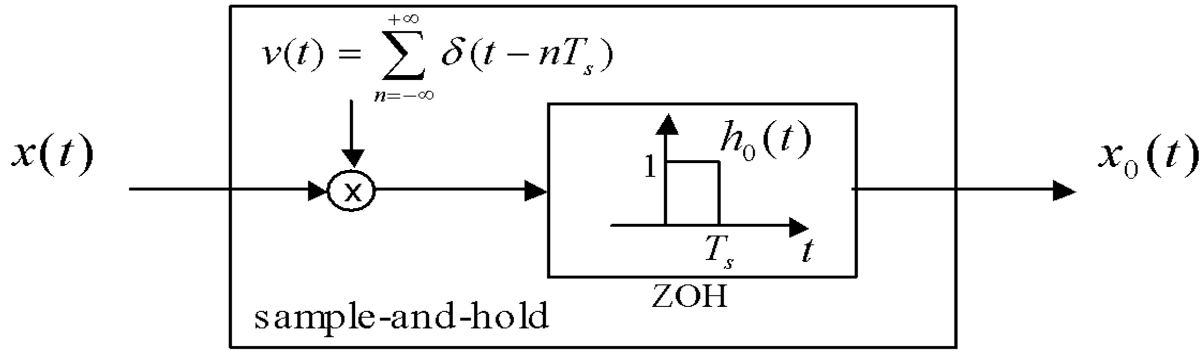
The sample-and-hold (SH) retains the value of the signal sample up until the following sampling instant.





Sample-and-hold (SH) produces a "staircase" signal from x(t).

The frequency response of ZOH



The zero-order-hold (ZOH) has an impulse response $h_0(t)$.
The frequency response of ZOH is given by the FT of $h_0(t)$:

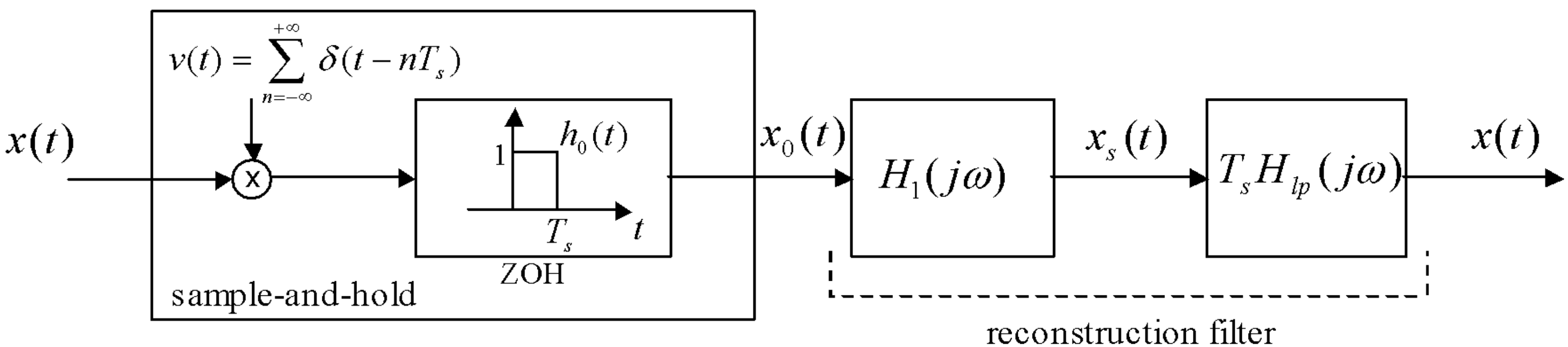
$$H_0(j\omega) = T_s e^{-j\omega\frac{T_s}{2}} \mathrm{sinc}\left(\frac{T_s}{2\pi}\omega\right) = 2e^{-j\omega\frac{T_s}{2}} \frac{\sin(\omega\frac{T_s}{2})}{\omega}$$

This is a sinc function, but multiplied by $e^{-j\omega\frac{T_s}{2}}$ because of the time delay of $T_s/2$ seconds.

