

**6.003: Signals and Systems — Spring 2004**

TUTORIAL 9

Monday, April 12 and Tuesday, April 13, 2004

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## Announcements

- There is no problem set due this week.
- Quiz 2 will be held on **Thursday, April 15**, 7:30–9:30 p.m. in Walker Memorial. The quiz will cover material in Chapters 1–7 of O&W through Section 7.4, Lectures and Recitations through April 2, Problem Sets #1–6, and that part of Problem Set #7 involving problems from Chapter 7.
- The TAs will jointly hold office hours from 2–8 p.m. on Wednesday, April 14 and again from 10 a.m.–3 p.m. on Thursday, April 15. A schedule is posted on the 6.003 web site.
- A quiz review package is available on the 6.003 web site. TAs will hold two identical optional quiz review sessions on Monday, April 12 and Tuesday, April 13, 7:30–9:30 p.m. in 34-101.
- Because of the Patriot’s Day holiday next week, there will be no tutorials next Monday and Tuesday, and no lecture on Tuesday.

## Today’s Agenda

- Fourier Transform Pitfalls
  - $2\pi$  factors
- Sampling Pitfalls
  - Impulses in the frequency domain
  - Do we really have to sample at the Nyquist rate?

# 1 Fourier Transform Pitfalls

## 1.1 $2\pi$ factors

We've seen  $2\pi$  and  $1/2\pi$  factors appear all over the place in Fourier transform formulae and got headaches trying to remember them. They all stem from the fact that we use *angular* frequency  $\omega$  instead of *cyclic* frequency,  $f$ , where  $\omega = 2\pi f$ . We view angular frequency as being more "natural," but many practical problems use cyclic frequency, so we need to remember when to add in factors of  $2\pi$ . With this convention, we saw that the synthesis and analysis equations for the CT and DT Fourier transforms become:

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega && \text{(CT synthesis, inverse CTFT)} \\ X(j\omega) &= \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt && \text{(CT analysis, CTFT)} \\ x[n] &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega && \text{(DT synthesis, inverse DTFT)} \\ X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} && \text{(DT analysis, DTFT)} \end{aligned}$$

The  $2\pi$  factor is manifested in the following FT pairs and properties:

- Value of a signal at zero time

$$\begin{aligned} x(0) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) d\omega \\ x[0] &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) d\omega \end{aligned}$$

- Constant signal

$$\begin{aligned} x(t) = 1 &\xleftrightarrow{\mathcal{F}} X(j\omega) = 2\pi\delta(\omega) \\ x[n] = 1 &\xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l) \end{aligned}$$

- Complex exponentials

$$\begin{aligned} x(t) = e^{j\omega_0 t} &\xleftrightarrow{\mathcal{F}} X(j\omega) = 2\pi\delta(\omega - \omega_0) \\ x[n] = e^{j\omega_0 n} &\xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l) \end{aligned}$$

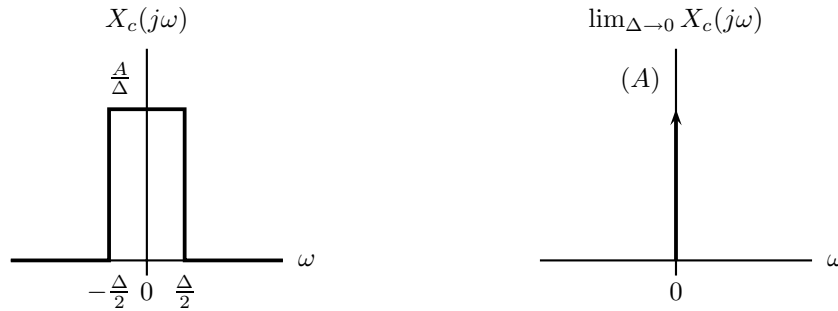
- Multiplication property

$$\begin{aligned} r(t) = s(t)p(t) &\xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(j\theta) P(j(\omega - \theta)) d\theta = \frac{1}{2\pi} \{S(j\theta) * P(j\theta)\} \\ r[n] = s[n]p[n] &\xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi} \int_{2\pi} S(e^{j\theta}) P(e^{j(\omega - \theta)}) d\theta = \frac{1}{2\pi} \{S(e^{j\theta}) \circledast P(e^{j\theta})\} \end{aligned}$$

