MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

Tutorial 7

Monday, March 29 and Tuesday, March 30, 2004

Announcements

- Problem set 6 is due this Friday.

Today's Agenda

- Additional Notes on Partial-Fraction Expansion
- Interpretation of Phase in the Frequency Response
  - Linear phase and constant time shift
  - Nonlinear phase and group delay
- First- and Second-Order Systems
- Bode Plots
  - First-order system Bode plots
  - Second-order system Bode plots

1 Additional Notes on Partial-Fraction Expansion

Last week, we saw that to do partial-fraction expansion on rational functions with repeated roots in the denominator, there was a complicated formula we could apply. However, this is confusing and non-intuitive. Fortunately, there is another method that is easy to follow.

Let's say we want to do a partial-fraction expansion on the following function with repeated roots in the denominator:

$$F(x) = \frac{2x+1}{(x+3)^3(x+2)}.$$

The partial-fraction expansion will be in the form

$$\frac{2x+1}{(x+3)^3(x+2)} = \frac{A_0}{(x+3)^3} + \frac{A_1}{(x+3)^2} + \frac{A_2}{(x+3)} + \frac{B}{(x+2)}.$$

We can determine that $A_0 = 5$ and $B = -3$ using last week's method (i.e., the cover up method). However, determining A_1 and A_2 using cover up results in the left side becoming $\frac{2x+1}{0} = \infty$. Fortunately, we can easily fix this by subtracting the highest-order term from both sides:

$$\frac{2x+1}{(x+3)^3(x+2)} - \frac{5}{(x+3)^3} = \frac{A_1}{(x+3)^2} + \frac{A_2}{(x+3)} + \frac{-3}{(x+2)}.$$

Simplifying the left side, we see that the order of the denominator has been reduced:

$$\frac{-3}{(x+3)^2(x+2)} = \frac{A_1}{(x+3)^2} + \frac{A_2}{(x+3)} + \frac{-3}{(x+2)}.$$

We can now determine that $A_1 = 3$ using cover up as before. How do we determine A_2? You guessed it. Just as before, we can subtract the highest-order term from both sides to reduce the order of the denominator. Using cover up, we can now determine that $A_2 = 3$. Therefore, the partial-fraction expansion of $F(x)$ is

$$\frac{2x+1}{(x+3)^3(x+2)} = \frac{5}{(x+3)^3} + \frac{3}{(x+3)^2} + \frac{3}{(x+3)} + \frac{-3}{(x+2)}.$$

So in general, to do partial-fraction expansion on rational functions with repeated roots in the denominator, determine all the highest-order residuals (A_0, B_0, C_0, etc.) using the cover up method, subtract the highest-order terms from both sides, simplify, and then recurse.

2 Interpretation of Phase in the Frequency Response

In past weeks we've seen that examining the *magnitude* of a system's frequency response $|H(j\omega)|$ allows us to intuit the behavior of the system by matching it to one of several filter types. (For review, those filter types were lowpass, highpass, bandpass, bandstop, and all-pass.) We'll now see that we can also interpret the *phase* of a system's frequency response $\angle H(j\omega)$.

2.1 Linear phase and constant time shift

We know from the Fourier transform representation of signals that certain CT signals can be expressed as superpositions of complex exponentials of the form $e^{j\omega t}$ (without loss of generality, we'll use CT notation here). Suppose such a signal $x(t)$ has Fourier transform $X(j\omega)$. We know from the transform tables that the FT of the shifted version of the signal $x(t-t_0)$ is $e^{-j\omega t_0}X(j\omega)$. But if we lost our tables and even forgot the formula for the Fourier transform, how can we re-derive this? And what does it mean?