The inverse system of ZOH

To compensate the distortion due to the "staircase" effect of the ZOH, we construct an inverse system of $H_0(j\omega)$:

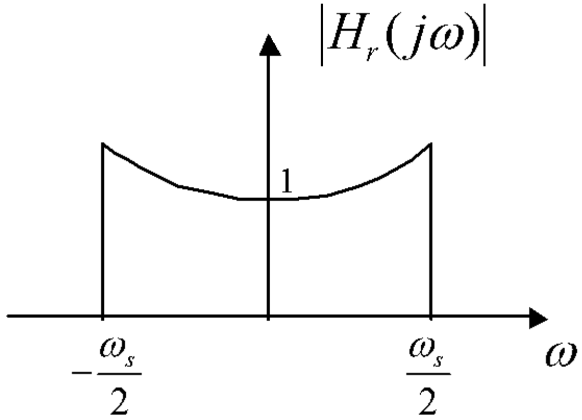
$$H_1(j\omega) = H_0^{-1}(j\omega) = \frac{1}{2} e^{j\omega \frac{T_s}{2}} \frac{\omega}{\sin(\omega \frac{T_s}{2})} .$$

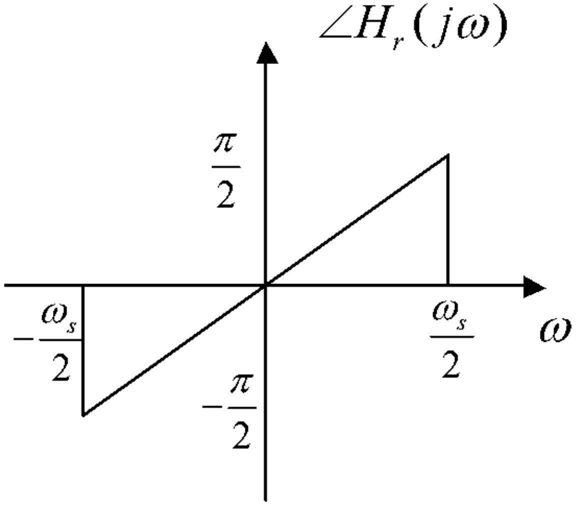


The *reconstruction filter* is the cascade of the inverse filter and the lowpass filter

$$H_r(j\omega) = T_s H_{lp}(j\omega) H_1(j\omega)$$

whose magnitude and phase plots are shown below.





- This frequency response cannot be realized exactly in practice, but it can be approximated with a causal filter.

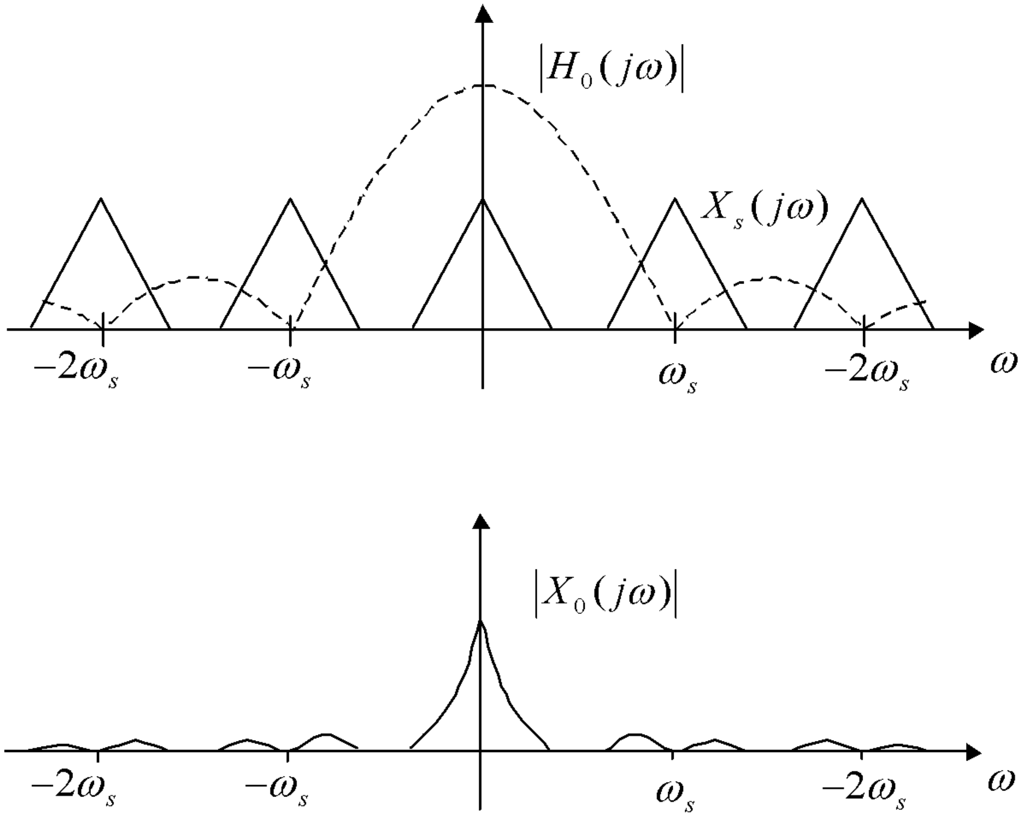
- In fact, in many practical situations, it is often sufficient to use a simple (nonideal) lowpass filter with a relatively flat magnitude in the passband to recover a good approximation of the signal.

