MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

TUTORIAL 8

Monday, April 5 and Tuesday, April 6, 2004

Announcements

- Problem set 7 is due this Friday.
- Quiz 2 will be held on **Thursday, April 15**, 7:30–9:30 p.m. in Walker Memorial. The quiz will cover material in Chapters 1–7 of O&W through Section 7.4, Lectures and Recitations through April 2, Problem Sets #1–6, and the part of Problem Set #7 involving problems from Chapter 7.
- The TAs will jointly hold office hours from 2–8 p.m. on Wednesday, April 14 and again from 10 a.m.–3 p.m. on Thursday, April 15. A schedule will be posted on the 6.003 website.
- A quiz review package will be available on the 6.003 website this Thursday. TAs will hold two identical optional quiz review sessions on Monday, April 12 and Tuesday, April 13, 7:30–9:30 p.m. in 34-101.

Today's Agenda

- Introduction to Sampling
- Sampling of CT Signals
  - CT impulse sample train
  - DT sample sequence
- Reconstruction of CT Signals: The Nyquist Sampling Theorem
  - Reconstruction from the CT impulse sample train
  - Reconstruction from the DT sample sequence
  - Viewing reconstruction in the time domain
- Summary and Subtleties of DT Sampling of CT Signals
- DT Processing of CT Signals
- CT Processing of DT Signals
  - Non-integer time shifting of DT signals
- Zero- and First-Order Holds
- DT Sampling and Interpolation
  - Sampling (decimation, downsampling)

- Interpolation (upsampling)
- Changing the sampling rate by a rational noninteger factor
- Changing the sampling rate by general real factor

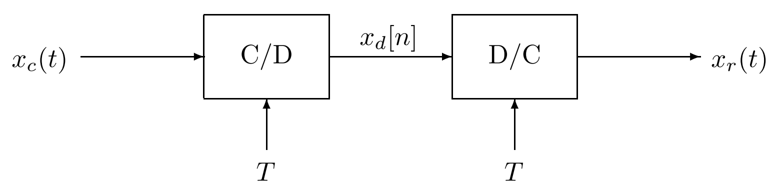
- Sampling in the Frequency Domain and Sampling Dualities (Optional)
  - Continuous-to-discrete dualities
  - The ultimate duality: the dual of the Nyquist sampling theorem
  - Discrete-to-discrete dualities
- Dimensional Analysis (Optional)
  - Units of time and signal values
  - Units of the signal value of the impulse response
  - Units of the Fourier transform
  - Fourier transform of the impulse train
  - Converting between CT and DT during the sampling process
  - Scaling by 1/T

1 Introduction to Sampling

Sampling is our first introduction to combining CT and DT signals and systems. So far, we were able to distinguish between CT and DT signals because the variables for time were different: CT uses t and DT uses n. Although both used ω for frequency, this was never a problem because we were always clear as to whether we were in CT or DT. However, we will be dealing with both CT and DT signals now, so it will be useful to use lower-case omega (ω) for CT frequency and upper-case omega (Ω) for DT frequency. Unfortunately, this convention is not standard. Sometimes[1], the notation is reversed!

2 Sampling of CT Signals

Consider the following block diagram, which consists of a *continuous-to-discrete*, or *C/D* converter and a corresponding *discrete-to-continuous*, or *D/C* converter, each with sampling period T and corresponding sampling (angular) frequency $\omega_s = 2\pi/T$. We have a CT signal $x_c(t)$ being converted into the DT signal $x_d[n]$, which is then converted into the reconstructed CT signal $x_r(t)$:



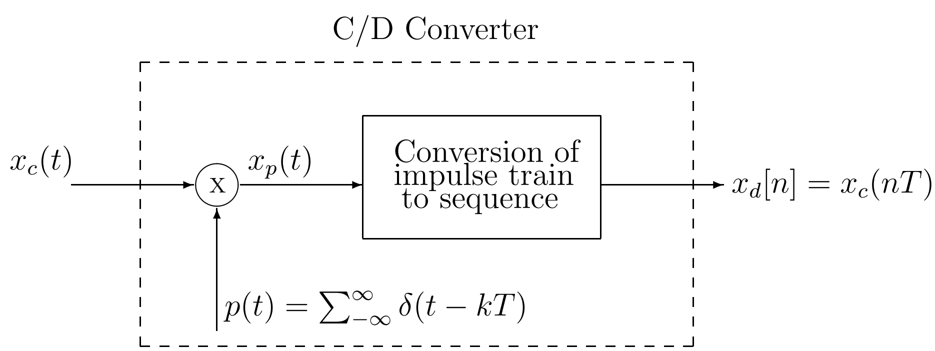
The C/D converter sets the value of the DT signal to the value of the CT signal at integer multiples of T:

$$x_d[n] \quad = \quad x_c(t)|_{t=nT} = x_c(nT).$$

We will look later at how the D/C converter works.

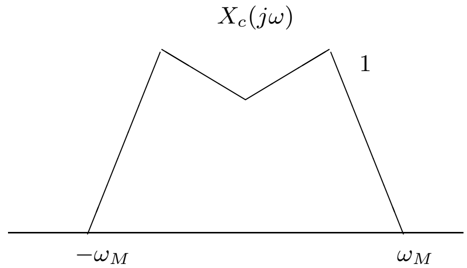
2.1 CT impulse sample train

Let's decompose the C/D converter into a muliplication with an impulse train $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$ and an *impulse-to-sequence*, or *I/S*, component. We call the intermediate CT signal $x_p(t) = x_c(t)p(t)$. Keep in mind that this decomposition is strictly a *mathematical idealization*. This helps us understand how the signals and their transforms evolve through the system, even though the actual physical implementation of such systems may be different:



[1] such as in 6.341 Discrete-Time Signal Processing.

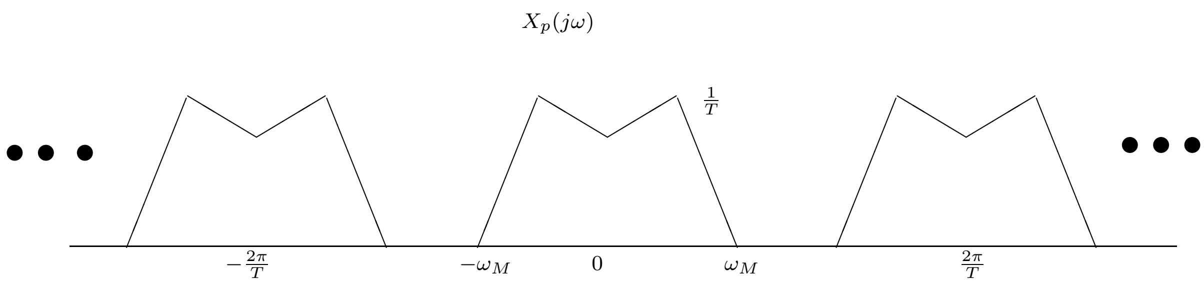
If $x_c(t)$ is *bandlimited*, meaning it only contains Fourier components within a finite frequency interval, then its CTFT $X_c(j\omega)$ may look like:



The CTFT of an unit impulse train with spacing T is also an impulse train, but with spacing ω_s and area $2\pi/T$ for each impulse. Since multiplication in time is convolution in frequency:

> $X_p(j\omega)$, the FT of $x_p(t)$, is repeated versions of $X_c(j\omega)$ scaled down by T and spaced by ω_s.