Each of the complex exponentials $e^{j\omega t}$ has a cosine in the real component and a sine in the imaginary component. If we time delay the entire signal $x(t)$ by t_0, then each exponential would also be delayed by the same amount in *time*. However, they would *not* by shifted by the same amount in *phase*. Let's consider the cosine component $x_c(t) = \cos(\omega t)$. A time shift produces $x_c(t - t_0) = \cos(\omega(t - t_0)) = \cos(\omega t - \theta)$, where $\theta = \omega t_0$ is the *phase shift* for that signal. Note that the phase added is proportional to the frequency of the cosine. This is consistent with our idea that a constant time shift corresponds to a larger phase shift for high-frequency components than it does for low-frequency ones (recall the diagrams drawn in recitation and tutorial). So, the complex exponential $e^{j\omega t}$ gets mapped to $e^{j\omega(t-t_0)} = e^{j\omega t}e^{-j\omega t_0}$. Since the Fourier transform $X(j\omega)$ of a signal is simply the scale factor (ignoring the 2π factor...) that sits in front of the complex exponentials when they are integrated together to form $x(t)$, this means that a time shift multiplies the FT by $e^{-j\omega t_0}$. But we know that the phase of a complex number is the θ in the $e^{j\theta}$ portion of the number in the magnitude-phase representation $z = re^{j\theta}$. Multiplying complex numbers adds their phases, so we have:

A constant shift in time corresponds to the addition of a linear phase in frequency.

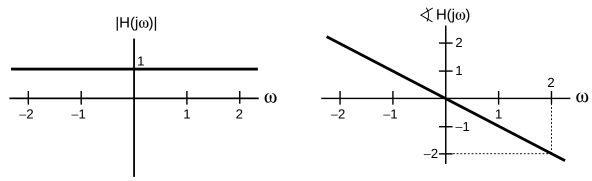
Problem 7.1

The magnitude and phase of the frequency response of six different all-pass systems are plotted below. Given the input

$$x(t) = \cos(t) + \cos(2t),$$

find an expression for the output of each system and summarize (in words) the behavior of each system.

(a)



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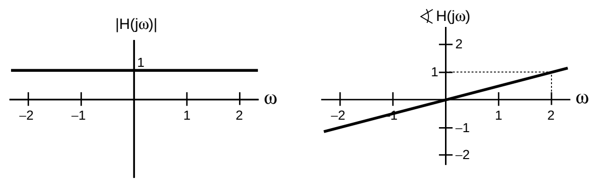

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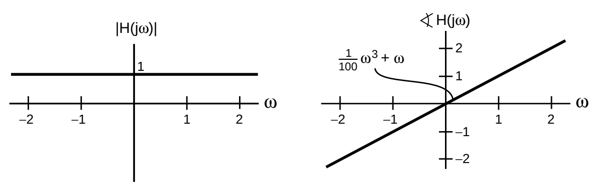

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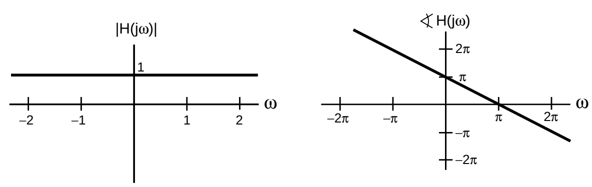

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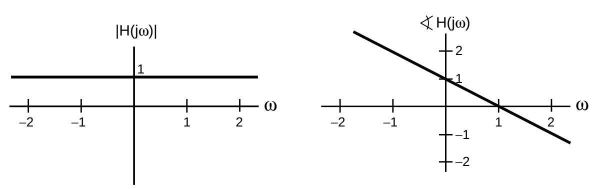
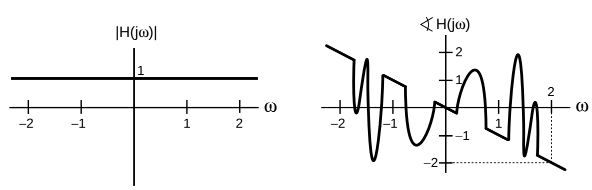


(f)



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2.2 Nonlinear phase and group delay

As we saw in Problem 7.1, when the phase of the frequency response $\angle H(j\omega)$ is linear, the system simply shifts the input by an amount equal to the slope, $t_0 = -\frac{d}{d\omega}\{\angle H(j\omega)\}$. When the phase of the frequency response is *nonlinear*, however, the behavior of the system is difficult to the interpret because the system really messes up the signal. In engineering lingo, one would say, "Systems with nonlinear phase cause distortion."