The spectrum of the ZOH output

Consider the case if we don't use the inverse of ZOH. The frequency response of the ZOH is:

$$H_0(j\omega) = e^{-j\omega T_s/2} T_s \text{sinc}(T_s\omega/2\pi)$$

Note: increasing the sampling frequency ω_s can reduce distortion due to the "staircase" effect, as did in some digital audio systems.



The spectrum of the ZOH output.

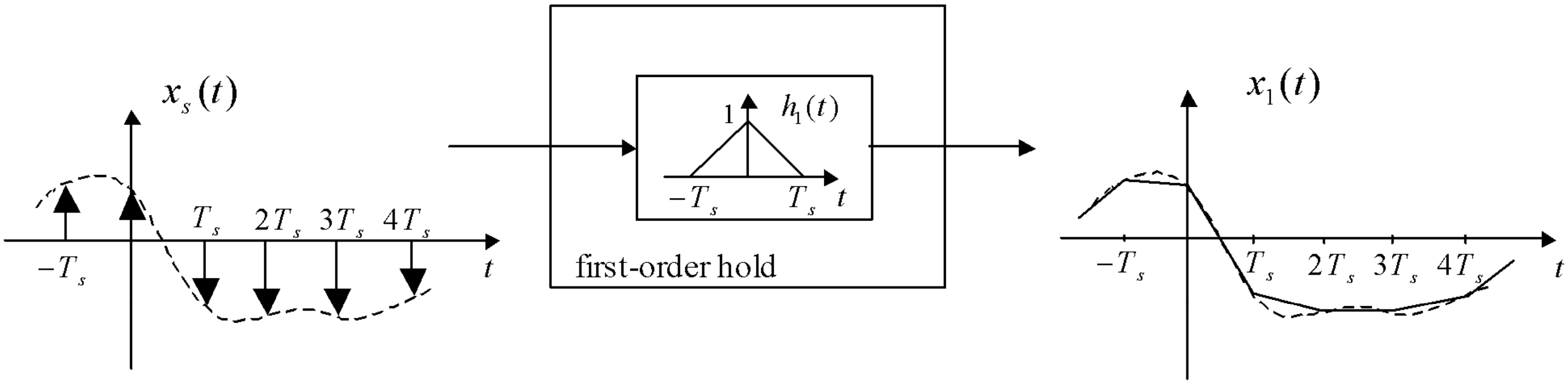
First-Order Hold (linear interpolation)

Instead of being a rectangular pulse, the first-order hold is a triangular impulse response $h_1(t)$, which can be viewed as the convolution of two $u(t)$ signals.