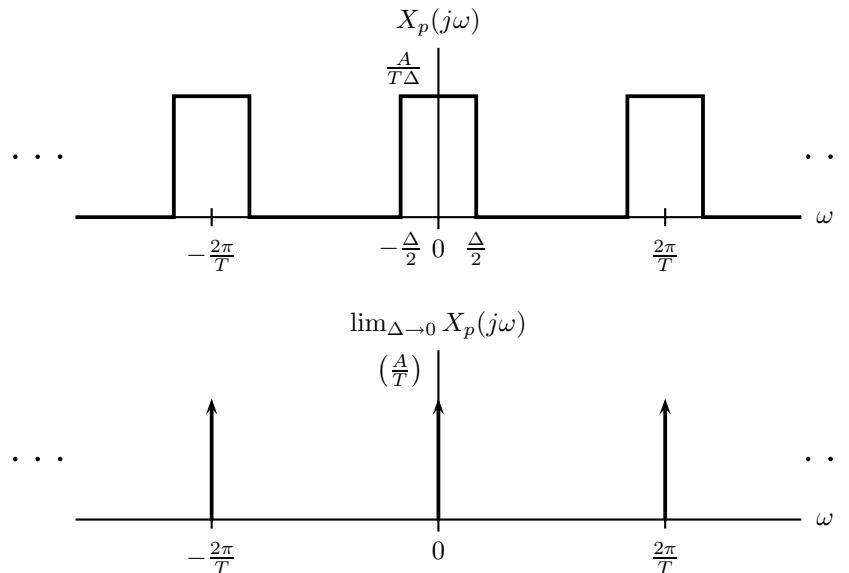
## 2 Sampling Pitfalls

### 2.1 Impulses in the frequency domain

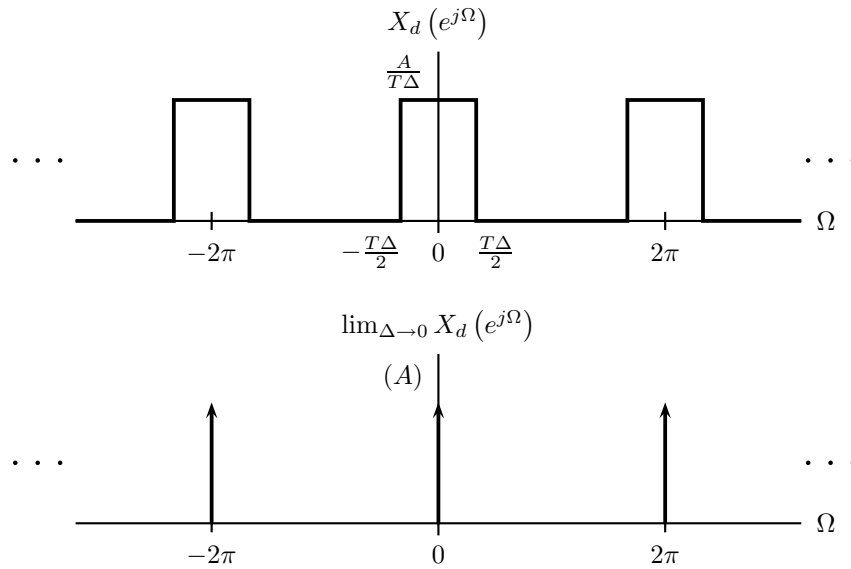
We need to be careful when there are impulses in the frequency domain. Let's consider the simplest case where  $X_c(j\omega) = A\delta(\omega)$  (so  $x_c(t) = A/(2\pi)$ ) and we sample  $x_c(t)$  with period  $T$ . Recall that we label impulses by the *area* obtained when they are integrated. I shall follow the convention of surrounding that value with a pair of parentheses to remind us of this. To aid us in the process, let's consider the impulse  $A\delta(\omega)$  to be the limit as  $\Delta \rightarrow 0$  of a rectangular pulse with width  $\Delta$  and height  $A/\Delta$ . For our purposes here, it does not matter how this pulse is centered:



Thus,  $X_p(j\omega)$  has the height  $A/(T\Delta)$ , retains the width  $\Delta$ , and is replicated; the area is now  $A/(T\Delta) \cdot \Delta = A/T$ :



Finally,  $X_d(e^{j\Omega})$  retains the height  $A/(T\Delta)$ , but its width is  $T\Delta$ ; the area is now  $A/(T\Delta) \cdot T\Delta = A$ :