So at this point, its appears that $\angle H(j\omega)$ can be interpreted as either a simple shift or a messy distortion. Surely there is some middleground that might appear on an exam, right? Well, there is a special case of interest when the input is narrowband. In short, when the input is narrowband, nonlinear phase can be approximated as linear phase. Part (f) of Problem 7.1 hints at how this narrowband approximation works. More specifically, if the frequency content of the input is tightly grouped around some frequency $\omega = \omega_0$ (i.e., the input is narrowband with center frequency $\omega = \omega_0$), $\angle H(j\omega)$ over the range of frequencies we're interested in can be accurately approximated by the line tangential to $\angle H(j\omega)$ at $\omega = \omega_0$. Therefore, the system simply shifts the input by an amount equal to the slope at $\omega = \omega_0$, $\tau_g(\omega_0) = -\frac{d}{d\omega}\{\angle H(j\omega)\}|_{\omega=\omega_0}$. Since $\tau_g(\omega_0)$ tells us how much the *group* of frequencies near ω_0 gets *delayed*, $\tau_g(\omega)$ is referred to as "group delay."

The following defines the terms *phase delay* and *group delay* of CT frquency responses and explains their interpretation for narrowband signals:

Phase Delay and Group Delay in Narrowband Signals:

The *phase delay* of a CT frequency response $H(j\omega)$ is:

$$\tau_p(\omega) = -\frac{\angle H(j\omega)}{\omega}.$$

The *group delay* of that system is:

$$\tau_g(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\angle H(j\omega).$$

Let $x(t)$ be a narrowband signal of the form:

$$x(t) = s(t)\cos(\omega_0 t + \phi),$$

where $s(t)$ is a sufficiently slowly varying signal (*i.e.* its frequency componenents are at frequencies much smaller than ω_0). If $x(t)$ is the input to a LTI system with frequency response $H(j\omega)$, then a good approximation of the output signal $y(t)$ is:

$$y(t) \approx |H(j\omega_0)| s(t - \tau_g(\omega_0)) \cos(\omega_0(t - \tau_p(\omega_0)) + \phi).$$

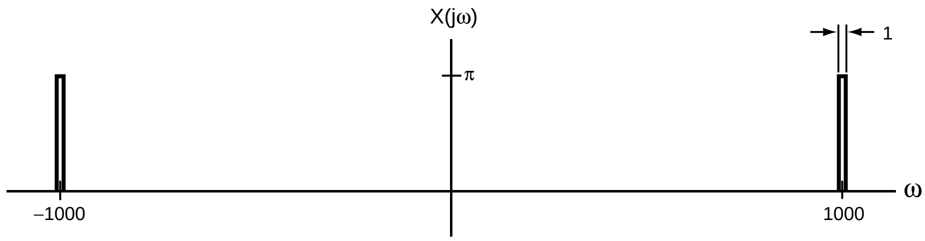
In other words, the *envelope* $s(t)$ is delayed by the group delay at ω_0, the *carrier* is delayed by the phase delay at ω_0, and the overall *amplitude* is scaled by the magnitude of the frequency response at ω_0.

Problem 7.2

Let the input signal $x(t)$ be a sinusoid whose amplitude is slowly varied:

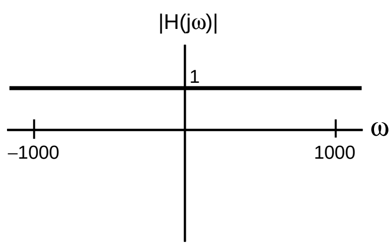
$$x(t) = v(t)\cos(1000t),$$

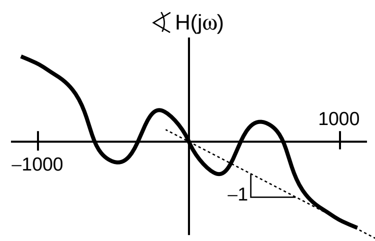
where $v(t)$ is called the "envelope" of the sinusoid. $X(j\omega)$ is plotted below. Note that $x(t)$ can be considered a narrowband signal.



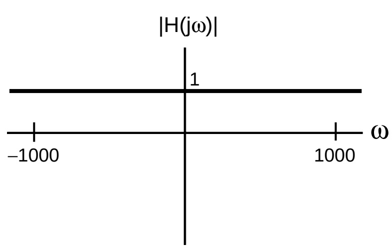
(a) Sketch $x(t)$.