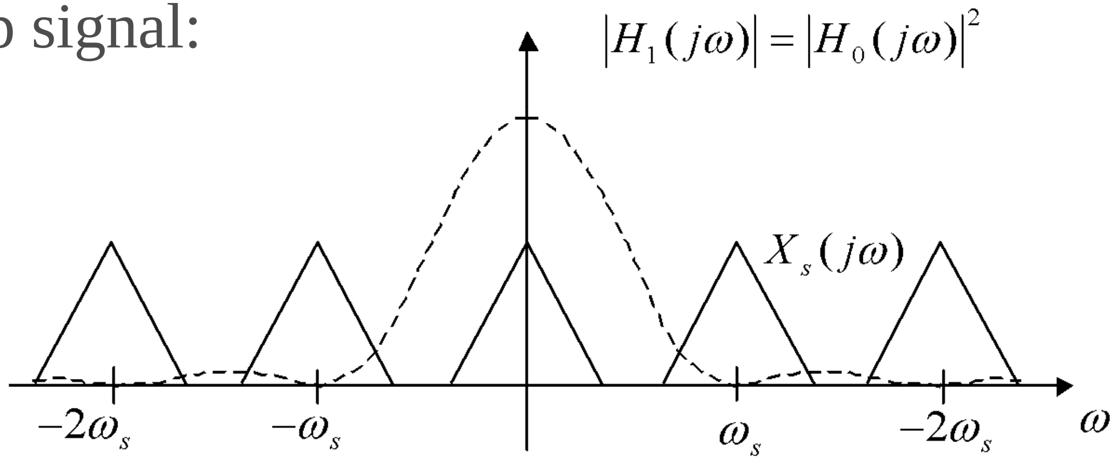
The interpolation using a first-order-hold is linear function of t between two adjacent samples. Linear interpolation is closer to the signal $x(t)$ than a “staircase” signal given by a ZOH output.

In the frequency domain, the Fourier transform of a triangular impulse response $h_1(t)$ is also a better approximation to the ideal lowpass filter than $H_0(j\omega)$ is.