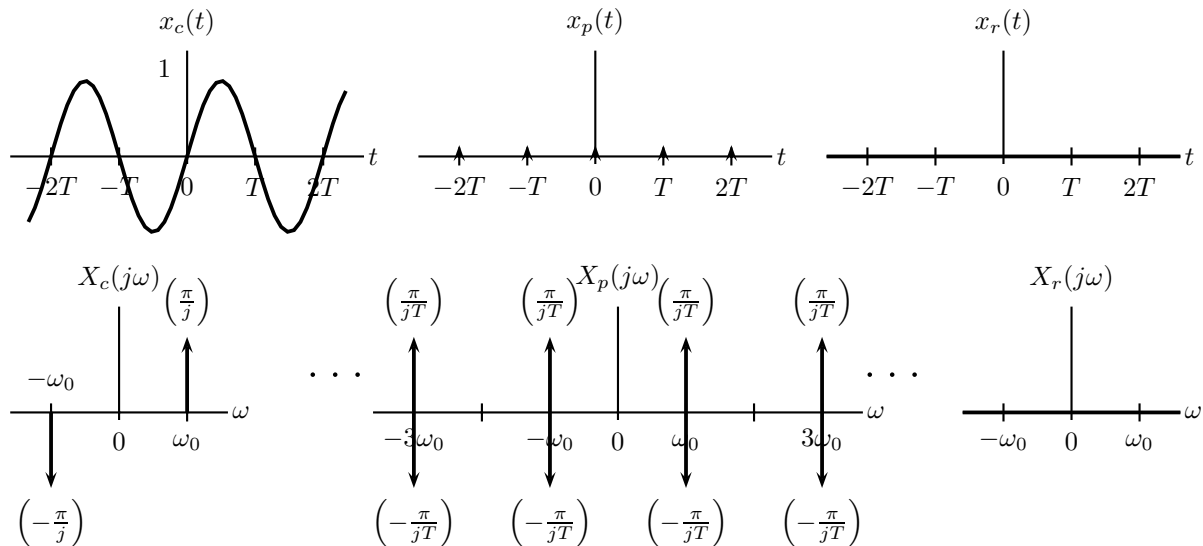
But wait a minute... didn't we say earlier that C/D conversion scaled down the FT by  $T$ ? The impulse apparently breaks our rules. But it actually follows them, because *we label a impulse by its area, NOT by its height, which is infinite*. Since C/D conversion preserves the Fourier area:

**Area-Invariance of Impulses in the Frequency Domain:**

The labeling of an impulse in the frequency domain does not change when a CT signal is converted into a DT sample sequence.

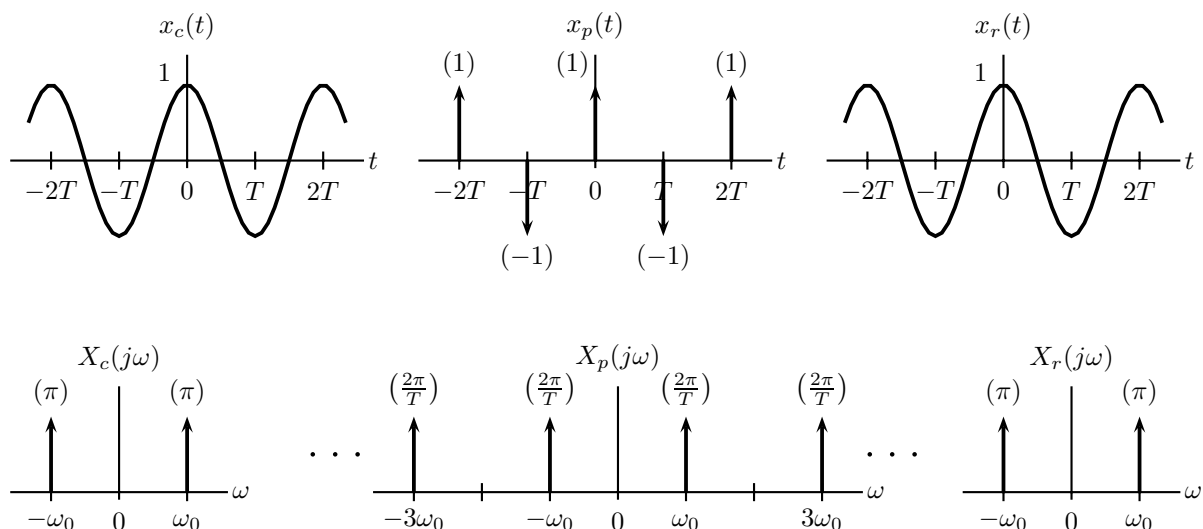
## 2.2 Do we really have to sample at the Nyquist rate?

