MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

Tutorial 6

Monday, March 15 and Tuesday, March 16, 2004

Announcements

- Problem set 5 is due this Friday.
- Spring break is next week.

Today's Agenda

- DT Fourier Transform and Inverse Fourier Transform
- Summary of the Four Fourier Series and Transforms
  - All Fourier stuff in one table
  - Basis signals
  - Fourier transform of periodic signals
  - Discreteness-periodicity duality
  - What's up with the 2π factors?
- Partial-Fraction Expansion
  - Top-heavy rationals
  - Repeated roots in the denominator
  - Canonical expansions
- Inverse CT Fourier Transform of Rational Functions of $j\omega$
  - How far do we need to go?
  - Summary of finding the inverse Fourier transform
- Inverse DT Fourier Transform of Rational Functions of $e^{j\omega}$
- Fourier Transforms and LTI Systems
  - Differential and difference equations
- Time-Frequency Uncertainty Principle (Optional)

1 DT Fourier Transform and Inverse Fourier Transform

Like continuous time signals, we can also write discrete time signals as the superposition of exponentials of a continuum of frequencies. One major difference between the CT and DT synthesis-analysis equations is that the transform $X(\mathrm{e}^{j\omega})$ is periodic with period 2π, so the integral in the synthesis is carried out for any interval of length 2π:

The Discrete-Time Fourier Transform:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{j\omega})e^{j\omega n}\,\mathrm{d}\omega && \text{(Synthesis, inverse DTFT)} \\
X(\mathrm{e}^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} && \text{(Analysis, DTFT)}
\end{aligned}
$$

The convolution-multiplication properties are:

- **DT convolution property**

$$y[n] = h[n] * x[n] \quad \xleftrightarrow{\mathcal{F}} \quad Y(\mathrm{e}^{j\omega}) = H(\mathrm{e}^{j\omega})X(\mathrm{e}^{j\omega})$$

- **DT multiplication property**

$$r[n] = s[n]p[n] \quad \xleftrightarrow{\mathcal{F}} \quad R(j\omega) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(\omega-\theta)})d\theta$$

Note that since the DTFT is periodic, we only convolve over a period of 2π, a process called *periodic convolution.*

Problem 6.1

Compute the Fourier transform of the following signals:

(a) $x_a[n] = 2^n \sin(\frac{\pi}{4}n)u[-n]$.

(b) $x_b[n] = (\frac{\sin(\pi n/5)}{\pi n})\cos(\frac{7\pi}{2}n)$.

(Work space)

Problem 6.2

Find $x[n]$ for the following DTFTs:

(a) $X_a(e^{j\omega}) = \cos 2\omega + \sin^2 \omega$.

(b) $X_b(e^{j\omega}) = \dfrac{1}{\frac{5}{4} + \cos\omega}$.

(Work space)

2 Summary of the Four Fourier Series and Transforms

We've seen four types of Fourier analysis: CT Fourier series, DT Fourier series, CT Fourier transform and the DT Fourier transform. Here they are:

2.1 All Fourier stuff in one table

The Continuous-Time Fourier Series:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{jk\omega_0 t} && \text{(Synthesis equation)} \\ a_k &= \frac{1}{T}\int_T x(t)\mathrm{e}^{-jk\omega_0 t}\,\mathrm{d}t && \text{(Analysis equation)} \end{aligned}$$

The Discrete-Time Fourier Series:

$$\begin{aligned} x[n] &= \sum_{k=\langle N\rangle} a_k \mathrm{e}^{jk\omega_0 n} && \text{(Synthesis equation)} \\ a_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\mathrm{e}^{-jk\omega_0 n} && \text{(Analysis equation.} \end{aligned}$$

The Continuous-Time Fourier Transform:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\mathrm{e}^{j\omega t}\,\mathrm{d}\omega && \text{(Synthesis, inverse CTFT)} \\ X(j\omega) &= \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-j\omega t}\,\mathrm{d}t && \text{(Analysis, CTFT)} \end{aligned}$$

The Discrete-Time Fourier Transform:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}\,\mathrm{d}\omega && \text{(Synthesis, inverse DTFT)} \\ X(\mathrm{e}^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]\mathrm{e}^{-j\omega n} && \text{(Analysis, DTFT)} \end{aligned}$$

Why have all these series and transforms? It's so confusing! Let's take a step back and think about everything again. Since complex exponentials are the eigenfunctions of LTI systems, we are strongly motivated to express signals as superpositions of these exponentials. In other words, we wanted to *change basis*.