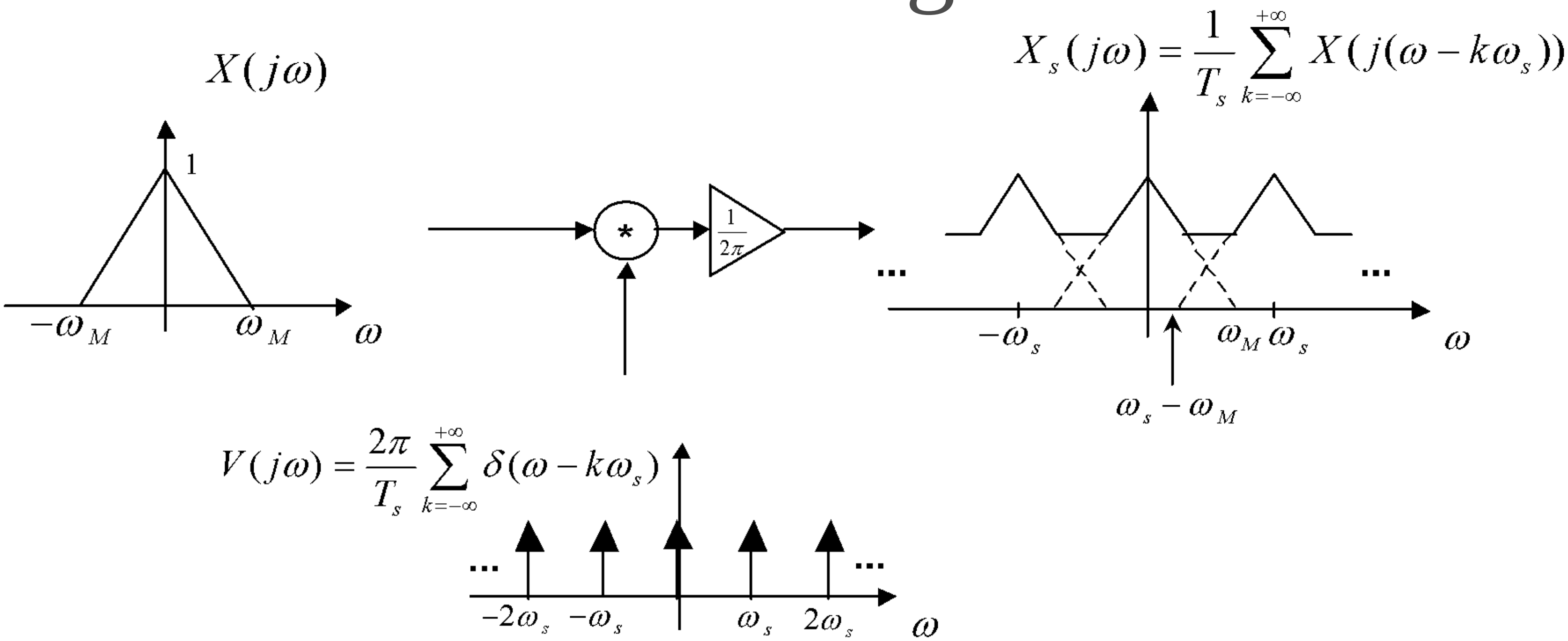
The output of the first-order-hold



$x_1(t)$ is a piece-wise linear function of time t. $h_1(t)$ is the convolution of two unite step signals, and the spectrum of $h_1(t)$ is squared spectrum of the unite step signal:



Aliasing



Aliasing occurs if $\omega_s < 2\omega_M$, where ω_M is the bandwidth of the signal, ω_s is the sampling frequency.

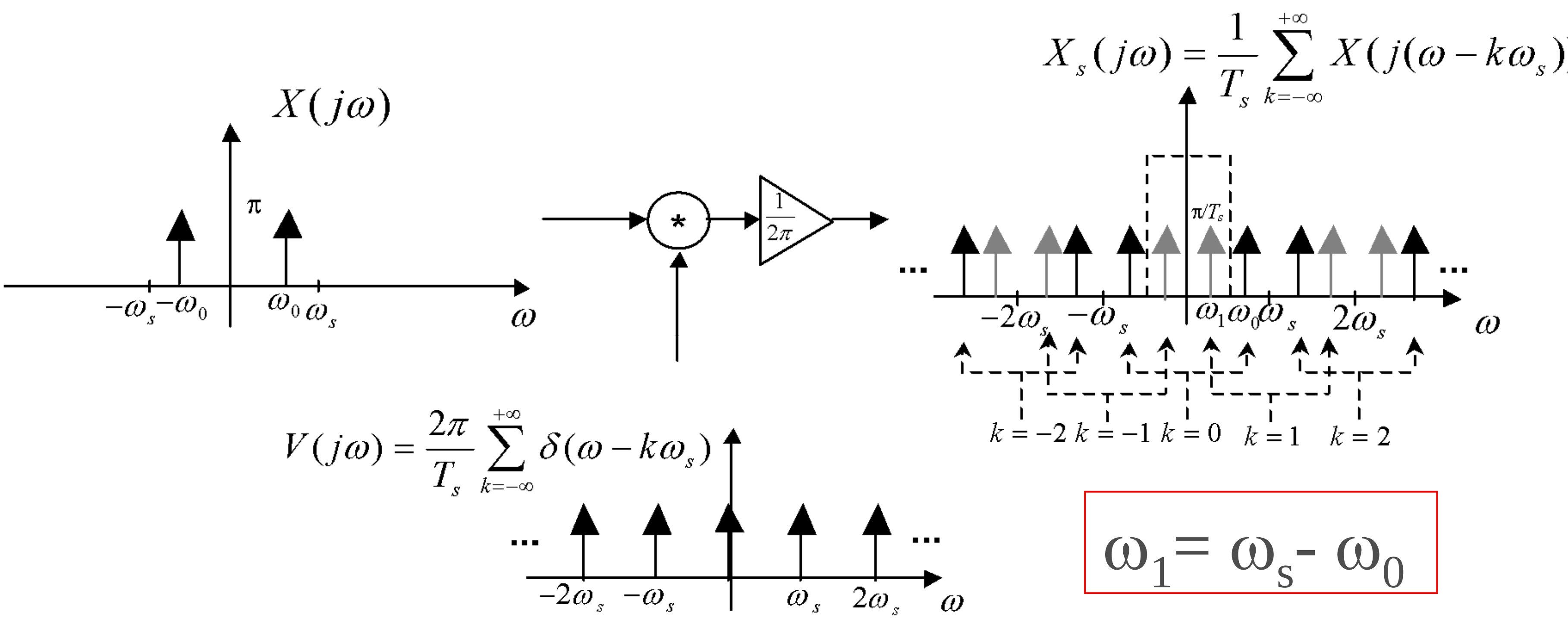
With aliasing, the original signal cannot be recovered by lowpass filtering.