This looks like:



2.2 DT sample sequence

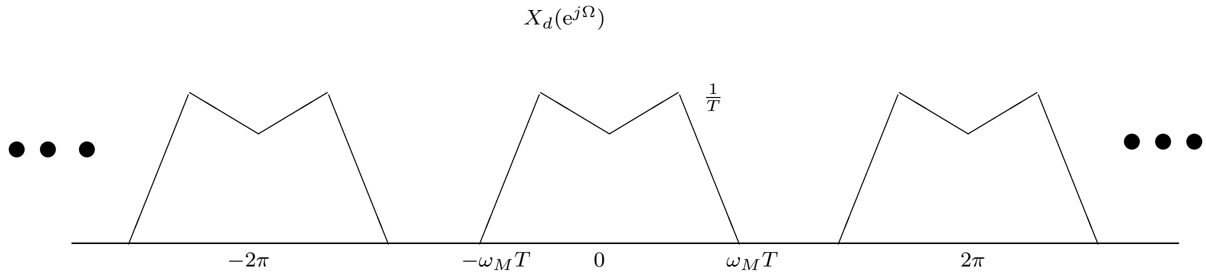
We can then translate the CT impulse samples into DT samples using the I/S converter so that $x_d[n] = x_c(nT)$. It can be shown (see textbook) that the DTFT of the DT sample sequence is related to the CTFT of the CT impulse sample train by:

$$X_d(e^{j\Omega}) = X_p(j\Omega/T)$$

So:

> $X_d(e^{j\Omega})$, the FT of the DT sample sequence, is $X_p(j\omega)$, the FT of the CT impulse sample train, evaluated at $\omega = \Omega/T$. This is simply a scaling of the frequency axis by T.

This looks like:

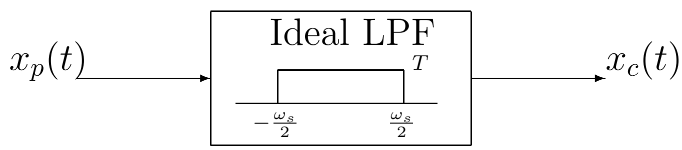


3 Reconstruction of CT Signals: The Nyquist Sampling Theorem

We now have two types of samples of the original CT signal $x_c(t)$: a CT impulse sample train $x_p(t)$ and a DT sample sequence $x_d[n]$. In general, we cannot recover a signal from its samples, for an infinite number of signals share identical sample sequences. However, if we restrict ourselves to *bandlimited* signals, then if we sample at a sufficiently fast rate, the original signals can be recovered exactly. Let's look at how we can produce a CT reconstruction $x_r(t)$ from each type of samples.

3.1 Reconstruction from the CT impulse sample train

Inspection of the frequency representation of the impulse sample train suggests using a low-pass filter with gain T to retain the original FT pattern and restore its magnitude. This produces the following block diagram:



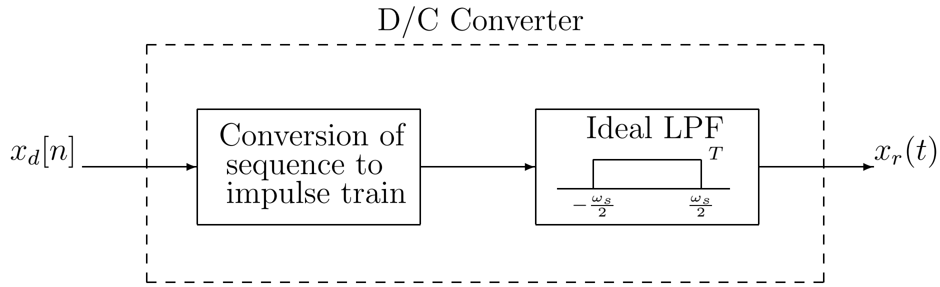
However, this works only if there is no overlap, or *aliasing*, of the original FT spectrum, which occurs when ω_s is too low. This leads to the *Nyquist Sampling Theorem*:

> **Nyquist Sampling Theorem:**
>
> Let $x_c(t)$ be a bandlimited signal such that it has no frequency components for $|\omega| > \omega_M$. Then $x_c(t)$ can be uniquely reconstructed from its samples $x_c(nT)$ (or $x_d[n]$) if the sampling frequency satisfies $\omega_s > 2\omega_M$.

The sampling frequency must be *strictly* greater than twice the highest frequency present in the signal. The limit $2\omega_M$ is called the *Nyquist rate*.

3.2 Reconstruction from the DT sample sequence

Now that we solved the problem of obtaining $x_r(t)$ from the CT impulse sample train, we can just convert the DT sample sequence back into that CT impulse sample train and continue the reconstruction process as done before. To do this, we conceptually decompose the D/C converter into a *sequence-to-impulse*, or *S/I*, converter followed by the ideal reconstruction filter above, namely the low-pass filter. This entire block is also known as the *ideal reconstruction system*:



All the steps we went through before now occur in reverse, so, if we sample the bandlimited signal fast enough, the reconstruction $x_r(t)$ is the same as the original signal $x_c(t)$.

3.3 Viewing reconstruction in the time domain

We found it straightforward to analyze bandlimited reconstruction in the frequency domain. But what does it look like in the time domain? Since the reconstruction filter $H_r(j\omega)$ is a low-pass filter (LPF), or a square pulse in frequency, its impulse response $h_r(t)$ is a sinc in time.

We have "pseudo-convolution" in time between the DT sample sequence $x_d[n]$ and the impulse response $h_r(t)$ to obtain $x_r(t)$. We'll call it "pseudo-convolution" because we are "convolving" a DT signal with a CT one:

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x_d[n] h_r(t - nT)$$

where

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

This leads to a relationship between the time representations of the DT sample sequence and the bandlimited reconstructed signal:

The Time-Domain Interpolation Formula:

The CT bandlimited interpolation $x_r(t)$ is constructed from the DT sample sequence $x_d[n]$ with sample period T using the formula:

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x_d[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}$$

This is one of the most interesting results in 6.003! Let's examine it further to see what it really means. Suppose $x_d[n]$ were a shifted impulse $\delta[n - n_0]$. Then, the summation would only "sift out" the value of this expression at $n = n_0$. So:

A DT shifted impulse $\delta[n - n_0]$ interpolates with sampling period T to the CT sinc function $\frac{\sin(\pi(t-n_0T)/T)}{\pi(t-n_0T)/T}$, which has its peak at $t = Tn_0$ and is zero when t is any other integer multiple of T.

