6.003: Signals and Systems — Spring 2004

TUTORIAL 9 SOLUTIONS

Tuesday, April 13, 2004

Problem 9.1

(a) 4.

(b) $H\left(e^{j0}\right) = \frac{3}{2}$.

(c) Yes, it is bandlimitted to $2\omega_m$.

(d) Highpass.

(e) Odd.

Problem 9.2

$$k = nT, \quad n \in \mathbb{Z}$$

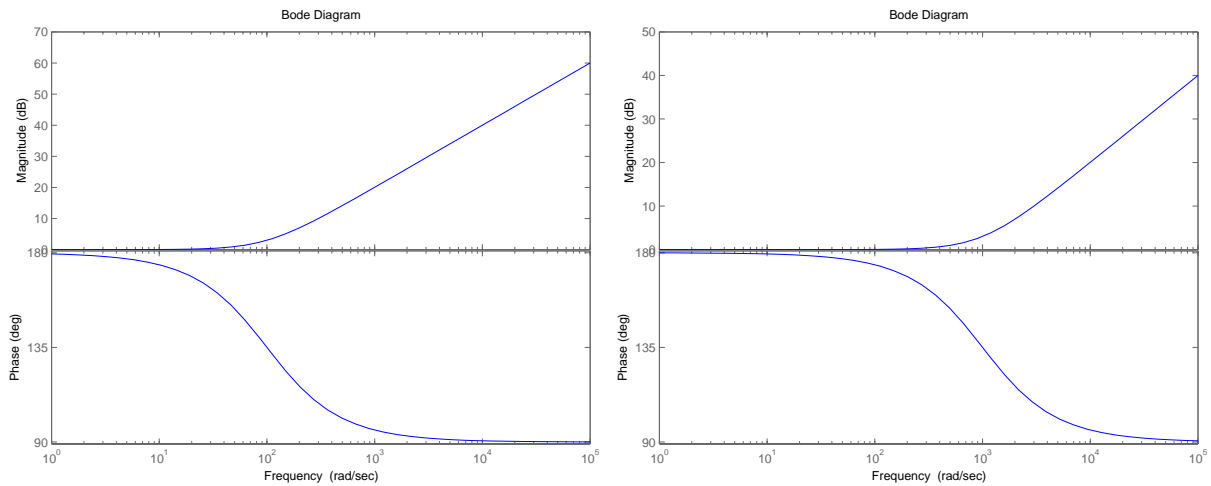**Problem 9.3**   We can plot each part of the Bode plot separately and then add them up.

Figure 1: Bode plot for $\frac{j\omega}{10^2} - 1$ (left) and $\frac{j\omega}{10^3} - 1$ (right).
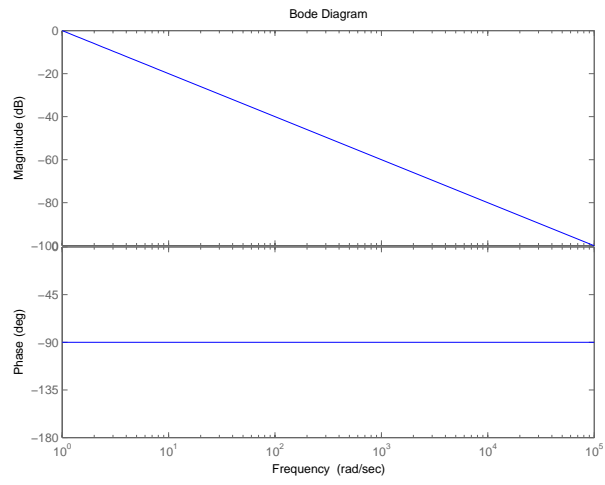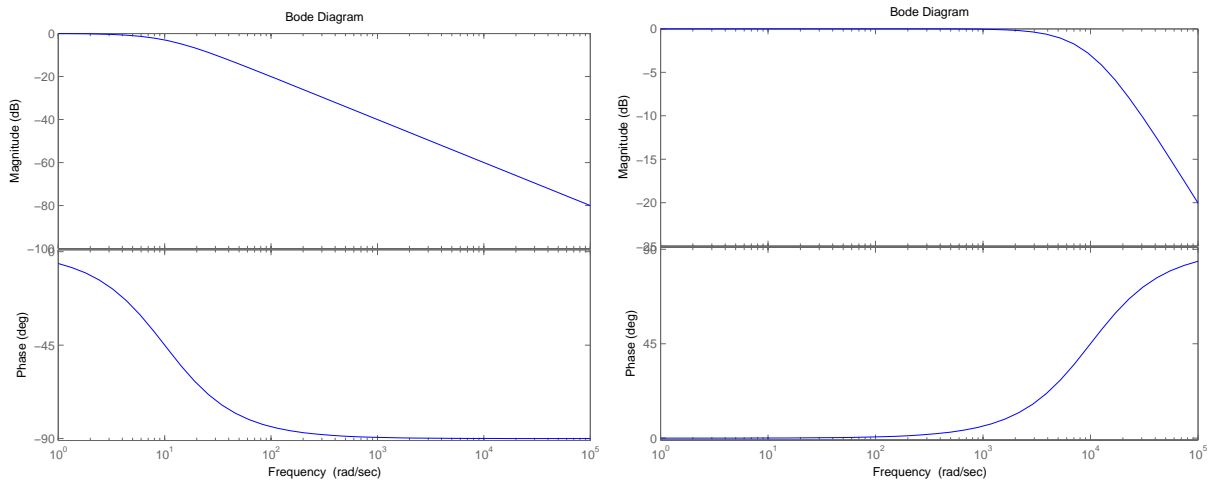
Figure 2: Bode plot for $\frac{1}{j\omega}$.