2.2 Basis signals

In Tutorial 3, we found that by setting the basis signals to be harmonically related exponentials (all with period $T = 2\pi/\omega_0$, or $N = 2\pi/\omega_0$) indexed by the *discrete* variable k:

$$\begin{aligned}\phi_k(t) &= \mathrm{e}^{jk\omega_0 t},\\ \phi_k[n] &= \mathrm{e}^{jk\omega_0 n},\end{aligned}$$

we could represent *periodic signals* as scaled superpositions of these basis signals:

$$\begin{aligned}x(t) &= \sum_{k=-\infty}^{+\infty} a_k \phi_k(t),\\ x[n] &= \sum_{k=\langle N\rangle} a_k \phi_k[n].\end{aligned}$$

Note that for CT, we need to use a countably infinite number of such basis signals (k takes values at all integers), whereas for DT, we only need to use N of them.

In Tutorial 4 and earlier in this tutorial, we found that by setting the basis signals to be exponentials indexed by the *continuous* variable ω:

$$\begin{aligned}\phi_\omega(t) &= \mathrm{e}^{j\omega t},\\ \phi_\omega[n] &= \mathrm{e}^{j\omega n},\end{aligned}$$

we could represent *aperiodic signals* as scaled superpositions of these basis signals. Since there the basis index is a continuous variable, we use integrals instead of sums:

$$\begin{aligned}x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\phi_\omega(t)\,\mathrm{d}\omega,\\ x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{j\omega})\phi_\omega[n]\,\mathrm{d}\omega.\end{aligned}$$

In both CT and DT, we use an uncountably infinite number of such basis signals. However, in CT we use the entire real number line as frequencies (ω spans the space of all real numbers), whereas for DT, we only need to use a segment of length 2π.

To extract the unique Fourier series coefficients a_k and the Fourier transform $X(j\omega)$ and $X(\mathrm{e}^{j\omega})$, we use the following orthogonality relationships:

$$
\begin{aligned}
\frac{1}{T}\int_T \phi_m^*(t)\phi_k(t)\,\mathrm{d}t &= \frac{1}{T}\int_T \mathrm{e}^{j(k-m)\omega_0 t}\,\mathrm{d}t &= \delta[m-k],\\
\frac{1}{N}\sum_{n=\langle N\rangle} \phi_m^*[n]\phi_k[n] &= \frac{1}{N}\sum_{n=\langle N\rangle} \mathrm{e}^{j(k-m)\omega_0 n} &= \delta[m-k],\\
\frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_{\omega'}^*(t)\phi_\omega(t)\,\mathrm{d}t &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{j(\omega-\omega')t}\,\mathrm{d}t &= \delta(\omega'-\omega),\\
\frac{1}{2\pi}\sum_{n=\infty}^{+\infty} \phi_{\omega'}^*[n]\phi_\omega[n] &= \frac{1}{2\pi}\sum_{n=\infty}^{+\infty} \mathrm{e}^{j(\omega-\omega')n} &= \delta(\omega'-\omega).
\end{aligned}
$$

The first two are quite straightforward to show; we are integrating or summing over a finite interval, and they result in the DT delta function. However, the second two are mysterious: we are integrating or summing over an infinitely long interval, so the $\omega' = \omega$ case would result in infinity. How do we know its exactly the Dirac delta function? And what's up with the 2π factor? The answer to these questions are beyond the scope of 6.003, and we will simply accept them as true.

2.3 Fourier transform of periodic signals

The Fourier series representation can only be used for periodic signals, but Fourier transform analysis can be applied to a broader set of signals, including aperiodic signals. We saw that periodic signals contain only *harmonically related* frequencies, namely those separated by increments of the fundamental frequency ω_0. Aperiodic signals, on the other hand, generally require a *continuum* of frequencies to be adequately represented.

So, what happens if we take the Fourier transform of a periodic signal? And what's its relationship to the Fourier series? Since only discrete frequencies are needed, we would expect delta functions to pop up at those frequencies (we need delta functions to keep the integral of the transform non-zero). First, let's write a CT periodic signal in terms of its Fourier series coefficients:

$$
x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{jk\omega_0 t}
$$

From the Fourier *transform* analysis equation, we have:

$$
X(j\omega) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-j\omega t}\,\mathrm{d}t.
$$

When we plug in expression for $x(t)$ into the above equation, we get:

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{+\infty}\left[\sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{jk\omega_0 t}\right]\mathrm{e}^{-j\omega t}\,\mathrm{d}t\\
&= \sum_{k=-\infty}^{+\infty} a_k \left[\int_{-\infty}^{+\infty} \mathrm{e}^{jk\omega_0 t}\mathrm{e}^{-j\omega t}\,\mathrm{d}t\right]\\
&= \sum_{k=-\infty}^{+\infty} a_k 2\pi \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{j(k\omega_0-\omega)t}\,\mathrm{d}t\right]
\end{aligned}
$$

By orthogonality, the term in the brackets is the Dirac delta function $\delta(\omega - k\omega_0)$. So:

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} a_k 2\pi\delta(\omega - k\omega_0).$$

So just showed the following:

Fourier Transform of Periodic Signals:

Periodic signals with Fourier *series* coefficients a_k, period T (N in DT), and fundamental frequency w_0 have the following Fourier *transforms*:

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$
$$X(\mathrm{e}^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0).$$

In other words, *the Fourier transform of a periodic signal is a train of impulses placed at the discrete harmonic frequencies where the area of the impulse for the frequency $k\omega_0$ is 2π times the corresponding Fourier series cofficient a_k*. We can now write the Fourier transforms of common periodic signals.