Because any DT signal can be represented as the linear combination of shifted impulses, linearity implies that the interpolation formula can be interpreted as:

> **The Time-Domain Interpolation Formula in Plain English:**
>
> The CT bandlimited interpolation of a DT sample sequence with sampling period T is done by replacing each DT impulse with a CT sinc of the same height with zeros at integer DT times and superimposing the sincs.

So now, we understand exactly what it means to interpolate a DT signal to a CT one. But why does it makes sense that the CT bandlimited interpolation of a DT impulse is a sinc? Why isn't the CT version of the DT impulse a CT impulse? Well, "bandlimited" means we want to use *as low frequencies as possible* to build the CT signal. The *only* way to get a CT signal that happens to hit zero at all the other integer multiples of T *and still remain bandlimited* is to use a sinc. Another way of thinking about this is to go back to the frequency domain. The FT of an impulse (in both DT and CT) is a constant, so it contains all frequencies in equal amounts. To translate this DTFT into the CTFT of a bandlimited signal, we need to truncate the high-frequency parts, leaving a centered low-frequency rectangular pulse, which we know is a sinc in time.

4 Summary and Subtleties of DT Sampling of CT Signals

We looked at the following system, which converts a CT signal $x_c(t)$ into a DT signal $x_d[n]$ (sampling), then back into a reconstructed CT signal $x_r(t)$ (interpolation).

Sampling was a straightforward operation: $x_d[n] = x_c(t)$. However, it was not obvious how to reconstruct a CT signal from a DT sample sequence. We found, according to the Nyquist sampling theorem, that if $x_c(t)$ is bandlimited, or its Fourier transform is zero for frequencies above some ω_M, and if we sampled at at least twice the highest frequency present, then we could reconstruct $x_c(t)$ perfectly using *bandlimited interpolation*. This process was most easily visualized in the frequency domain. To aid in this process, we *mathematically decomposed* the C/D and D/C processes into two steps each: C/D is a CT impulse sampling followed by impulse-sequence (I/S) conversion and D/C is a S/I conversion followed by lowpass filtering.

We can summarize the frequency picture:

1. **CT bandlimited signal to CT impulse sample train:** CTFT is replicated at every integer multiple of ω_s and is scaled down by T.

2. **CT impulse train to DT sample sequence:** Frequency axis is scaled by T: $\Omega = \omega T$. Height is unchanged.

3. **DT sample sequence to CT impulse sample train:** Frequency axis is scaled by $1/T$: $\omega = \Omega/T$. Height is unchanged.

4. **CT impulse train to CT bandlimited signal:** CTFT is lowpass filtered with cutoffs at $\pm\omega_s/2$ and gain T.

Since the breakdown of the C/D and D/C blocks were simply aids to help us develop the previously mentioned relationships, *we can skip the CT impulse sample train completely and jump straight from a CT bandlimited signal to the DT sample sequence, and vice versa.*

Another thing to watch out for:

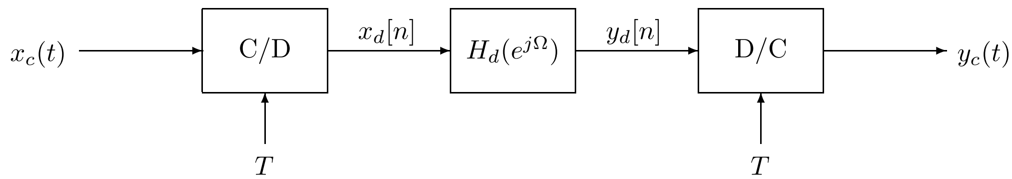
Linearity and Time-Variance of DT Processing of CT Signals:

DT processing of CT signals is *always a linear process*, but is, in general, *time-varying* (not time-invariant). However, if the input CT signals are restricted to be bandlimited and the sampling rate is faster than Nyquist, then the overall CT system is LTI. These two are *sufficient but not necessary* conditions.

Thus, it is possible to sample slower than Nyquist yet still have an overall LTI system.

5 DT Processing of CT Signals

By inserting a DT system in our original C/D-D/C block diagram, we obtain a mechanism to perform *DT processing of CT signals*:



Although this is in general a time-varying system, under certain conditions, then the entire system is a CT LTI system. The following describes this remarkable result:

The Effective CT Frequency Response of DT Systems:

If $x_c(t)$ is bandlimited and the sampling rate is above the Nyquist rate, then the entire CT system is LTI, and the *effective* frequency response of the CT system $H_c(j\omega)$ above is related to the frequency response of the DT system $H_d(e^{j\Omega})$ by:

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| \leq \pi/T = \omega_s/2 \\ 0, & |\omega| > \pi/T \end{cases}$$

Thus, we replace Ω in the DT frequency response with ωT and restrict the frequency range to the frequency spectrum of the bandlimited CT signal. It also turns out that:

Impulse-Imvariance of CT and DT Systems:

The impulse response $h_d[n]$ of the DT system is a scaled, sampled version of the impulse response $h_c(t)$ of the CT system:

$$h_d[n] = Th_c(nT)$$

In other words, the DT system is an *impulse invariant* version of the CT system.

Many of the problems you'll likely encounter on sampling will involve DT processing of CT signals. When working them out, try to keep the dicreteness-periodicity duality in your mind. We'll be switching back and forth between CT and DT, and also between time and frequency domains, so it's helpful to use rules we developed before to double-check your progress.

As a reminder, here is the duality:

The Dicreteness-Periodicity Duality:

Something that is discrete in one domain is periodic in the other domain, and something that is continuous in one domain is aperiodic in the other domain.

Problem 8.1

Let $x_c(t) = \sin(75\pi t)/(\pi t)$. The signal $x_c(t)$ is sampled every T seconds to produce the discrete time signal $x_d[n] = x_c(nT)$. The signal $x_d[n]$ is the input to a discrete LTI system with impulse response $h_d[n]$, where $h_d[n] = \sin(\pi n/4)/(\pi n)$. The output of the discrete time system $y_p(t)$ is converted into an impulse train $y_p(t) = \sum_{m=-\infty}^{\infty} y_d[m]\delta(t - mT)$. Finally the signal $y_p(t)$ is the input to a low pass filter with impulse response $h_L(t) = \sin(100\pi t)/(\pi t)$. The output of this low pass filter is $y_c(t)$. For each of the cases below, find $y_c(t)$.

(a) T = 1/50

(b) T = 1/100

(c) T = 1/200

(d) T = 1/125

(Workspace)

Problem 8.2 (From the 6.003 Spring 1999 Final Exam.)