It was said earlier that we must sample at *strictly* greater than twice the highest frequency present in a signal to recover it. However, there exist signals where sampling at *exactly* twice that frequency is sufficient. Let's look at two related examples, which will not only illustrate this point, but will also highlight the fact that we cannot ignore phase when we analyze signals. First, let's try to sample  $x_c(t) = \sin(\omega_0 t)$  at  $\omega_s = 2\omega_0$ .



As we can see in both the time and frequency domains, the interpolation is zero; the two impulses in the sine have opposing signs and exactly cancel each other out by aliasing. Since we did not satisfy Nyquist, we should not be surprised that this happened.

Now let's consider  $x_c(t) = \cos(\omega_0 t)$ :



Something subtle happened here. As usual, we performed bandlimited interpolation by lowpass filtering with cutoff frequency  $\omega_s/2$ , which is  $\omega_0$  in this case. But what happens *exactly* at that frequency: do we filter out the whole thing, or do we keep the whole impulse (area  $2\pi$ )? Our figures for filters were always ambiguous as to what the value of the filter was at the cutoff frequency. We never really cared about this sort of boundary case before, but here it is important. The time domain picture resolves this issue: it shows that we end up with the cosine again. Since the frequency domain and time domain pictures must agree and produce the same result, this means that we compromise and keep *half* of each impulse on either side! This example confirms that:

**The Value of Signals at Discontinuities:**

Whenever Fourier analysis is used, we should consider the value of a signal at discontinuities (in both time and frequency) to be the mean of its limits from both sides. However, we can usually ignore this fact.

### Problem 9.1

(From the 6.003 Fall 2002 Quiz 2)

Answer each of the following short answer questions. All parts of this problem are independent.

- (a) Evaluate  $\int_{-\infty}^{\infty} \frac{\sin^2(4\pi t)}{\pi^2 t^2} dt$ .
- (b) Consider a discrete-time system with input  $x[n]$  and output  $y[n]$  which are related by the following difference equation:

$$y[n] = 3x[n] - x[n-1] - \frac{1}{2}x[n-2].$$

What is the value of the frequency response at  $\omega = 0$ ?

- (c) Suppose that two continuous-time signals  $x(t)$  and  $y(t)$  have Fourier transforms  $X(j\omega)$  and  $Y(j\omega)$  that are bandlimited (i.e.  $X(j\omega) = Y(j\omega) = 0$  for  $|\omega| \geq \omega_m$  where  $\omega_m$  is a given real number). Is  $x(t)y(t)$  bandlimited?
- (d) If  $h[n]$  is the impulse response of a lowpass filter, does  $h_1[n] = (-1)^n h[n]$  correspond to the impulse response of a lowpass, highpass, or bandpass filter?
- (e) Suppose that  $x[n]$  is a purely imaginary discrete-time signal with Fourier transform  $X(e^{j\omega})$ . Is  $\text{Re}\{X(e^{j\omega})\}$  an even function of  $\omega$ , an odd function of  $\omega$ , or neither?

(Work space)



**Problem 9.2**

You want to design a system whose impulse response has the form:

$$h(t) = u(t) - u(t - k).$$

Find one possible value of  $k > 0$  such that when we input  $x(t) = \cos(\pi t)$ , the output will be  $y(t) = 0$ .  
Hint: Think about the zero crossings of a sinc function.

(Work space)

**Problem 9.3** Sketch the Bode plot for the magnitude and phase of the following CT LTI system

$$H(j\omega) = \frac{\left(\frac{j\omega}{10^2} - 1\right) \left(\frac{j\omega}{10^3} - 1\right)}{j\omega \left(\frac{j\omega}{10} + 1\right) \left(-\frac{j\omega}{10^4} + 1\right)}$$