Let's derive the result a different way that may be more intuitive. First, note that the Fourier transform of pure exponentials are:

$$\mathrm{e}^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$
$$\mathrm{e}^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l).$$

Now, for periodic signals with Fourier *series* coefficients a_k, period T (N in DT), and fundamental frequency w_0, we can write the signals as:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k \mathrm{e}^{jk\omega_0 t},$$
$$x[n] = \sum_{k=\langle N \rangle} a_k \mathrm{e}^{jk\omega_0 n}.$$

If we take the Fourier transform of both equations, we get the same result.

2.4 Discreteness-periodicity duality

So far, we have discussed Fourier series and Fourier transforms of CT and DT signals. There were major differences between these four transformations, such as periodicity and discreteness. But how are we supposed to remember which ones are periodic, which ones are discrete, *etc.*, without looking at tables or formulae? We saw that *periodic* signals can be represented by a set of *discrete* frequencies (Fourier *series sum*), and that *aperiodic* signals require a set of *continuous frequencies* (Fourier *transform integral*). Likewise, *continuous* time signals have *aperiodic* representations in the frequency domain, and *discrete* time signals have *periodic* representations in the frequency domain, regardless of whether we use Fourier series or transform. These observations lead to the following pair of mnemonics:

Discreteness-Periodicity Duality:

If a signal is discrete in one domain, then it is periodic in the other domain. Similarly, if a signal is continuous in one domain, then it is aperiodic in the other domain.

If you ever find yourself lost in dealing with all sorts of time-frequency relationships, just keep this in mind. There is a table on page 396 of the textbook that summarizes this concept. Section 5.7 of the textbook discusses *duality*, or the existence of symmetries between the time and frequency representation of signals. Dualities between two (not necessarily distinct) transformations only exist when *swapping time and frequency leads to the discreteness-periodicity properties of the other.*

2.5 What's up with the 2π factors?

We've seen 2π and $1/2\pi$ factors appear all over the place in Fourier transform formulae and got headaches trying to remember them. They all stem from the fact that we use *angular* frequency ω instead of *cyclic* frequency f, where $\omega = 2\pi f$. We view angular frequency as being more "natural," but many practical problems use cyclic frequency, so we need to remember when to add in factors of 2π. With this convention, we saw that the synthesis and analysis equations for the CT and DT Fourier transforms become:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\mathrm{e}^{j\omega t}\,\mathrm{d}\omega && \text{(CT synthesis, inverse CTFT)}\\
X(j\omega) &= \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-j\omega t}\,\mathrm{d}t && \text{(CT analysis, CTFT)}\\
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}\,\mathrm{d}\omega && \text{(DT synthesis, inverse DTFT)}\\
X(\mathrm{e}^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]\mathrm{e}^{-j\omega n} && \text{(DT analysis, DTFT)}
\end{aligned}$$

We also noted that the synthesis-analysis pair becomes symmetric (no 2π factor out in front for the synthesis equation) when we used cyclic frequency f:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{+\infty} X(f)\mathrm{e}^{j2\pi ft}\,\mathrm{d}f && \text{(CT cyclic synthesis, inverse CTFT)}\\
X(f) &= \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-j2\pi ft}\,\mathrm{d}t && \text{(CT cyclic analysis, CTFT)}\\
x[n] &= \int_{2\pi} X(\mathrm{e}^{j2\pi f})\mathrm{e}^{j2\pi ft}\,\mathrm{d}f && \text{(DT cyclic synthesis, inverse DTFT)}\\
X(\mathrm{e}^{j2\pi f}) &= \sum_{n=-\infty}^{+\infty} x[n]\mathrm{e}^{-j2\pi fn} && \text{(DT analysis, DTFT)}
\end{aligned}
$$

The 2π factor is manifested in the following FT pairs and properties:

- **Value of a signal at zero time**

$$
\begin{aligned}
x(0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,\mathrm{d}\omega\\
x[0] &= \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{j\omega})\,\mathrm{d}\omega
\end{aligned}
$$

- **Constant signal**

$$
\begin{aligned}
x(t) = 1 \quad &\stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = 2\pi\delta(\omega)\\
x[n] = 1 \quad &\stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(\mathrm{e}^{j\omega}) = 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - 2\pi l)
\end{aligned}
$$