Consider the following sampling system:

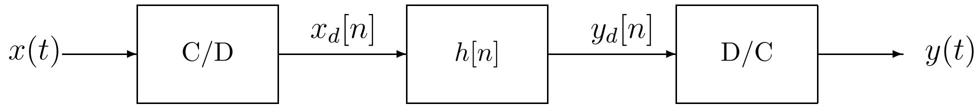
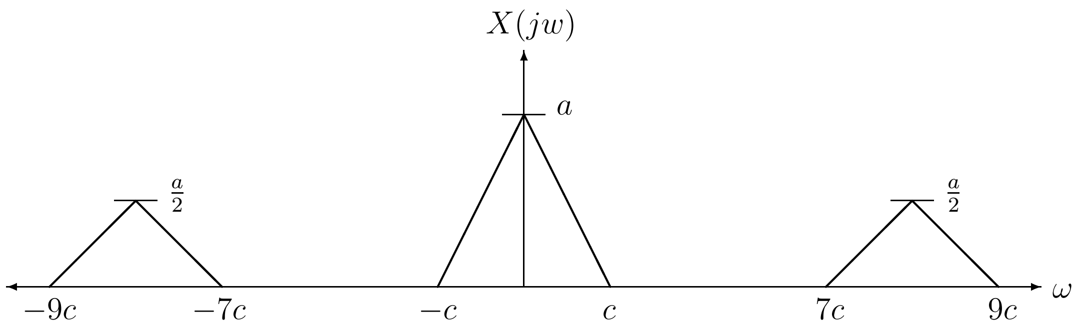
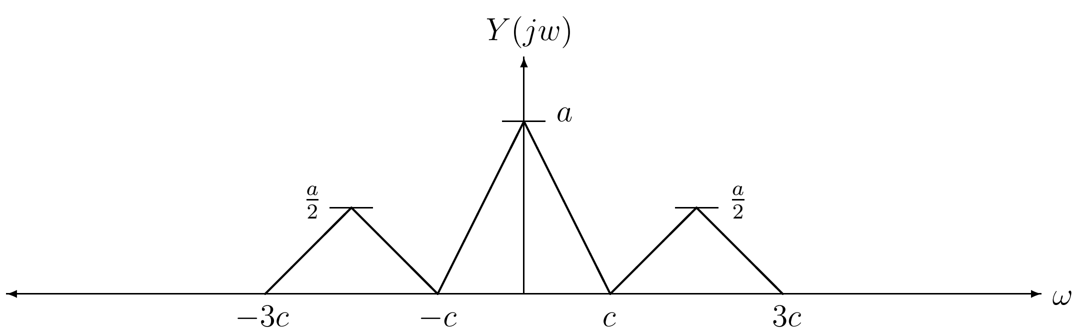


Figure 1: Sampling system.

The input $x(t)$ has real-valued Fourier Transform $X(j\omega)$ shown below.



Is it possible to adjust the sampling period T so that the Fourier transform $Y(j\omega)$ of the output $y(t)$ has the following shape?



If yes, determine the sampling period T and the impulse response $h[n]$ as a function of the parameters of $X(j\omega)$. If no, explain why not.

(Workspace)

Problem 8.3

Consider the standard system of DT processing of CT signals as shown in the previous problem, where the sampling period is T. Say the DT system satisfies the difference equation:

$$y_d[n] - \frac{1}{2}y_d[n-1] + 3y_d[n-2] = x_d[n].$$

(a) Find the frequency response $H(e^{j\Omega})$ of the DT system.

(b) Under what conditions does the entire system behave as a CT LTI system?

(c) Assuming these conditions are met, find the frequency response $H(j\omega)$ of the overall CT system.

(d) Assuming these conditions are met, find an equation relating $x(t)$ and $y(t)$. How does this equation relate to the difference equation above?

(Workspace)

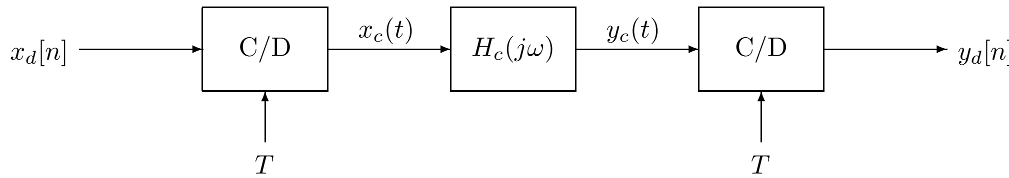
Problem 8.4 (From the 6.341 Fall 2002 Background Exam)

Consider a system that performs DT processing of CT signals where the sampling frequency is 16 kHz. The DT filter is an ideal lowpass filter with a cutoff of $\pi/2$ rad/sample. What should the input $x_c(t)$ be bandlimited to so that the overall system is LTI?

(Workspace)

6 CT Processing of DT Signals

Likewise, we can also perform CT processing of DT signals by reversing the order of the C/D and D/C converters and inserting a CT system between them. This is not a practical thing to do in real-life, but it is an interesting exercise to think about to help us understand certain problems:



However, a conceptual difficulty that doesn't happen in the other direction arises: In DT processing of CT signals, we "took away" information by sampling. But in CT processing of DT signals, we are "adding in" extra information. What does it mean to create a CT signal from a DT one? After all, that DT signal is not a sequence of samples of some CT signal, so we are not reconstructing that CT signal. But the D/C converter doesn't know that! So:

> The CT signal $x_c(t)$ produced by a D/C converter from the DT signal $x_d[n]$ is the bandlimited CT signal of which $x_d[n]$ is the DT sample sequence. We can *pretend* that $x_d[n]$ came from a bandlimited CT signal, and that the D/C converter outputs the reconstruction of this CT signal.

We saw that DT processing of CT signals are linear and time-varying, but the corresponding result is more straightforward for CT processing of DT signals:

> **Linearity and Time-Invariance of CT Processing of DT Signals:**
>
> CT processing of DT signals is *always* a linear time-invariant (LTI) process.

The impulse responses are also related: the DT impulse response is a scaled, sampled version of the CT impulse response, and the CT impulse response is a scaled, bandlimited interpolation of the DT impulse response, a relationship known as *impulse invariance*.[2] We'd like to emphasize that these statements reflect the *equivalence* of processing with and without C/D and D/C converters; there is *no mathematical difference whatsoever*.

Despite the usefulness of that equivalence, tricky problems tend to have different T's for C/D and D/C and may also violate the Nyquist condition, so one would have to follow the transformation of the process step-by-step.

[2]For the second part, we need to add the condition for CT processing DT signals that the embedded CT impulse response is appropriately bandlimited.

Similarly, we have:

Effective DT Frequency Response of CT Systems:

The *effective* frequency response of the entire DT system $H_d(e^{j\Omega})$ above is related to the frequency response of the CT system $H_c(j\omega)$ by:

$$H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right), \quad |\Omega| \leq \pi,$$

with $H_d(e^{j\Omega})$ periodic with period 2π (like all DTFTs).

This is a powerful result: this means that we can decompose *any* DT LTI system into a D/C converter followed by some CT LTI system followed by a C/D converter. This can help us understand using CT how some DT systems work. Non-integer time delay is a beautiful example of this.