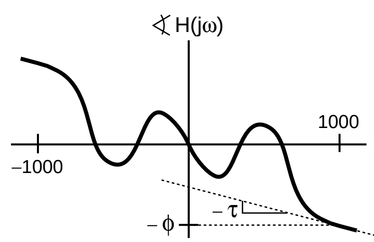
(b) Given the all-pass system whose frequency response is plotted below, find an approximate expression for the output by incorporating a group delay. (Note the straight-line approximation. Assume $h(t)$ is real.)





(c) Given the all-pass system whose frequency response is plotted below, find an approximate expression for the output by incorporating a group delay and a phase delay. (Note the straight-line approximation. Assume $h(t)$ is real.)





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3 First- and Second-Order Systems

We can decompose a large LTI system into a combination of small LTI systems for which we know how to analyze. Our canonical systems are 1st and 2nd order systems. The following are the sets of descriptions for 1st and 2nd order systems:

1. **1st-order CT system**

$$\begin{aligned} \text{LCCDE} \qquad \tau\dot{y}(t) + y(t) &= x(t), \\ \text{Freq. Resp.} \qquad H(j\omega) &= \frac{1}{j\omega\tau + 1}, \\ \text{Impulse Resp.} \qquad h(t) &= \frac{1}{\tau}e^{-\frac{t}{\tau}}u(t). \end{aligned}$$

   τ is known as the *characteristic time constant* for the system. An example of a first-order system is an RC circuit.

2. **1st-order DT system**

$$\begin{aligned} \text{LCCDE} \qquad y[n] - ay[n-1] &= x[n], \\ \text{Freq. Resp.} \qquad H(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}}, \\ \text{Impulse Resp.} \qquad h[n] &= a^n u[n]. \end{aligned}$$

   In a first-order DT system, a plays an analogous role to τ.

3. **2nd-order CT system**

$$\begin{aligned} \text{LCCDE} \qquad \omega_n^2 x(t) &= \ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t), \\ \text{Freq. Resp.} \qquad H(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}, \\ \text{Impulse Resp.} \qquad h(t) &= \begin{cases} \dfrac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_n\sqrt{1-\zeta^2})t\right]u(t) & c_1 \neq c_2, 0 < \zeta < 1 \\ M\left[e^{c_1 t} - e^{c_2 t}\right]u(t) & c_1 \neq c_2, \zeta \neq 1 \\ \omega_n^2 t e^{-\omega_n t}u(t) & c_1 = c_2 \end{cases} \end{aligned}$$

   where $c_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ and $M = \dfrac{\omega_n}{2\sqrt{\zeta^2 - 1}}$. ω_n is called *undamped natural frequency* and ζ *damping ratio*. When $\zeta > 1$, both c_1 and c_2 are real, and we can decompose the system into two first-order systems, which we already know how to solve. The impulse response is then two decaying exponentials; we called this the *overdamped* case. When $\zeta = 1$, $c_1 = c_2$, and we have the *critically damped* case. Finally, when $\zeta < 1$, both c_1 and c_2 are complex, and we have the *underdamped* case, which has damped oscillations in its impulse response. Examples of second-order systems include an RLC circuit and a mass-spring-damper system.

4 Bode Plots

Let $H(j\omega)$ be the product of two transforms:

$$\begin{aligned} H(j\omega) &= H_1(j\omega)H_2(j\omega) \\ &= |H_1(j\omega)||H_2(j\omega)|e^{\angle H_1(j\omega)+\angle H_2(j\omega)} \end{aligned}$$

We can represent the magnitude and phase of $H(j\omega)$ separately:

$$\begin{aligned} |H(j\omega)| &= |H_1(j\omega)||H_2(j\omega)| \\ \angle H(j\omega) &= \angle H_1(j\omega) + \angle H_2(j\omega) \end{aligned}$$

If we take the logarithm of both sides of the magnitude equation and multiply by 20, we get:

$$\begin{aligned} 20\log|H(j\omega)| &= 20\log|H_1(j\omega)| + 20\log|H_2(j\omega)| \\ \angle H(j\omega) &= \angle H_1(j\omega) + \angle H_2(j\omega) \end{aligned}$$

These equations represent the log magnitude and phase of a frequency response as the sum of the corresponding components of the factors of the frequency response. Adding plots is easy, so this is a convenient way for use to build pictures of the frequency response if we can break it down into simpler components. The units of 20 log are *decibels* (dB). In CT, we also use a log scale for frequency. We call plots of $20\log|H(j\omega)|$ and $\angle H(j\omega)$ vs. $\log\omega$ *Bode plots*. We will almost always deal with real systems, so $H(j\omega)$ is conjugate-symmetric. Therefore, we will only plot $H(j\omega)$ for positive ω.