- **Complex exponentials**

$$
\begin{aligned}
x(t) = \mathrm{e}^{j\omega_0 t} \quad &\stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = 2\pi\delta(\omega - \omega_0)\\
x[n] = \mathrm{e}^{j\omega_0 n} \quad &\stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(\mathrm{e}^{j\omega}) = 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - \omega_0 - 2\pi l)
\end{aligned}
$$

- **Multiplication property**

$$
\begin{aligned}
r(t) = s(t)p(t) \quad &\stackrel{\mathcal{F}}{\longleftrightarrow} \quad R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))\,\mathrm{d}\theta = \frac{1}{2\pi}\left\{S(j\theta) * P(j\theta)\right\}\\
r[n] = s[n]p[n] \quad &\stackrel{\mathcal{F}}{\longleftrightarrow} \quad R(j\omega) = \frac{1}{2\pi}\int_{2\pi} S(\mathrm{e}^{j\theta})P(\mathrm{e}^{j(\omega-\theta)})\,\mathrm{d}\theta = \frac{1}{2\pi}\left\{S(\mathrm{e}^{j\theta}) \circledast P(\mathrm{e}^{j\theta})\right\}
\end{aligned}
$$

Since the cyclic frequency representation is free of 2π's in the first place, we know that:

$$\boxed{\text{If we wrote all the above formulae using cyclic frequency } f \text{ instead of angular frequency } \omega\text{, then all the } 2\pi \text{ factors would disappear.}}$$

In the cyclic frequency domain, the multiplication property would have an integral over cyclic frequency $\mathrm{d}f$ without a 2π. But because $2\pi f = \omega$ so that $\mathrm{d}f = \mathrm{d}\omega/2\pi$, a factor of $1/2\pi$ appears in the angular frequency domain. This leads to the following rule:

Whenever we integrate over angular frequency, we need to divide by 2π.

We sometimes forget to do this when we use the shorthand notation of convolution because we don't see an integral. But the definition of convolution of a continuous variable contains an integral, so:

Convolution over a continuous variable is an implicit integration.

Now what about the 2π factors in front of the impulses? Keep in mind that:

Dirac delta functions (CT impulses) are *always* intended for use under an integral.

This statement is supposed to remind us that integrals can be (and should be) used to examine properties of impulses. Say we were looking at the FT of a constant CT signal $x(t) = 1$, which is an impulse. From the "value of a CT signal at zero time" formula, we know that 1 equals the integral of the impulse, and since we are integrating over angular frequency, we need to divide by 2π. To compensate for this, the impulse must have a *multiplicative* factor of 2π. So:

Canonical impulses in the frequency domain have a 2π factor.

"Canonical" here refers to the Fourier transform of the complex exponential and other "standard" signals. Thus, we can extend this principle to sines and cosines, which have factors of π in their transforms.

3 Partial-Fraction Expansion

Before we jump into the inverse Fourier transform, let's discuss partial-fraction expansion, which is useful for rational Fourier transforms. Suppose $F(x)$ is a rational function of x with the degree of the numerator strictly less than the degree of the denominator. That is:

$$F(x) = \frac{(x - z_1)(x - z_2)\cdots(x - z_m)}{(x - p_1)(x - p_2)\cdots(x - p_n)},$$

where $m < n$ and z_i, p_j complex. If $p_i \neq p_j$ for all i, j, then $F(x)$ may be written as:

$$F(x) = \frac{A_1}{x - p_1} + \frac{A_2}{x - p_2} + \cdots + \frac{A_n}{x - p_n}.$$

This is known as *partial-fraction expansion*, or *partial-fraction decomposition*. One way of finding the coefficients A_k is by multiplying through and matching terms. For instance, suppose:

$$F(x) = \frac{2x + 1}{(x + 3)(x + 2)}$$

Then,

$$
\begin{aligned}
\frac{2x+1}{(x+3)(x+2)} &= \frac{A_1}{x+3} + \frac{A_2}{x+2} \\
2x+1 &= A_1(x+2) + A_2(x+3) \\
&= (A_1 + A_2)x + (2A_1 + 3A_2)
\end{aligned}
$$

Matching the two coefficients gives:

$$
\begin{aligned}
A_1 + A_2 &= 2 \\
2A_1 + 3A_2 &= 1
\end{aligned}
$$

Solving these two equations produces the coefficients:

$$A_1 = 5, \quad A_2 = -3$$

However, solving systems of equations, even linear ones, can get rather tedious. A much faster method of finding the coefficients is a follows:

Formula for the ith Expansion Coefficient:

The coefficient A_i in Eq. 3 can be determined by:

$$A_i = [(x - p_i)F(x)]|_{x=p_i}$$

Let's apply this analysis to our example and see why it works:

$$
\begin{aligned}
A_1 &= [(x-p_1)F(x)]\,|_{x=p_1} \\
&= \left.\left[(x+3)\frac{2x+1}{(x+3)(x+2)}\right]\right|_{x=-3} \\
&= \left.\frac{2x+1}{x+2}\right|_{x=-3} \\
\Longrightarrow A_1 &= 5
\end{aligned}
$$

We can proceed likewise for A_2. Note that the factors in the numerator and denominator cancel *before* evaluation, which would have made that factor zero. Why does it work? Let's multiply both sides of Eq. 6.1 by $(x+3)$:

$$
\begin{aligned}
\frac{2x+1}{(x+3)(x+2)}(x+3) &= \frac{A_1}{x+3}(x+3) + \frac{A_2}{x+2}(x+3) \\
\frac{2x+1}{x+2} &= A_1 + \frac{A_2}{x+2}(x+3)
\end{aligned}
$$

Now, we set $(x+3)$ to zero, so that *all terms except the one we want become zero.* Then, we have the value of A_1. This is the algebra behind this trick. When we do it mechanically, it is known affectionately as "the cover up method." This is because we tend to use our fingers to "cover up" the terms that disappear.

3.1 Top-heavy rationals

What do we do if the rational has a numerator whose order is equal to or higher than that of the denominator? We can use long division! For instance, suppose:

$$F(x) = \frac{x^2+4x+3}{x^2+4x-5}$$

Then,

$$\begin{aligned} F(x) &= \frac{(x^2+4x-5)+8}{x^2+4x-5} \\ &= 1 + \frac{8}{x^2+4x-5} \end{aligned}$$

We can do a PFE on the right part of the sum:

$$\begin{aligned} F(x) &= 1 + \frac{8}{x^2+4x-5} \\ &= 1 - \frac{4/3}{x+5} + \frac{4/3}{x-1} \end{aligned}$$

3.2 Repeated roots in the denominator

Sure, but what if we have repeated roots in the denominator? Specifically:

$$F(x) = \frac{(x-z_1)(x-z_2)\cdots(x-z_m)}{(x-p_1)^{k_1}(x-p_2)^{k_2}\cdots(x-p_n)^{k_n}},$$

where $m < \sum_{i=1}^{n} k_i$ and z_i, p_j complex. Then, $F(x)$ may be written as:

$$\begin{aligned} F(x) = & \frac{A_{10}}{(x-p_1)^{k_1}} + \frac{A_{11}}{(x-p_1)^{k_1-1}} + \cdots + \frac{A_{1i}}{(x-p_1)^{k_1-i}} + \cdots + \frac{A_{1k_1-1}}{x-p_1)} \\ & + \frac{A_{20}}{(x-p_2)^{k_2}} + \frac{A_{21}}{(x-p_2)^{k_2-1}} + \cdots + \frac{A_{2j}}{(x-p_2)^{k_2-j}} + \cdots + \frac{A_{2k_2-1}}{x-p_2)} \\ & + \cdots \\ & + \frac{A_{n0}}{(x-p_n)^{k_n}} + \frac{A_{n1}}{(x-p_n)^{k_n-1}} + \cdots + \frac{A_{nl}}{(x-p_n)^{k_n-l}} + \cdots + \frac{A_{nk_n-1}}{x-p_n)} \end{aligned}$$

The cofficients are:

Formula for the Expansion Coefficient of the mth Factor and the $(k_m - n)$th Power:

The coefficient A_{mn} in Eq. 6.1 can be determined by:

$$A_{mn} = \frac{1}{n!}\left[\frac{d^n}{dx^n}(x-p_m)^{k_m}F(x)\right]\bigg|_{x=p_i}$$

We'll look at some examples of this later.

3.3 Canonical expansions

Particular partial fraction expansions come up so often that we should write them down once and for all and not have to resort to the cover up method each time. You may find the following expansions useful to put on an quiz sheet:

Canonical Partial Fraction Expansions:

$$\begin{aligned}
\frac{1}{(x+a)(x+b)} &= \frac{\frac{1}{b-a}}{x+a} + \frac{\frac{1}{a-b}}{x+b} \\
\frac{x}{(x+a)(x+b)} &= \frac{\frac{a}{a-b}}{x+a} + \frac{\frac{b}{b-a}}{x+b} \\
\frac{cx+d}{(x+a)(x+b)} &= \frac{\frac{ac-d}{a-b}}{x+a} + \frac{\frac{bc-d}{b-a}}{x+b}
\end{aligned}$$

Since this is a linear procedure, scaling the expressions on the left-hand side would also scale the expansions on the right-hand side by the same factor. Also, beware of the signs of a and b! You may want to jot down your own versions using $-a$ and $-b$.