Problem 8.5

Consider the standard system of CT processing of DT signals, where the sampling period is T. Say the CT system satisfies the differential equation:

$$\frac{d^2}{dt^2}y_c(t) + 2\frac{d}{dt}y_c(t) - y_c(t) = x(t)$$

- Find the frequency response $H(j\omega)$ of the CT system.
- Find the frequency response $H(e^{j\Omega})$ of the overall DT system. How can we interpret this?

(Workspace)

6.1 Non-integer time shifting of DT signals

The most elegant application of CT processing of DT signals is how to perform a non-integer time shift in DT signals. Back in chapter 1, we saw how to time delay a DT signal $x[n]$ by some integer amount n_0 to obtain $x[n-n_0]$. This process makes no sense when n_0 is not an integer. However, it would be nice if we had some way of *interpreting* a non-integer time delay of Δ to obtain "$x[n-\Delta]$." The expression is in quotes because it is sloppy notation, but we know its intended meaning. As a first pass, we can try using *zero- and first-order holds.*

In this cases, these forms of non-integer delay are acceptable. However, let's try to build the concept of non-integer delay by looking at a CT system that implements time delay in DT systems and is embedded between a D/C and a C/D converter. We begin with the frequency response we'd like to implement, which we can obtain by extrapolating the frequency for integer delay, which is:

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta}, \quad |\Omega| \leq \pi.$$

where Δ is an integer that indicates the shift in time. Thus, we *expect* the DT frequency response of a system that shifts a DT signal by non-integer Δ to be the same. Thus, the the frequency response $H_c(j\omega)$ of the embedded CT system is:

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta_c}, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

where

$$\Delta_c = T\Delta.$$

This is a familiar system to us: it is the CT system that delays $x_c(t)$ by Δ_c, when $x_c(t)$ is bandlimited to ω_c! This leads to the following way of understanding non-integer delay:

Non-integer time delay in DT signals: We can interpret the time shifting of a DT signal $x[n]$ by any *real* (not necessarily integer) amount Δ to $y[n] =$ "$x[n-\Delta]$" as the DT sample sequence of the CT signal resulting from a time shift by Δ_c in the bandlimited CT interpolation of $x[n]$, where the CT time shift Δ_c is $\Delta_c = T\Delta$ and T is the sampling rate. In other words, we can conceptually break down non-integer time delay of a DT signal into three steps:

1. Do a bandlimited interpolation into CT.
2. Shift the CT signal by the corresponding CT time delay.
3. Sample back into DT.

Mathematically, we have:

$$y[n] = x[n] * h[n],$$

where

$$h[n] = \frac{\sin \pi(n-\Delta)}{\pi(n-\Delta)}.$$

When $\Delta = n_0$ is an integer, then this reduces to $h[n] = \delta[n - n_0]$, which we recognize as the impulse response of a DT time shift system. Note that the impulse response of such a system is a DT *sampled* version of the CT sinc (impulse invariance).

At this point, one may ask, "Why use bandlimited interpolation? Why can't I use zero- or first-order hold (discussed below) as my interpolation in the process of obtaining non-integer delay?" Well, *only* the bandlimited interpolation leads to a mathematically consistent analysis using Fourier transforms.

It can be shown that the Fourier transform of $h[n]$ is $H_d(e^{j\Omega})$, which is *precisely* what we need for this interpretation to be the "correct" one! In fact, this is a beautiful approach to finding the inverse Fourier transform of $e^{-j\Omega\Delta}$ for non-integer Δ. In fact, this proves that our interpretation of non-integer time shifting as the DT sample sequence of the CT signal resulting from a time shift in the bandlimited CT interpolation of the original signal is the *true* method of performing this operation.

There a slight conceptual difference in the approaches followed by the 6.003 textbook (pages 543 - 545) and the 6.341 textbook (pages 164 - 165) in developing this idea of non-integer delay. The former uses the DT processing of CT signals model. It asks what impulse response for the embedded DT system would produce a shift in the CT input by a non-integer multiple of T. The latter uses the CT processing of DT signals model. It asks what overall DT impulse response would give rise to the desired frequency response. Although it is not essential in 6.003 to know both, we encourage studying both to deepen your understanding of converting between CT and DT.

7 Zero- and First-Order Holds

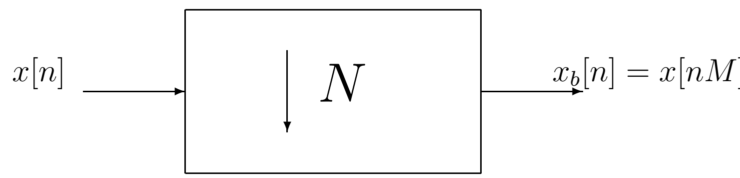
We mentioned before that bandlimited interpolation is not the only conceivable method of producing a CT signal out of a DT sample sequence. The two simplest methods, the zero-order and first-order holds, are used all the time in some form or another. The zero-order hold "holds" the value of the sample for the duration of the sample period following the sample. Digital pictures and pixels are a form of zero-order hold. First-order hold is when we "connect the dots."

This is what we did when we were told by our kindergarten teachers to complete the outline of drawings. Both types of holds introduce a distortion in the frequency picture of the signal, and it is possible to recover from it. In problem 8 (problem 7.50 from the textbook) of problem set 8, you were asked to try this.

8 DT Sampling and Interpolation

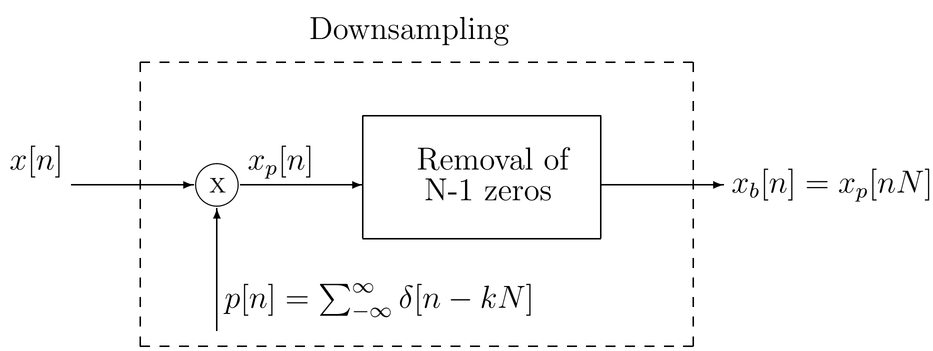
8.1 Sampling (decimation, downsampling)

We can also sample, or *decimate*, DT signals; this is process by which we take every Nth sample in a DT signal $x[n]$ to obtain $x_b[n] = x[nN]$. We will use the following block, known as a *compressor*, to indicate DT sampling by a factor of N:

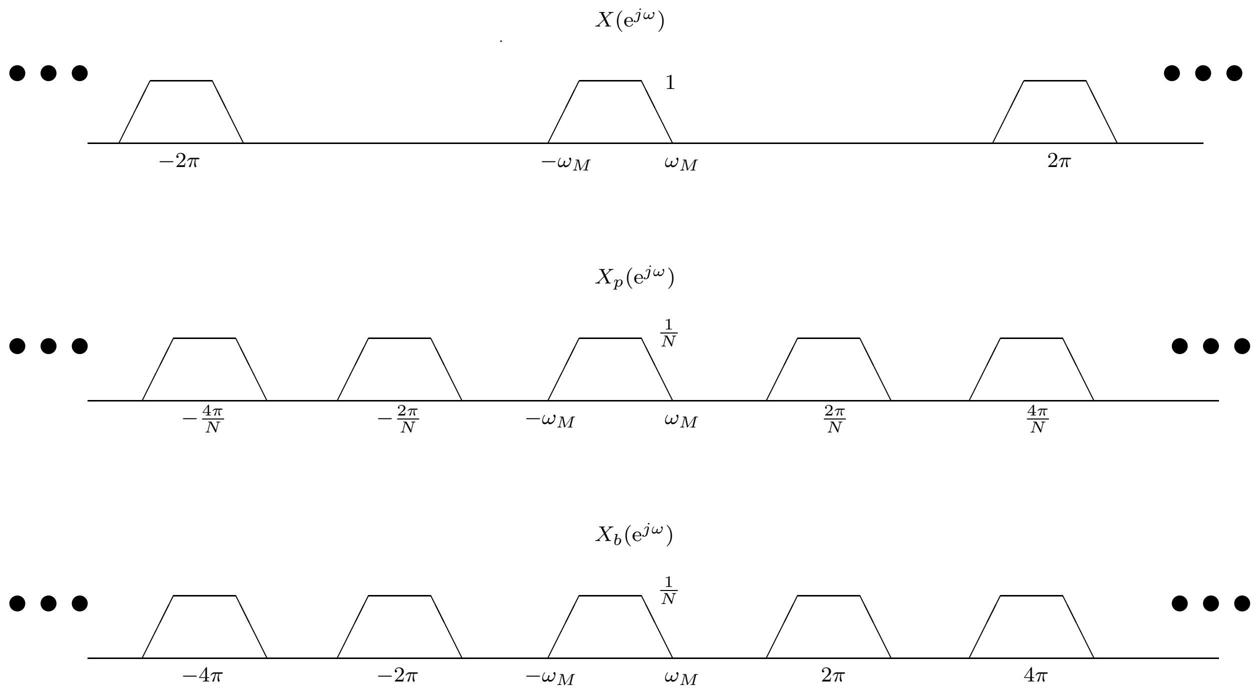


We know that even if the DT signal $x[n]$ were not obtained from sampling a CT signal, we could perform a bandlimited interpolation using sampling period T to produce the CT signal $x_c(t)$ and *pretend* that $x[n]$ came out by sampling $x_c(t)$. Thus, we can interpret decimation by N as *building a CT bandlimited interpolation using sampling period T and sampling it using a new sampling period $T' = NT$*. This picture motivates the term *downsampling* as yet another synonym for DT sampling and decimation.

As in the CT case, we can view compression as the combination of two steps: multiplication of $x[n]$ by an impulse train $p[n]$ to obtain $x_p[n] = x[n]p[n]$ and removing the $N-1$ zeroes between each multiple of N to obtain $x_b[n] = x_p[nN]$:



The frequency picture is:



We see the same mechanics as before coming up; there's replication of transforms at $\Omega_s = 2\pi/N$ accompanied by a reduction in the height by N, then there's a rescaling of the frequency axis by N. Note that since the values of all three signals $x[n]$, $x_p[n]$, and $x_b[n]$ at zero time are the same, the areas under the transforms over 2π are also the same, which parallels the intuition we developed with CT sampling. As is always the case, there may be aliasing, which would prevent perfect reconstruction of the original $x[n]$ from the samples $x_b[n]$. We see that aliasing does not occur, and therefore reconstruction is possible (using a lowpass filter), when *Nyquist's sampling theorem for DT signals* holds:

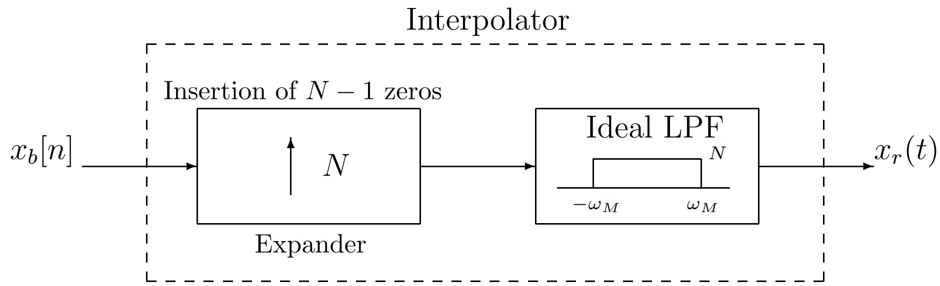
Nyquist Sampling Theorem for DT Signals:

Let $x[n]$ be a bandlimited DT signal such that it has no frequency components for $|\Omega| > \Omega_M$. Then $x[n]$ can be uniquely reconstructed from its samples $x[nN]$ (or $x_b[n]$) if the sampling frequency $\Omega_s = 2\pi/N$ satisfies $\Omega_s > 2\omega_M$.

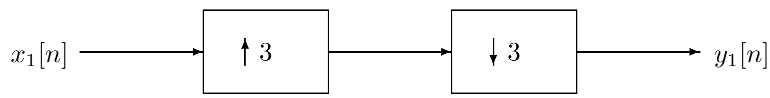
There's something that may bother some students about the time and frequency pictures. In the time domain, it seems as though the operation of going from $x[n]$ to $x_b[n]$ is exactly the same as going from $x_p[n]$ to $x_b[n]$; both take samples of the signals at multiples of N. But the frequency operations are totally different. Why? This issue is resolved by the realization that *the first transformation takes out information whereas the second one takes out samples known to be zeroes.* The second one is the only one of the two that can truly be interpreted as rescaling the time axis, which leads to the inverse scaling in the frequency axis.