4.1 First-order system Bode plots

Let's find the Bode plot for a first-order system:

$$H(j\omega) = \frac{1}{j\omega\tau + 1}.$$

As ω runs from zero to infinity, the magnitude of the vector increases monotically to infinity and the angle runs from zero to π. Analytically, we have:

$$\begin{aligned} 20\log|H(j\omega)| &= -10\log((\omega\tau)^2+1) \\ \angle H(j\omega) &= -\tan^{-1}(\omega\tau). \end{aligned}$$

When $\omega\tau << 1$, the log magnitude is near zero, and when $\omega\tau >> 1$, the log magnitude is

$$\begin{aligned} 20\log|H(j\omega)| &= -10\log((\omega\tau)^2+1) \\ &\approx -10\log((\omega\tau)^2) \\ &= -20(\log\omega) - 20(\log\tau), \end{aligned}$$

which is linear in $\log\omega$ with slope -20. This line is called the *20-dB-per-decade* asymptote. Thus, we can represent the log magnitude of the frequency response by two asymptotes: two straight lines for each regime that meet at the *break frequency* $1/\tau$. The actual plot deviates from the asymptotic approximation the most at the break frequency itself, when it is:

$$-20\log\sqrt{2} = -10\log 2 \approx -3\,\text{dB}.$$

Likewise, we approximate the angle as being 0 for $\omega << 0.1/\tau$, $-\pi/2$ for $\omega >> 10/\tau$, and a line connecting the two extremes in between. At the break frequency, the asymptote and the actual value coincide: $\angle H(j/\tau) = -\pi/4$.

If we can factor a frequency response into first-order polynomials with real roots on both top and bottom, then we can apply the analysis above to each and add up them up to form the Bode plot of the frequency response. However, it is important to rewrite the frequency response from pole-zero form:

$$H(j\omega) = M\frac{(j\omega)^L(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_Q)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_P)},$$

to *time-constant* form:

$$H(j\omega) = K\frac{(j\omega)^L(\tau_{z1}j\omega + 1)(\tau_{z2}j\omega + 1)\cdots(\tau_{zR}j\omega + 1)}{(\tau_{p1}j\omega + 1)(\tau_{p2}j\omega + 1)\cdots(\tau_{pP}j\omega + 1)}.$$

The advantage is we can read off right away what the DC gain (or asymptote) is. The Bode plots can easily be constructed from each factor:

$$\begin{aligned}
H(j\omega) &= K\frac{(j\omega)^L(1 + \tau_{z1}j\omega)\cdots(1 + \tau_{zR}j\omega)}{(1 + \tau_{p1}j\omega)\cdots(1 + \tau_{pP}j\omega)} \\
\Rightarrow \log|H(j\omega)| &= \log|K| + L\log|\omega| + \sum_{i=1}^{R}\log|(1 + \tau_{zi}j\omega)| - \sum_{i=1}^{P}\log|(1 + \tau_{pi}j\omega)| \\
\Rightarrow \angle H(j\omega) &= \angle K + L\angle(j\omega) + \sum_{i=1}^{R}\angle(1 + \tau_{zi}j\omega) - \sum_{i=1}^{P}\angle(1 + \tau_{pi}j\omega)
\end{aligned}$$

From this form, we use the following procedure:

Building the CT Bode Plot from First-Order Polynomials:

Consider a rational transfer function in time-constant form, where each of the time constant is positive and real:

$$H(j\omega) = K\frac{(j\omega)^L(\tau_{z1}j\omega+1)(\tau_{z2}j\omega+1)\cdots(\tau_{zR}j\omega+1)}{(\tau_{p1}j\omega+1)(\tau_{p2}j\omega+1)\cdots(\tau_{pP}j\omega+1)}.$$

The Bode plot of the frequency response $H(j\omega)$ can be produced with the following rules:

- The constant gain K shifts the log magnitude up by $20\log K$ dB.
- Factors in the numerator of the transfer function
  - A factor of $j\omega$ contributes to the log magnitude a line that is 0 dB at $\omega = 1$ rad/s with slope 20 dB/dec and to the phase a constant of π (90 degrees).
  - A factor of $(\tau_{z1}j\omega+1)$ contributes to the log magnitude a bent line that is 0 dB for $\omega < 1/\tau_{z1}$ (ω is less than the break frequency) and has slope 20 dB/dec for $\omega > 1/\tau_{z1}$ and to the phase a bent line that is 0 for $\omega < (1/\tau_{z1})/10$ (one decade below the break frequency), $\pi/2$ (90 degrees) for $\omega > 10/\tau_{z1}$ (one decade above the break frequency), and is linear with slope $\pi/4$ rad/decade (45 degrees/decade) in between.
- Factors in the denominator of the transfer function
  - A factor of $j\omega$ contributes to the log magnitude a line that is 0 dB at $\omega = 1$ rad/s with slope -20 dB/dec and to the phase a constant of $-\pi$ (-90 degrees).
  - A factor of $(\tau_{z1}j\omega+1)$ contributes to the log magnitude a bent line that is 0 dB for $\omega < 1/\tau_{z1}$ (ω is less than the break frequency) and has slope -20 dB/dec for $\omega > 1/\tau_{z1}$ and to the phase a bent line that is 0 for $\omega < (1/\tau_{z1})/10$ (one decade below the break frequency), $-\pi/2$ (-90 degrees) for $\omega > 10/\tau_{z1}$ (one decade above the break frequency), and is linear with slope $-\pi/4$ rad/decade (-45 degrees/decade) in between.

Notice that this summary in words did not include the possibility of negative time constants. They work exactly the same way, except that the phase is flipped in sign. Generally, we will not see such factors in the denominator, for these would make a causal system unstable.

Problem 7.3

Draw the Bode plot for the following frequency response:

$$H(j\omega) = \frac{(5 \times 10^6)(j\omega + 10)}{j\omega(j\omega + 100)(j\omega + 5 \times 10^4)}.$$

(Work space)

4.2 Second-order system Bode plots

The standard form for the transfer function of a second-order system is

$$H(s) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{\omega_n^2}{(j\omega - p_+)(j\omega - p_-)},$$

where ω_n is the *undamped natural frequency* and ζ is the *damping ratio*. These terms come from typical second-order systems, such as RLC circuits and mass-spring-dashpot mechanical systems. The roots of the denominator (which we will refer to as *poles* after studying the Laplace transform) are

$$p_\pm = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$$

When $\zeta > 1$, the roots are real, and we decompose the system into two first-order systems, which we already know how to solve. The impulse response is then two decaying exponentials; we called this the *overdamped* case. When $\zeta = 1$, the roots are equal, and we have the *critically damped* case. Finally, when $\zeta < 1$, both roots are complex, and we have the *underdamped* case, which has damped oscillations in its impulse response. The textbook shows some Bode plots for various values of the damping ratio, which greatly affects the shape of the plots. For $\zeta < \sqrt{2}/2$, there is a maximum peak in the log magnitude of the frequency response at

$$\omega_{max} = \omega_n\sqrt{1 - 2\zeta^2},$$

where the value is

$$|H(j\omega_{max})| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}.$$

The "quality factor" Q is commonly used to describe the sharpness of the peak at $\omega = \omega_{max}$ and is given by

$$Q = \frac{1}{2\zeta}.$$

For a high-Q second-order system (such as a RLC circuit or a spring-mass-dashpot system) that begins with some energy and is allowed to evolve without further input, a useful energy interpretation of Q is 2π times the inverse of the proportion of energy lost per oscillation. Note that if $\zeta << 1$ (the *undamped* case, very high-Q),

$$\begin{aligned} \omega_{max} &\approx \omega_n, \\ |H(j\omega_{max})| &\approx \frac{1}{2\zeta}. \end{aligned}$$

If ω moves to $\omega_n \pm \zeta\omega_n$, magnitude is decreased by $1/\sqrt{2}$. Power is magnitude squared, so power is cut in half (3 dB lower). Hence, the margin $2\zeta\omega_n$ is called the *half-power bandwidth*, or the *full-width half maximum* when power is plotted. We also see that the angle in this range falls from $-\pi/4$ to $-3\pi/4$. For very low frequencies, the magnitude is 1 (0 dB) and the angle is zero. For very high frequencies, the magnitude becomes a -40 dB/dec line and the angle becomes $-\pi$. This is best illustrated by an example problem.