4 Inverse CT Fourier Transform of Rational Functions of $j\omega$

Ok, so now why did we take a detour and talk about partial-fraction expansion? Well, they are useful for finding inverse Fourier transforms. We saw in Tutorial 4 that *stable CT LTI systems described by linear constant-coefficient differential equations have frequency responses that are rational functions of $j\omega$*. Thus, we want to be able to take the inverse Fourier transform of the frequency responses of such systems to get the impulse response.

In the previous section, we used x as the generic variable name. Since we are now looking at rational functions of $j\omega$, we replace x with $j\omega$. Keep in mind that this is only a *bookkeeping* step to make partial-fraction expansion easier.

4.1 How far do we need to go?

Decaying sinusoids occur in many systems that we study. Straightforward calculation shows that their Fourier transforms are:

Fourier Transforms of Decaying Sinusoids:

For real and positive α, we have the following Fourier transform pairs:

$$\left[\mathrm{e}^{-\alpha t}\cos\omega_0 t\right]u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2}$$
$$\left[\mathrm{e}^{-\alpha t}\sin\omega_0 t\right]u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2}$$

Believe it or not, this pair of transforms will actually affect how far we carry out partial-fraction expansion. Consider the following Fourier transform:

$$X(j\omega) = \frac{(j\omega)^2+4j\omega+11}{(j\omega+3)((j\omega)^2+2j\omega+5)}$$

Applying our usual partial fraction expansion produces:

$$X(j\omega) = \frac{(j\omega)^2+4j\omega+11}{(j\omega+3)((j\omega)^2+2j\omega+5)} = \frac{1}{j\omega+3} + \frac{j/2}{j\omega+(1+2j)} + \frac{-j/2}{j\omega+(1-2j)}$$

Calculating this was a little tedious, even using the cover-up method. Is it really necessary? Since the Fourier transform techniques are normally applied to systems in the physical world, we will usually see only *rational Fourier transforms with real coefficients*. We know we can factor a nth order polynomial into a product of first-order polynomials each containing a root. Polynomials with real coefficients have complex conjugate roots; those factors can be pairwise combined to form quadratic polynomials (second-order) with real coefficients. So:

Factorization of Polynomials with Real Coefficients:

A polynomial with real coefficients can be factored into first- and second-order polynomials with real coefficients.

Second-order polynomials are in the tables, so we don't have to factor those into first-order polynomials.

Sufficiency of Factorization:

For the purposes of finding the inverse Fourier transform, it is sufficient to factor the denominator of a rational Fourier transform, which is a polynomial in s with real coefficients, into first- and second-order polynomials, which have real coefficients.

Thus, we can expand the Fourier transform to just two terms:

$$X(j\omega) = \frac{(j\omega)^2 + 4j\omega + 11}{(j\omega + 3)((j\omega)^2 + 2j\omega + 5)} = \frac{1}{j\omega + 3} + \frac{2}{(j\omega)^2 + 2j\omega + 5}$$

We can complete the square on the second term to get:

$$X(j\omega) = \frac{(j\omega)^2 + 4j\omega + 11}{(j\omega + 3)((j\omega)^2 + 2j\omega + 5)} = \frac{1}{j\omega + 3} + \frac{2}{(j\omega + 1)^2 + 2^2}$$

Ah ha! We know how to handle the second term; it's just a decaying sinusoid. To, the corresponding signal is:

$$x(t) = e^{-3t}u(t) + \left[e^{-t}\sin(2t)\right]u(t).$$

We didn't actually have to factor the denominator completely; having second-order polynomials (quadratics) was fine. These are in the square-completed form:

$$\frac{1}{(j\omega + \alpha)^2 + \omega_0^2}.$$

Thus, we didn't have to use the quadratic formula at all; completing the square is an easier and more useful procedure.

4.2 Summary of finding the inverse Fourier transform

If we also take top-heavy and multiple roots into account, the partial faction decompostion process in the context of Fourier transforms can be summarized:

Finding the Inverse Fourier Transform of Rational Transforms with Real Coefficients:

1. If the order of the numerator is equal to or greater than that of the denominator, use synthetic division to express the transform as a polynomial plus a rational polynomial with the order of the numerator less than that of the denominator.
2. Factor the denominator into first- and second-order polynomials.
3. Do partial fraction expansion (watch for multiple roots).
4. Complete the square for second-order polynomials.
5. Use the tables to find the inverse Fourier transform of each term.

5 Inverse DT Fourier Transform of Rational Functions of $\mathrm{e}^{-j\omega}$

Similarly to CT, *stable DT LTI systems described by linear constant-coefficient difference equations have frequency responses that are rational functions of* $\mathrm{e}^{-j\omega}$. The standard form of the rational functions for DT systems is different from that of CT systems, so partial-fraction expansion is a little different. Okay, now here's an interesting question: should we set the generic variable x to be $\mathrm{e}^{j\omega}$, or should it be $\mathrm{e}^{-j\omega}$? We can actually use either! But, as we will see when we do z-transforms, the $x = \mathrm{e}^{j\omega}$ convention is preferred.