8.2 Interpolation (upsampling)

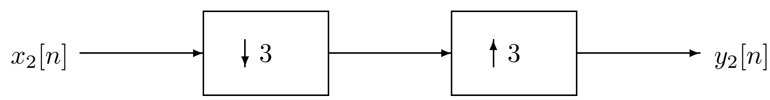
We can run the downsampling steps in reverse in a process called *interpolation*. Like decimation, we can interpret this as resampling a CT bandlimited interpolation of the original DT signal at a higher sampling rate, so we also call the process *upsampling*. The following diagram shows an *interpolater*, which consists of an *expander* and a lowpass filter. The operation of the expander is deceiving; it simply inserts zeroes between the samples. We need the lowpass filter to turn those zeroes into values of the interpolation:



Be careful about how the expander and compressor work. For example, the following system has the output equal to the input for all inputs:



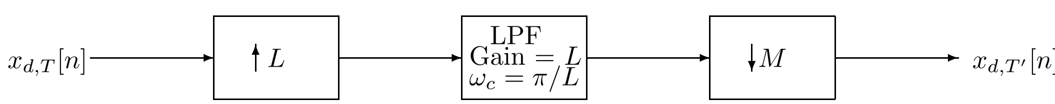
but the following system does not have this property:



In the latter system, there is no lowpass filter after the expander, so the output has a bunch of zeros between non-zero values. Furthermore, x_2 may not be bandlimited properly, so it may be impossible to restore it after downsampling, even with a lowpass filter after the expander.

8.3 Changing the sampling rate by a rational noninteger factor

Suppose we had a DT signal $x_{d,T}[n]$ that is the sample sequence of a bandlimited CT signal $x_c(t)$ with sampling period T, where T is small enough to satisfy Nyquist. Can we to obtain the samples of $x_c(t)$ (call it $x_{d,T'}[n]$) using a different sampling period $T' = T \cdot M/L$ from $x_{d,T}[n]$ alone (*i.e.* without going back to $x_c(t)$), where M and L are integers? Yes! We can accomplish this by upsampling with factor L and then downsampling with factor M:



8.4 Changing the sampling rate by general real factor

What if we wanted to change the sampling rate by any real (not necessarily rational) factor? It is impossible to implement this with upsampling and downsampling alone; we would have to convert to continuous time and resample at the new frequency.

9 Sampling the Frequency Domain and Sampling Dualities (Optional)

We looked at sampling a signal in *time*; let's explore sampling in *frequency*. This means that we take samples of the Fourier transform of the signal instead. Consider the CT signal $x_c(t)$ with corresponding Fourier transform $X_c(j\omega)$. As with sampling in time, we can sample $X_c(j\omega)$ at intervals of ω_s by multiplying it by an impulse train $P(j\omega)$ with spacing ω_s and impulse area $2\pi/T$. The corresponding signal $p(t)$, of course, is the impulse train in time with spacing $T = 2\pi/omega_s$ and unit impulse area. Multiplication in frequency is convolution in time.

9.1 Continuous-to-discrete dualities

Notice that many of these steps look just like those in sampling in time, only with time and frequency swapped. The sampling step in both types of sampling creates replicas of the signal represented in the *other* domain so that it becomes periodic. If there is no aliasing, we can recover the original countinuous-variable signal from the discrete-variable sequence by keeping only one of the replicas, or by *windowing* (which is lowpass filtering for frequency). This leads to the following dualities:

The Sampling-"Periodization" Duality:

Sampling (discretization) in one domain is "periodization" in the other domain.

The counterpart of this statement is:

The Interpolation-"De-Periodization" Duality:

Bandlimited interpolation ("continuous-ization") in one domain is windowing ("de-periodization") in the other domain.

In light of what we know already, these dualities are consistent with the discreteness-periodicity duality: something that is discrete in one domain is periodic in the other and something that is continuous in one domain is aperiodic in the other. Thus, when we sample the *continous*-variable (ω) Fourier *transform* of an *aperiodic* signal, we obtain a *discrete*-variable (k) Fourier *series* of a *periodic* signal. Time-domain sampling is similar: Sampling the *continous*-variable (t) signal with an *aperiodic* transform produces a *discrete*-variable (n) signal with a *periodic* transform.

Actually, we have seen sampling and interpolation in frequency before, though we didn't realize it at the time. In the beginning of chapter 4, in the development of the concept of the Fourier transform, we began with at the Fourier series of a periodic signal, and took the limit as the period went infinity. This had the effect of the Fourier series coefficients representing closer and closer samples of what turned out be the Fourier transform. This is sampling!

Why are these dualities true anyway? They stem from the fact that *the Fourier transform of an impulse train is another impulse train.* The dualities exist only when the operation we are interested in (sampling in this case) involve working with signals whose transforms are of the same form as the signals themselves. The Gaussian is the only other signal that has this property. Thus, sampling is a very special operation, for it is one that exhibits these dualities.

9.2 The ultimate duality: the dual of the Nyquist sampling theorem

We can reconstruct the Fourier transform when there is no aliasing in the time domain. This happens when the following conditions are satisfied:

> **The Dual of the Nyquist Sampling Theorem:**
>
> The Fourier transform of a finite-length CT or DT signal can be exactly recovered from samples of the the Fourier transform, provided the sampling frequency $\omega_s = 2\pi/T$ exceeds the length of the signal.

9.3 Discrete-to-discrete dualities

We just looked at sampling a continuous variable (frequency ω) to a discrete one (k). We can also consider sampling a discrete variable to another discrete one, just like with downsampling. For the downsampling process, note that the transform of $x_p[n]$, which has zeros at locations that are not multiples of some integer N is periodic with period $2\pi/N$. However, like all DTFTs, we consider its period to be 2π. This means we're not using the proper fundamental period. Where have we seen this before? Earlier, we showed that the effect of using a non-fundamental period in *time* for periodic signals corresponded to zero insertion in the Fourier series. Thus, we have:

> **Duality of Zero Insertion and Nonfundamental Periods:**
>
> Zero-insertion in one (discrete) domain means using a non-fundamental period in the other (periodic) domain.
>
> The counterpart of this statement is:
>
> **Duality of Zero Deletion and Fundamental Periods:**
>
> Zero-deletion in one (discrete) domain means using a fundamental period in the other (periodic) domain.

Also:

> **Duality of Upsampling and Compression:**
>
> Discrete upsampling in one (discrete) domain means compression without changing the period in the other (periodic) domain.
>
> The counterpart of this statement is:
>
> **Duality of Downsampling and Expansion:**
>
> Discrete downsampling in one (discrete) domain means expansion without changing the period in the other (periodic) domain.

Problem 8.6 (Extended from the 6.003 Fall 2002 Quiz 2 Review Packet)

We are given a signal $x[n]$ that is symmetric about $n_0 = 4$, and 0 for $n \leq 1$ and $n \geq 7$.
Let $X\left(e^{j\omega}\right)$ denote the discrete-time Fourier transform of $x[n]$. We can write

$$X\left(e^{j\omega}\right) \quad = \quad A\left(e^{j\omega}\right) e^{j\phi(\omega)},$$

where $A\left(e^{j\omega}\right)$ is real (possibly positive or negative) and $\phi(\omega)$ is some function of ω. We found that

$$\phi(\omega) = -4\omega$$

Thus, $X\left(e^{j\omega}\right)$ has linear phase. Let's extend this concept and determine which of the following statements are true and which ones are false. Assume that all these signals are real in the time domain. (Hint: Think about bandlimited interpolation and non-integer delay...)