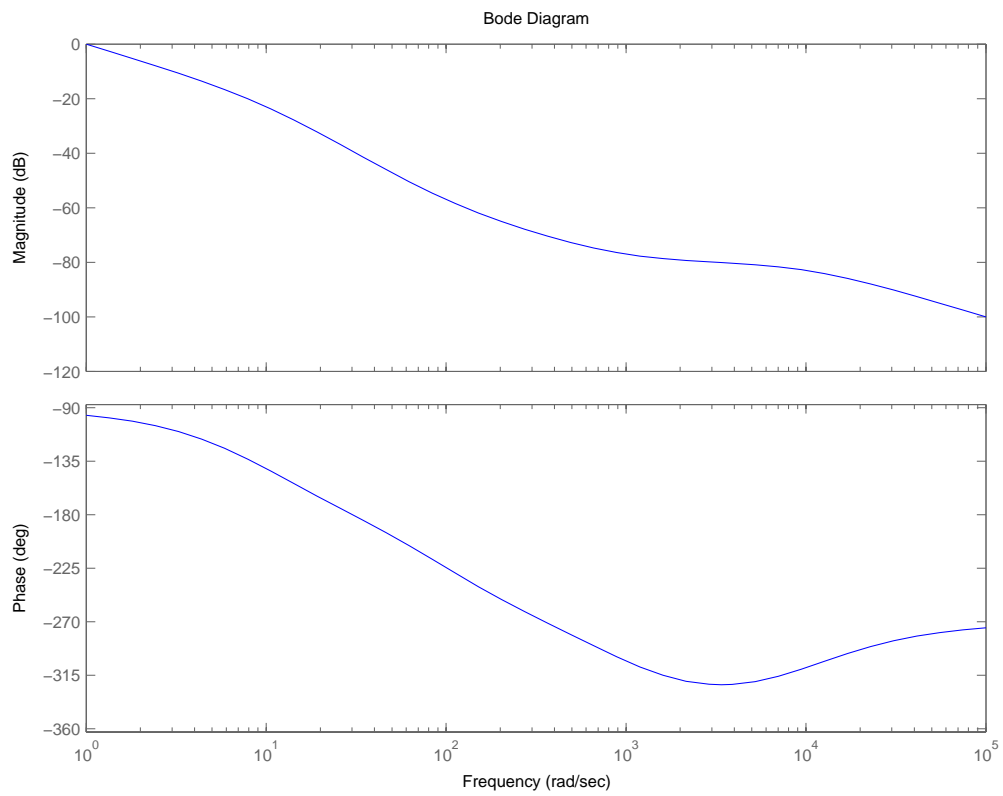Figure 3: Bode plot for $\frac{1}{\frac{j\omega}{10}+1}$ (left) and $\frac{1}{-\frac{j\omega}{10^4}+1}$ (right).

Bode Diagram

Figure 4: Bode plot for $H(j\omega)$.

Problem 9.4

(a) The Fourier transform of $x(t) = \text{sinc}(t) = \frac{\sin \pi t}{\pi t}$ is:

$$X(j\omega) = \left\{ \begin{array}{ll} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{array} \right.$$

Since $x(t)$ is bandlimited to $|\omega| < \pi$, we can use DT processing of CT signals and replace the CT system with C/D and D/C converters and an equivalent DT LTI system in between. According to the Nyquist sampling theorem, we need to sample such that $2\pi/T = \omega_s \geq 2\omega_M = 2\pi$. So, let's sample at $T = 1$ so that the DT input to the DT system is simply the unit impulse $x_d[n] = x(n) = \delta[n]$.