Problem 6.3

Find the inverse DT Fourier transform $x_i[n]$ for the following Fourier transform:

(a)

$$X_a(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

(b)

$$X_b(e^{j\omega}) = \frac{1}{(1 - \frac{1}{2}e^{-j\omega})^2(1 - \frac{3}{4}e^{-j\omega})}$$

Find the inverse CT Fourier transform $x_i(t)$ of the following Fourier transform:

(c)

$$X_c(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}.$$

(d)

$$X_d(j\omega) = \frac{(j\omega)^2 + 7}{(j\omega + 3)(j\omega + 4)}$$

6 Fourier Transforms and LTI Systems

In Tutorials 3 and 4, we found that complex exponentials are the eigenfunctions of LTI systems, and thus the Fourier series coefficients of the output signal of LTI systems is the Fourier series coeffficients of the input signal each scaled by the frequency response of the system evaluated at that frequency. This analysis is applicable only to periodic signals. By using the Fourier transform, however, we can extend this to aperiodic signals as well. We saw that convolution in time corresponds to multiplication in frequency. Thus, *the Fourier transform of the output signal is the product of the Fourier transform of the input signal and the frequency response of the system, which is the Fourier transform of the impulse response.*[1]

So, for a CT LTI system, instead of doing convolution to calculate the output $y(t)$ when the input is $x(t)$ and the impulse response is $h(t)$:

$$y(t) \quad = \quad x(t) * h(t),$$

the following analysis may be simplier or provide more insight. First, we take the Fourier transforms of $x(t)$ and $h(t)$ to get:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-j\omega t}\,\mathrm{d}t, \\ H(j\omega) &= \int_{-\infty}^{+\infty} h(t)\mathrm{e}^{-j\omega t}\,\mathrm{d}t. \end{aligned}$$

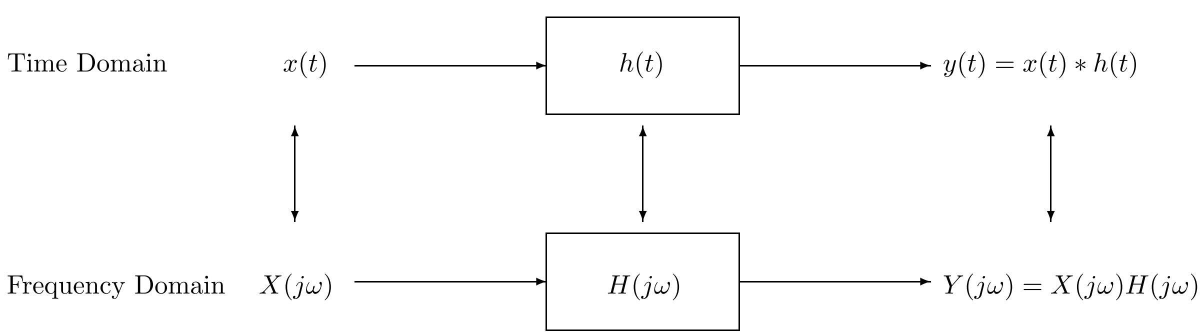
Then, we multiply the two together to get the Fourier transform of the output:

$$H(j\omega) \quad = \quad X(j\omega)H(j\omega).$$

Finally, we take the inverse Fourier transform of this to get the output signal $y(t)$:

$$y(t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)\mathrm{e}^{j\omega t}\,\mathrm{d}\omega.$$

Thus, we have two ways of looking at LTI systems: the time domain and the frequency domain.



The DT case is analogous.

[1] *N.B.:* Only *stable* LTI systems have a frequency response.

6.1 Differential and difference equations

In general, finding transforms is difficult. However, for LTI systems described by differential and difference equations, Fourier analysis is very straightforward for many inputs. The following steps should be used to find the output:

Computing the Output of a Stable System Described by a Diff Eq from the Input:

1. First compute the frequency response of the system. You can either take the Fourier transform of the impulse response or use the differential/difference equation via the method we used in Tutorial 4. Let this be $H(j\omega)$.
2. Take the Fourier transform of the input signal. You may have to use the Fourier transform properties and tables to do this. Let this be $X(j\omega)$.
3. Multiply $X(j\omega)$ by $H(j\omega)$. This will be the Fourier Transform of the output, $Y(j\omega)$.
4. Take the inverse transform of $Y(j\omega)$ to find $y(t)$. You may have to use Fourier Transform properties, polynomial long division and partial fraction expansion along with Fourier Transform tables.
   (a) If the expression for $Y(j\omega)$ has a higher order polynomial in the denominator vs. numerator, use partial fraction expansion.
   (b) If the expression for $Y(j\omega)$ has a lower order polynomial in the denominator vs. numerator, use long division and partial fraction expansion.
   (c) If the expression for $Y(j\omega)$ has a polynomial in the denominator raised to a power greater than 1, use the differentiation property in conjunction with one or more of the above steps.