(a) **TRUE** **FALSE** If a CT signal $x_a(t)$ is symmetric about t_0 (*i.e.*, $x_a(t_0 - t) = x_a(t_0 + t)$ for all t), then its FT $X_a(j\omega)$ has linear phase $-\omega t_0$.

(b) **TRUE** **FALSE** If the FT $X_b(j\omega)$ of a CT signal $x_b(t)$ has linear phase $-\omega t_0$, then $x_b(t)$ is symmetric about t_0.

(c) **TRUE** **FALSE** If a DT signal $x_c[n]$ is symmetric about n_0, where n_0 is an integer or half-integer, then its FT $X_c(e^{j\omega})$ has linear phase $-\omega n_0$.

(d) **TRUE** **FALSE** If the FT $X_d(e^{j\omega})$ of a DT signal $x_d[n]$ has linear phase $-\omega\Delta$, where Δ is a real number, then $x_d[n]$ is symmetric about Δ.

10 Dimensional Analysis (Optional)

What's a section on dimensional analysis doing in topics that are disconnected with physical reality? Well, I'm obsessive about units, and I find myself assigning physical units to quantities in the idealized math world for many problems I come across to understand the concepts and to provide some intuition. I found that this sort of thinking to be most useful for sampling and converting between CT and DT, so I'll introduce it this week. If you also have a dimensional analysis habit, you may find the following helpful.

10.1 Units of time and signals values

We call the the independent variable in signals *time*, so we allow it to represent physical time. Let's stick with SI units, and assign *seconds* (s) for the CT variable t. In DT, time is unitless, so we simply call a DT time unit a *sample*. These assignments lead to the frequency units: cycles per second, or *Hertz* (Hz), for CT f, cycles per sample for DT F, radians per second (s^{-1}) for CT ω, and radians per sample for DT Ω. Note that CT frequency is in "per time," but DT frequency is unitless.

We have treated the values of signals as unitless real numbers. We will keep it this way, though in real life, signals may take on units of meters, Volts, Amperes, *etc.*

10.2 Units of the signal value of the impulse response

In DT, the impulse has unit height, so it is unitless, just like DT signals. So, the DT impulse response $h[n]$ is also unitless. However, a CT impulse has unit *area*, not unit height. Since the "width" is in units of time, or seconds, the "height" must be in units of per time, or s^{-1}. Thus, the CT impulse $h(t)$ has units of s^{-1}. These unit assignments are consistent with the convolution formulae:

$$
\begin{aligned}
y(t) &= \int x(\tau)h(t-\tau)d\tau \\
y[n] &= \sum_k x[k]h[n-k]
\end{aligned}
$$

In DT, $x[n]$ and $y[n]$ are both unitless, so $h[n]$ is unitless. In CT, we also want $x(t)$ and $y(t)$ to be unitless, but we are integrating over time ($d\tau$), so $h(t)$ must be in units of per time.

10.3 Units of the Fourier transform

From the definition of the CT and DT Fourier transforms (see page 72 of the tutorial notes), the units of the CTFT $X(j\omega)$ must be the units of the signal $x(t)$ times time, and the units of the DTFT $X(e^{j\Omega})$ must be the same as the units of the signal $x[n]$. Since we will keep signals unitless, the CTFT normally has units of time, or s, and the DTFT normally is unitless.

When the signal is the impulse response, then the Fourier transform is the frequency response. Combining what we know already, the DT frequency response $H(e^{j\Omega})$ is unitless, and the CT frequency response $H(j\omega)$ has units of per time times time, which is also unitless. This is consistent with the fact that the FT of the output of a system is the product of the FT of the input with the frequency response.

10.4 Fourier transform of the impulse train

We saw earlier that the FT of the CT unit periodic impulse train with spacing T was also an impulse train, but with strength $2\pi/T$ and spacing $\omega_s = 2\pi/T$. If we only remembered the impulse-train-to-impulse-train relationship but forgot those constants, how can we re-derive them using dimensional analysis alone? Well, the canonical impulse train, which has unit area for each impulse, must have units of per time for the height. Thus, its CTFT has units of per time times time, so it is unitless. But we label the strength of impulses by their area. Since the "width" is in units of CT frequency, or per time, and the "height" is unitless, the area is also in units of per time. This is most easily done by setting it to $1/T$. Now, recall from the "loose ends from last week" section on page 73 that "canonical impulses in the frequency domain have a 2π factor." So, we need to tack on a 2π also, which gives us $2\pi/T$ as the strength of the FT impulses. What about the spacing? We know that the spacing must be in terms of angular frequency. The most natural way of creating this is by using $2\pi/T$. This whole analysis may sound a bit hand-wavy, and it is, but it just shows how far one can get by using dimensional analysis alone.

10.5 Converting between CT and DT during the sampling process

We can also "derive" several of the relationships between the Fourier transforms of the various stages of signals in the sampling process. For instance, why is the FT of the DT sample sequence scaled down by a factor of T? We concluded before that CTFTs were in units of time and DTFTs were unitless. Thus, we need to divide by time to maintain these units. The only "time factor" we have is T, so that's the scale factor. The frequency relationship $\Omega = \omega T$ also makes sense from the dimensional analysis point of view; Ω is in radians per samples and ω is in radians per time, so we need to multiply CT frequency by a time factor (once again, T) to get DT frequency.

10.6 Scaling by $1/T$

Why is there a scaling of the value of the FT by $1/T$ when we sample? Well, by dimensional analysis, we know that the frequency axis must be scaled by T: Ω is in units of radians per sample and ω is in units of radians per second, so the conversion factor must be in seconds per sample, or the amount of time between samples. This is, of course, T:

$$\begin{aligned} \Omega &= \omega \cdot T \\ \frac{\text{radians}}{\text{sample}} &= \frac{\text{radians}}{\text{second}} \cdot \frac{\text{seconds}}{\text{sample}} \end{aligned}$$

Now, recall the zero-time formulae, which state that *the value of a signal at zero time is the area under the Fourier transform (aperiodic for CT, periodic for DT) divided by 2π:*

$$\begin{aligned} x_c(0) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_c(j\omega)\, \mathrm{d}\omega \\ x_d[0] &= \frac{1}{2\pi} \int_{2\pi} X_d(e^{j\Omega})\, \mathrm{d}\Omega \end{aligned}$$

Since $x_d[n]$ is a sampled version of $x_c(t)$, we must have $x_c(0) = x_d[0]$. Therefore:

Area-Invariance in the Frequency Domain During Sampling:

The area under the Fourier transform of the CT signal is equal the the area under one period of the Fourier transforms of the corresponding DT sample sequence. In other words, the Fourier transform area is invariant under sampling.

This is the generalization of the subtlety we saw earlier with impulses in the frequency domain. Since we *multiplied* the frequency axis by T due to a change in units, we must correspondingly *divide* the value of the transform by T to preserve the area. This is the intuition behind the $1/T$ factor.

The same analogy holds for downsampling in discrete time.