(**Note:** The theorem says we should sample at *strictly* greater than twice the highest frequency in the system. However, according to Fourier theory, the FT of the sinc at the discontinuity $\omega = \pi$ is the average of its neighbors, or 0.5. In the sampling process, there is aliasing at that point, and the two 0.5's sum up to 1. Since we are sampling right at this point, the reconstruction lowpass filter also has its boundary there. As before, the value is the average of the value of its neighbors, which restores the FT value at that point to 0.5! Check out Tutorial 9, pages 195 to 196, for more details. This a rather technical point... don't worry about it too much. Just keep in mind that *it is okay to sample a sinc at its zeros and still have perfect reconstruction*.)
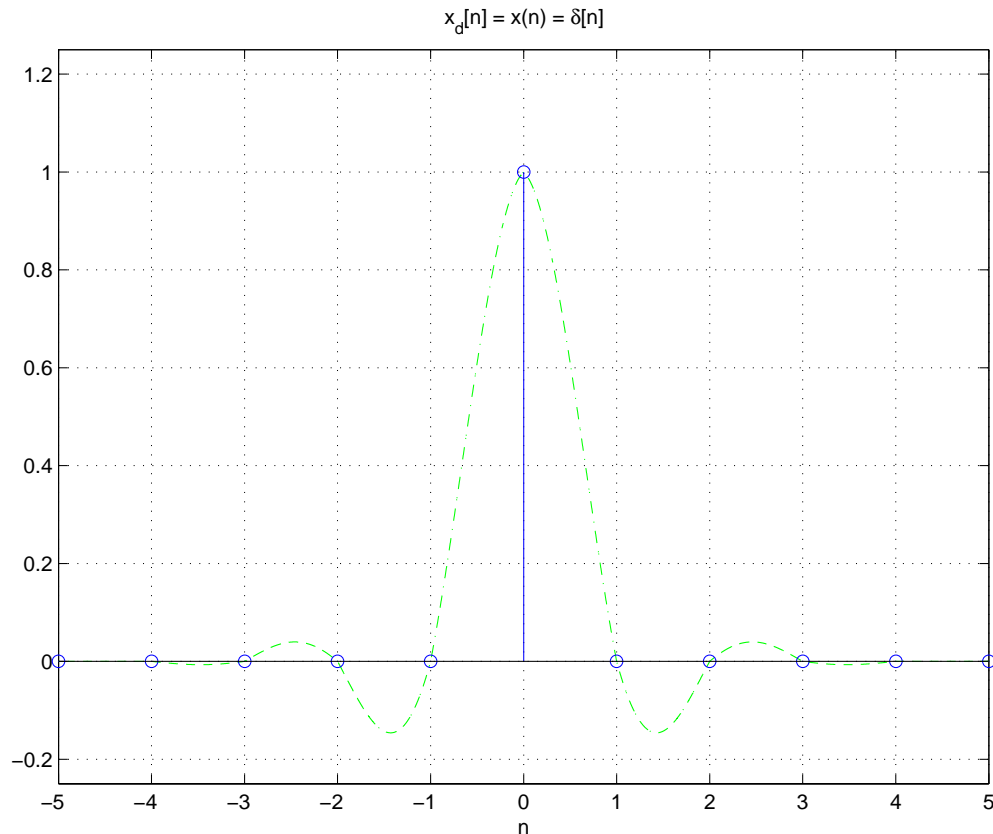

Figure 5: A CT sinc sampled at its zeros is a DT impulse.

As we saw in Tutorial 8, page 169, problem 8.3, we can translate the CT difference equation to a DT difference equation by replacing CT time shifts of t_0 with DT time shifts of $n_0 = t_0/T$:

$$y_d[n] - \frac{1}{2}y_d[n-1] = x_d[n].$$

We know from tutorial 4, page 43, problem 1 that the impulse response to this system is:

$$h_d[n] = \left(\frac{1}{2}\right)^n u[n].$$

Since the input to the DT system is the unit impulse, the output is the impulse response: $y_d[n] = h_d[n]$. The D/C converter produces the bandlimited interpolation of this DT signal, so the the output $y(t)$ is the CT bandlimited interpolation of $y_d[n] = \left(\frac{1}{2}\right)^n u[n]$.
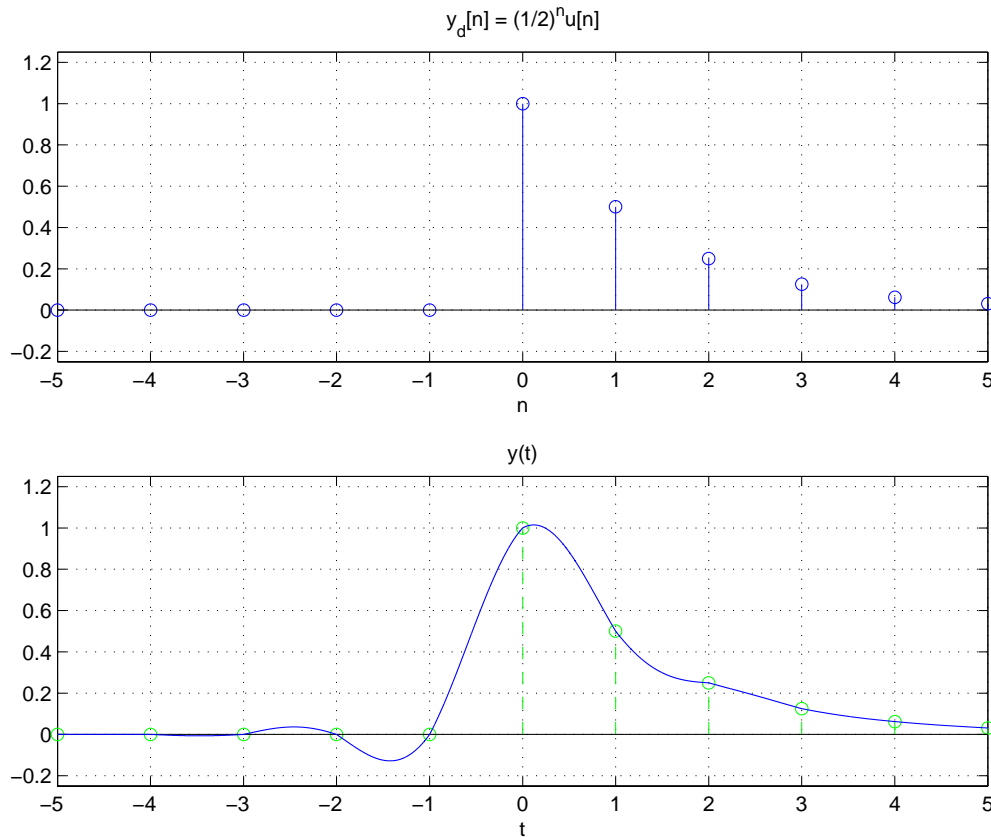

Figure 6: The upper graph is the impulse response and output of the DT processor. The lower graph is the output of the overall CT system, which is the bandlimited interpolation of the upper graph.

(b) Tutorial 8, page 160 has the formula for the CT bandlimited interpolation of a DT sample sequence:

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x_d[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}.$$

Setting $x_d[n] = y_d[n] = \left(\frac{1}{2}\right)^n u[n]$, $x_r(t) = y(t)$, and $T = 1$ produces the analytic form for $y(t)$:

$$y(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{\sin(\pi(t-n))}{\pi(t-n)}.$$

We can plug this expression for $y(t)$ into the difference equation to verify that it is the solution. We can also solve this problem without sampling. The CT difference equation can be rewritten as:

$$y(t) = \frac{1}{2}y(t-1) + x(t).$$

Since this holds for all time t, we can shift all the expressions in time by any arbitrary amount. For instance:

$$y(t-1) = \frac{1}{2}y(t-2) + x(t-1).$$

These produce:

$$
\begin{aligned}
y(t) &= \frac{1}{2}\left[\frac{1}{2}y(t-2) + x(t-1)\right] + x(t) \\
&= x(t) + \frac{1}{2}x(t-1) + \frac{1}{4}y(t-2).
\end{aligned}
$$

We can continue this process to obtain:

$$
\begin{aligned}
y(t) &= x(t) + \frac{1}{2}x(t-1) + \frac{1}{4}x(t-2) + \frac{1}{8}x(t-3) + \cdots \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n x(t-n).
\end{aligned}
$$

Since the system is stable, this series converges. Plugging in the expression for $x(t)$ produces the same analytic expression for $y(t)$:

$$y(t) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{\sin(\pi(t-n))}{\pi(t-n)}.$$

Problem 9.5

(a)
$$H_c(j\omega) = j\omega T$$

$H_c(j\omega)$ is a differentiator.

$$y_c(t) = T\frac{d}{dt}x_c(t)$$

(b)
$$x_c(t) = T\frac{\sin\left(\frac{\pi}{T}t\right)}{\pi t}$$

(c)
$$y_c(t) = T\frac{\pi t \cos\left(\frac{\pi t}{T}\right) - T\sin\left(\frac{\pi t}{T}\right)}{\pi t^2}$$

(d)
$$y_d[n] = y_c(nT) = T\frac{\pi n T \cos\left(\pi n\right) - T\sin\left(\pi n\right)}{\pi n^2 T^2} = \frac{(-1)^n}{n},$$

where we have used the fact that $\cos(\pi n) = (-1)^n$ and $\sin(\pi n) = 0$.