Note that this assumes that the Fourier transforms for the signal and the impulse response of the system exist.

Problem 6.4

Consider a stable CT LTI system given by the following differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 4\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t}x(t) + 2x(t).$$

(a) Find the frequency response $H(j\omega)$ of this system.

(b) Find the impulse response $h(t)$ of this system by taking the inverse Fourier transform of $H(j\omega)$. (See Problem 6.3(c).)

(c) Find the output of this system if the input is $x(t) = \mathrm{e}^{-2t}u(t)$.

(Work space)

Problem 6.5

When we learned about convolution, there was an issue we didn't address. If $y(t) = x(t) * h(t)$, then is it possible to find $x(t)$ given $y(t)$ and $h(t)$? Is this $x(t)$ unique? In other words, is it possible to "deconvolve?" A proper treatment of these questions requires other techniques (such as the Laplace transform), but in this problem, we will answer it for signals with Fourier transforms.

Consider a causal stable CT LTI system with impulse response:

$$h(t) \quad = \quad \mathrm{e}^{-3t}u(t).$$

Let the output be:

$$y(t) \quad = \quad \mathrm{e}^{-3t}u(t) - \mathrm{e}^{-4t}u(t).$$

Find the the input signal $x(t)$.

Problem 6.6

Consider an LTI system S with impulse response

$$h(t) = \frac{\sin(4(t-1))}{\pi(t-1)}.$$

Determine the output of S for each of the following inputs:

(a) $x_1(t) = \sum_{k=0}^{\infty} (\frac{1}{2})^k \sin(3kt)$

(b) $x_2(t) = (\frac{\sin(2t)}{\pi t})^2$

(Work space)

7 Time-Frequency Uncertainty Principle (Optional)

Note that a "narrow" signal tends to have a "wide" Fourier transform and *vice versa*. The time-frequency scaling property confirms this, namely that turning $x(t)$ into $x(at)$ corresponds to turning $X(j\omega)$ into $\frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$. The impulse and the constant signal represent the extreme cases of this phenomenon. We can quantify this concept by thinking in terms of probability. Let's consider $|x(t)|^2$ and $|X(j\omega)|^2$ to be *probability density functions (PDFs)*, normalized by the total energy:[2]

$$\begin{aligned} ||x||_2^2 &= \int_{-\infty}^{+\infty} |x(t)|^2 dt \\ ||X||_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega. \end{aligned}$$

So, the *first moments* (or "mean") of these PDFs are the expected values of t and ω:

$$\begin{aligned} \overline{t} &= \frac{1}{||x||_2^2}\int_{-\infty}^{+\infty} t|x(t)|^2 dt \\ \overline{\omega} &= \frac{1}{||X||_2^2}\int_{-\infty}^{+\infty} \omega|X(j\omega)|^2 \frac{d\omega}{2\pi}. \end{aligned}$$

We can define the "standard deviation" of these PDFs as:

$$\begin{aligned} \Delta t &= \frac{1}{||x||_2}\sqrt{\int_{-\infty}^{+\infty} (t-\overline{t})^2|x(t)|^2 dt} \\ \Delta\omega &= \frac{1}{||X||_2}\sqrt{\int_{-\infty}^{+\infty} (\omega-\overline{\omega})^2|X(j\omega)|^2 \frac{d\omega}{2\pi}}. \end{aligned}$$

It can be shown that if $x(t)$ falls off sufficiently quickly, specifically, if $\lim_{|t|\to\infty}\sqrt{t}x(t) = 0$, then:

$$\boxed{\Delta t \cdot \Delta\omega \geq \frac{1}{2}.}$$

This means that *the product of the "spread" of the signal in the time and frequency domains is always greater than or equal to one.* This is known as the *time-frequency uncertainty principle for Fourier transform pairs.* We saw in lecture that a Gaussian (or bell curve) has a Fourier transform that is also a Gaussian; this signal is the only one that achieves equality in the above relation. Thus, the Gaussian is known as the *minimum uncertainty wave packet.* In fact, we can use this theorem to prove the Heisenberg uncertainty principle in quantum mechanics, for the corresponding wavefunctions are related through the Fourier transform (with a factor of $\hbar$). For instance, the wavefunction in momentum space is the Fourier transform of that in position space. In this case, the Heisenberg uncertainty principle states that $\Delta x\Delta p \geq \frac{\hbar}{2}$, where Δx and Δp are the uncertainties in position and momentum, respectively.

[2] The integral of a PDF is 1, so we need to divide by the total energy.