Problem 7.4

Sketch the Bode magnitude and phase plots for the following frequency response:

$$H(j\omega) = \frac{101}{(j\omega)^2 + 2j\omega + 101}.$$

(Work space)

Problem 7.5

We are given the following frequency responses:

$$H_1(j\omega) = \frac{j\omega+100}{10(j\omega+10)} \qquad H_2(j\omega) = \frac{j\omega-100}{10(j\omega+10)}$$
$$H_3(j\omega) = \frac{10(j\omega+10)}{j\omega+100} \qquad H_4(j\omega) = \frac{100}{(j\omega)^2+4j\omega+100}$$
$$H_5(j\omega) = \frac{j\omega+100}{(j\omega)^2+4j\omega+100} \quad H_6(j\omega) = \frac{j\omega+100}{10j\omega(j\omega+10)}$$

(a) Match each of the frequency response functions above to its corresponding Bode magnitude plot.

Figure 1

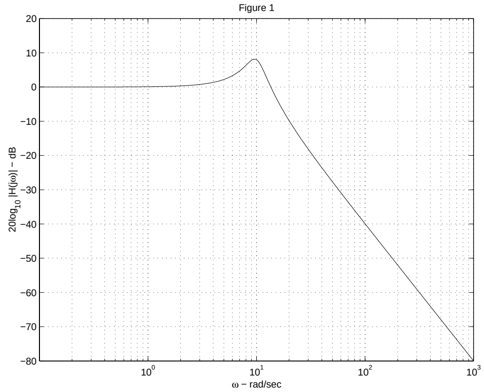


Figure 2

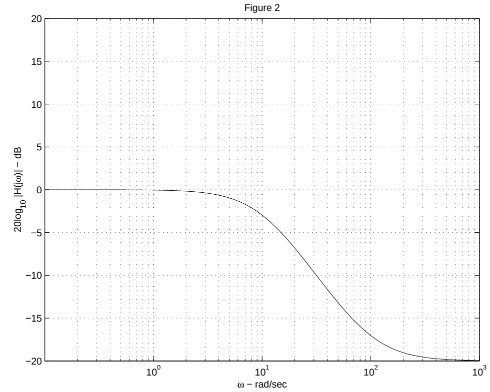


Figure 3

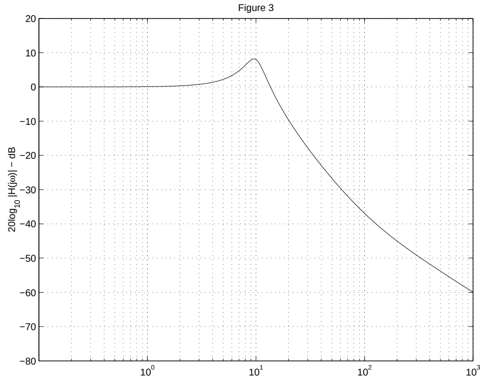
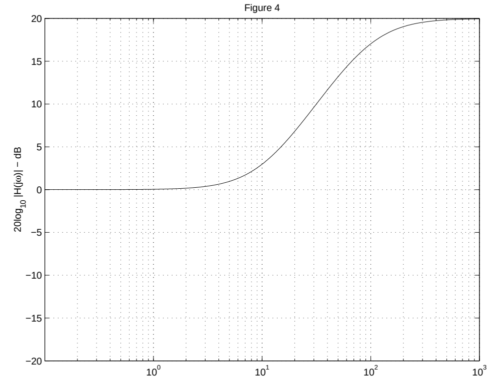


Figure 4



(b) Draw the Bode phase plots for $H_1(j\omega)$ and $H_2(j\omega)$.

(c) Does there exist $H_7(j\omega)$ such that $|H_7(j\omega)| = |H_1(j\omega)|$ and$|H_7(j\omega)| = |H_2(j\omega)|$, but $\angle H_7(j\omega)$ is not identical to $\angle H_1(j\omega)$ nor $\angle H_2(j\omega)$? If there exists, find an expression for $H_7(j\omega)$. If not, explain why.

(d) Draw the Bode plots for the frequency responses for which there are not corresponding magnitude plots in (a).

(Work space)

(Work space)