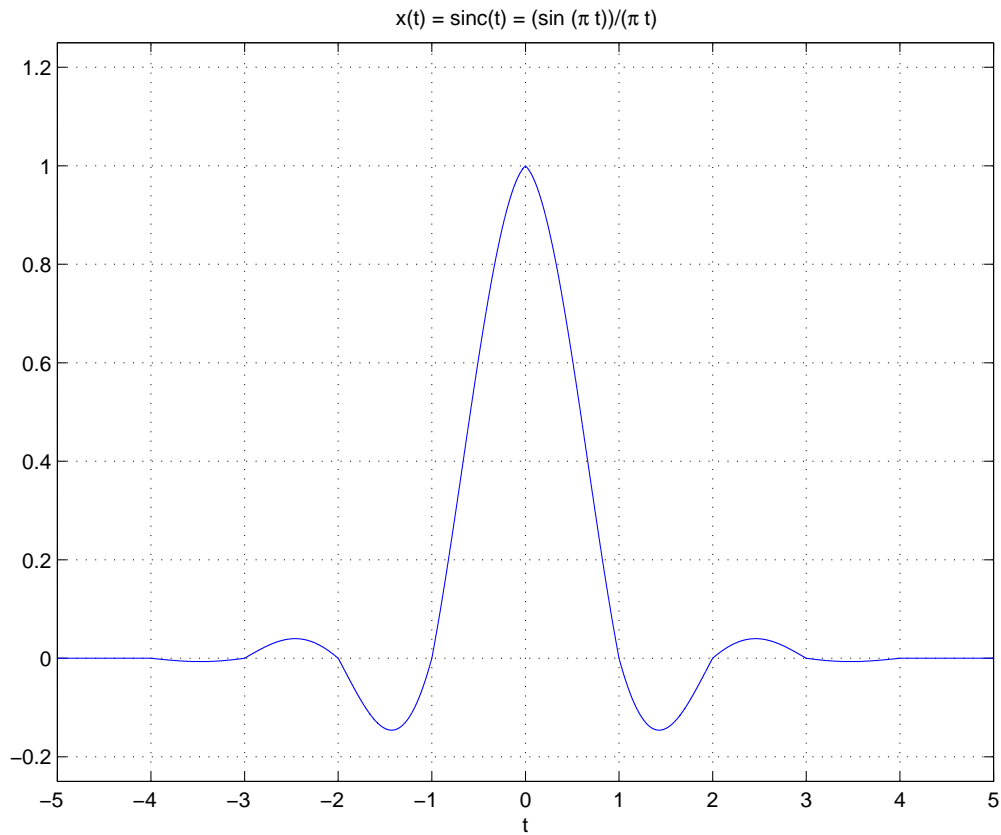
### Problem 9.4

Consider a stable and causal CT LTI system whose input  $x(t)$  and output  $y(t)$  are related by the following *difference* equation:

$$y(t) - \frac{1}{2}y(t-1) = x(t).$$

The input  $x(t)$  is:

$$x(t) = \text{sinc}(t) = \frac{\sin \pi t}{\pi t}.$$



- (a) Sketch the output  $y(t)$ .
- (b) Determine an analytic expression for the output  $y(t)$ .

Hint: It may be easier to think in terms of DT processing of CT signals by converting the CT input to DT, performing some equivalent DT operation, then converting the DT output to CT.

(Work space)

**Problem 9.5**

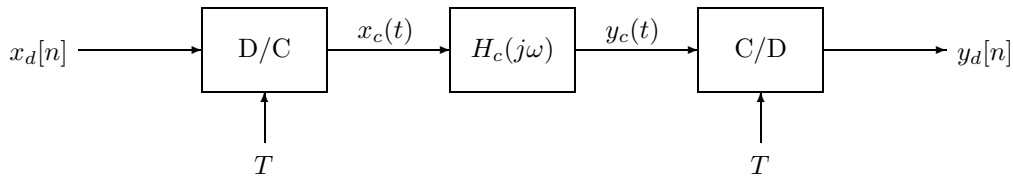
Let's look at how neat CT processing of DT signals is. Consider the following DT frequency response:

$$H_d(e^{j\Omega}) = \begin{cases} j\Omega, & |\Omega| < \pi \\ \text{periodic} & \end{cases}$$

It can be shown by taking the inverse Fourier transform of the frequency response (you should verify this yourself; takes integration by parts) that the impulse response  $h[n]$  is:

$$h[n] = \frac{(-1)^n}{n}.$$

Now, consider the following system:



Let's see how we can use this to find the DT impulse response  $h[n]$ . For the rest of the problem, let the DT input be a unit sample:

$$x_d[n] = \delta[n].$$

- Find the frequency response of the embedded CT system  $H_c(j\omega)$  so that the overall DT system is equivalent to the original DT system. What a common name for this system? Find a time-domain equation that relates  $x_c(t)$  and  $y_c(t)$ .
- Find the output of the D/C converter  $x_c(t)$ . Hint: With what do we replace DT impulses to get a CT interpolation?
- Find the output of the CT system  $y_c(t)$ . Hint: This is most easily done with the time-domain equation from the first part.
- Find the output of the C/D converter  $y_d[n]$ . Check that this is the same as  $\frac{(-1)^n}{n}$ .

(Work space)