Example: sampling a sinusoidal signal

A pure sinusoidal signal with frequency ω_0 is sampled at ω_s,

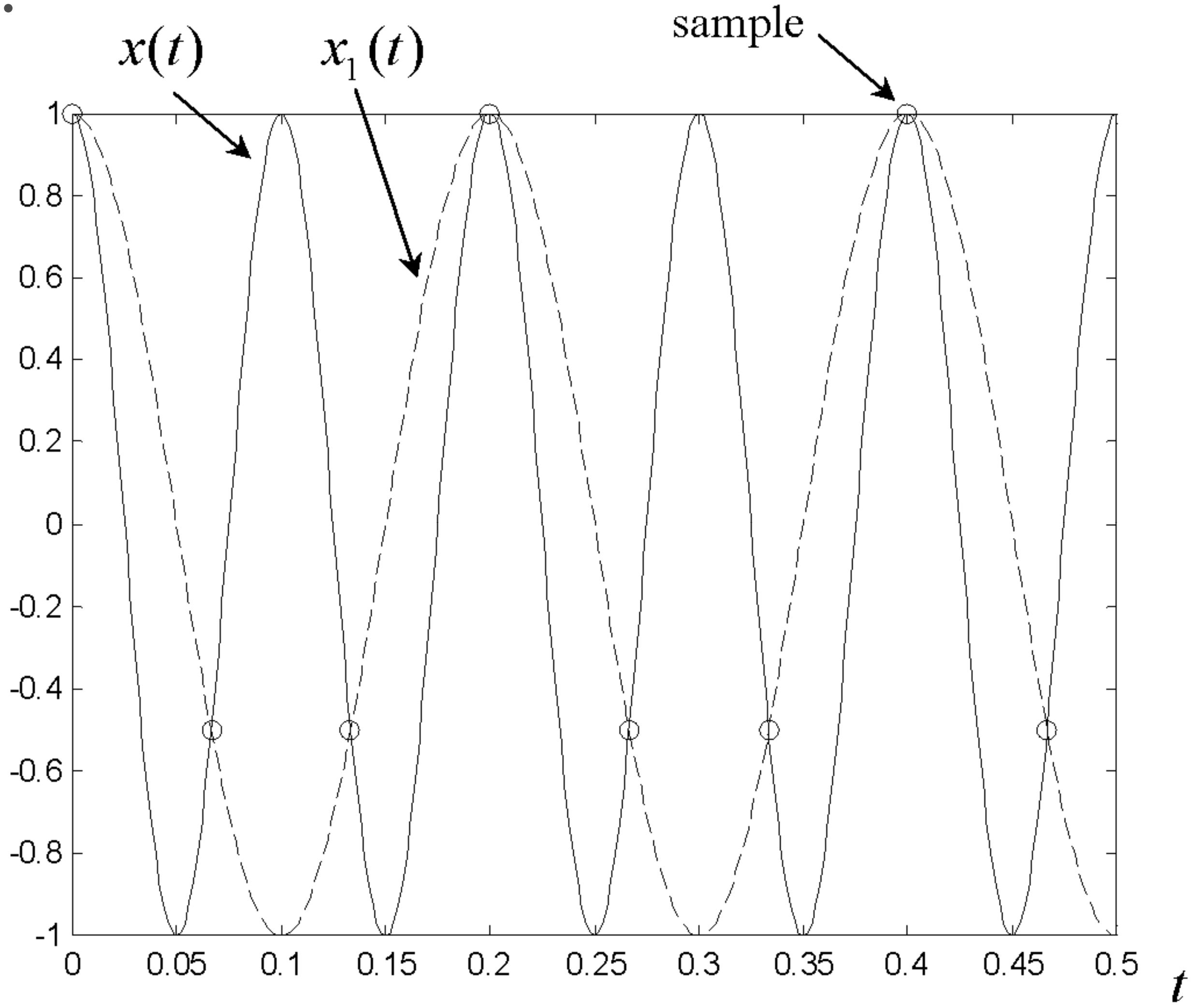
What is the spectrum of the signal?

What is the spectrum of the sampled signal?



$$\omega_1 = \omega_s - \omega_0$$

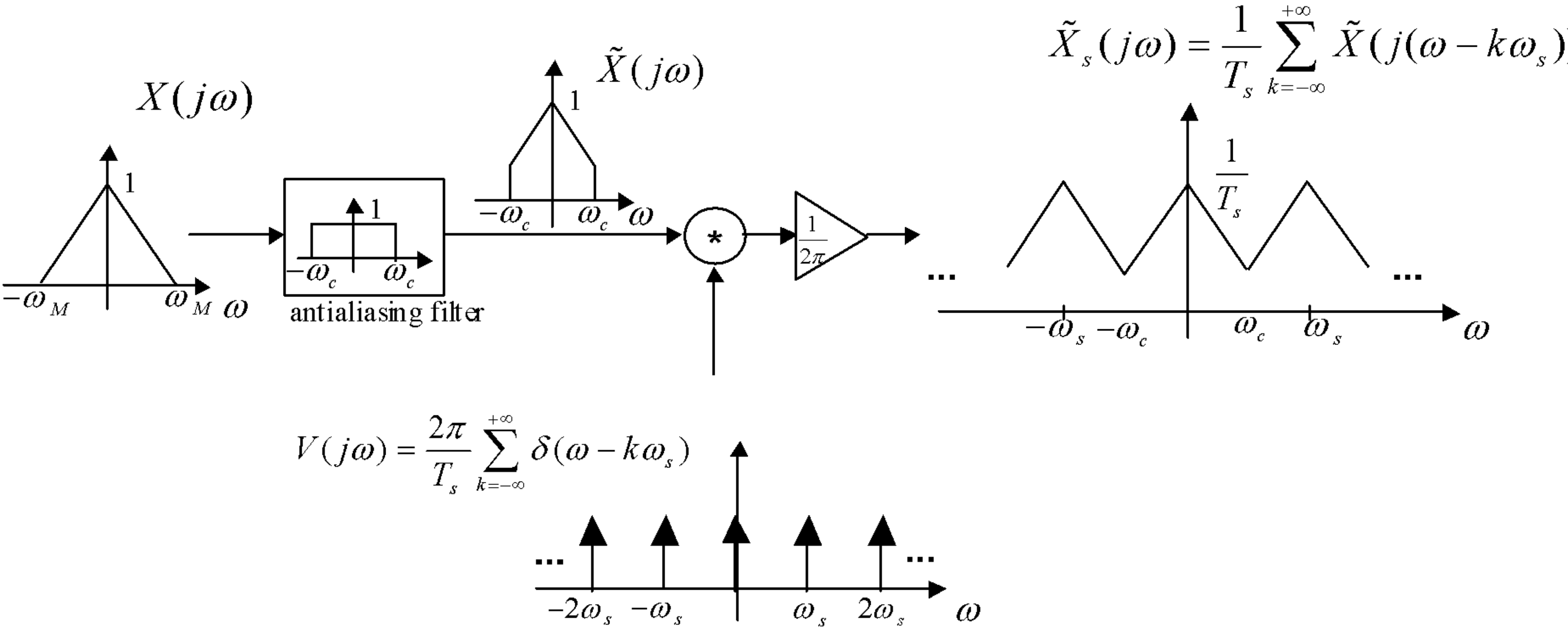
At the output of the LPF with cutoff frequency $\omega_s/2$, we get a sinusoidal signal with a different frequency than the original one.



Anti-aliasing

To ensure that the sampled signal don't contain alias noise, use an anti-aliasing filter to limit the bandwidth of the signal to be sampled.

Anti-aliasing filter



To prevent aliasing noise , an anti-aliasing low-pass filter with a cutoff frequency $\boldsymbol{\omega_c \leq \omega_s/2}$ **is** desired.