MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

Tutorial 5

Monday, March 8 and Tuesday, March 9, 2004

Announcements

- There is no problem set due this week.
- Quiz 1 will be held on **Thursday, March 11**, 7:30–9:30 p.m. in Walker Memorial. The quiz will cover material in Chapters 1–3 of O&W, Lectures and Recitations through February 27, Problem Sets #1–3, and that part of Problem Set #4 involving problems from Chapter 3.
- The TAs will jointly hold office hours from 2–8 p.m. on Wednesday, March 10 and again from 10 a.m.–3 p.m. on Thursday, March 11. A schedule is posted on the 6.003 web site.
- A quiz review package is available on the 6.003 web site. TAs will hold two identical optional quiz review sessions on Monday, March 8 and Tuesday, March 9, 7:30–9:30 p.m. in 34-101.

Today's Agenda

- The Big Picture Thus Far
- Complex Number Tricks
  - Evaluating the magnitude and phase of a sum of complex exponentials
  - Evaluating the magnitude and phase of a frequency response
  - Real/imaginary parts and even/odd parts
- Caveats About the Unit Impulse
  - Time scaling
  - Differentiation and the product rule
- Caveats About the CT and DT Fourier Series
  - Differentiation rule in CT
  - Time scaling
  - Frequency scaling
  - Multiplication by $(-1)^k$ and $(-1)^n$
  - Matrix view of the DTFS (optional)
- Eigenstuff

1 The Big Picture Thus Far

Let's summarize 6.003 so far without using any equations (English only). In Chapter 1 of the textbook, we defined a *signal* to be a set of complex numbers indexed by time, and the index can be continuous (CT) or discrete (DT). We then defined a *system* to be a deterministic mapping from *input* signals to *output* signals. There are four (independent) system properties of interest to us: linearity, time-invariance, causality and stability.

In Chapter 2, we restricted ourselves to systems that are both linear and time-invariant (LTI). LTI systems are associative, commutative and distributive. Four questions motivated us:

- How can we represent LTI systems?
- Given a representation and an input, how can we find the output?
- How can we convert one representation to another?
- Which representation is the best?

The first characterization of LTI systems was the *impulse response*, defined to the output of an LTI system when the input is a unit impulse. The impulse response is a *complete characterization* of an LTI system, namely there is a one-to-one correspondence between the set of LTI systems and the set of impulse responses. The I/O signals and the impulse response are related in a straightforward manner: the output is the *convolution sum* or *convolution integral* of the impulse response and the input signal. However, this computation is considered tedious and does not offer much insight into solving certain kinds of problems.

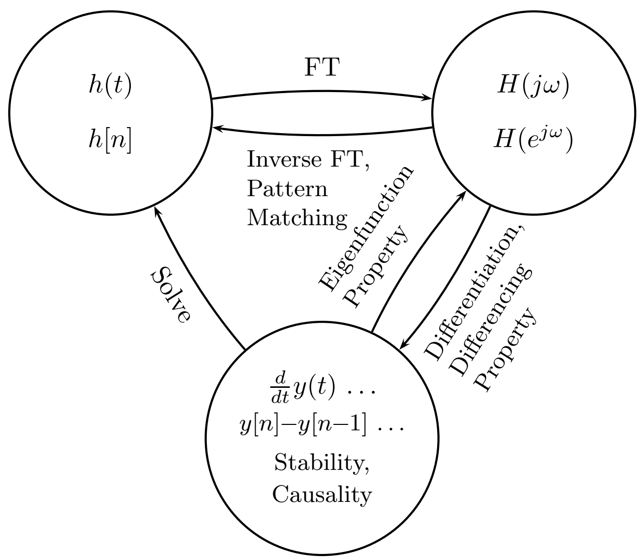
We then looked at another method of describing LTI systems: linear constant coefficient ordinary differential (or difference) equations. Keep in mind that only *some* LTI systems can be described in this manner, but a large number of the LTI systems that occur in nature fall into this class. The caveat to this description is that diff eq's do *not* uniquely specify LTI systems. Two or more LTI systems may be described by exactly the same diff eq. If we specify auxiliary properties, such as stability or causality, then the system is unique. However, finding the output given a particular input is just as tedious as convolution: we need to solve a differential equation. Can we convert from the diff eq (with additional conditions) representation to the impulse response representation? Two problems on Problem Set 2 were devoted to the forward conversion (which was tedious), and we haven't looked at going in the other direction.

In Chapter 3, we studied what is perhaps the most important idea in all of 6.003: *complex exponentials are the eigenfunctions of all LTI systems.* In other words, the output of an LTI system when the input is a complex exponential is the *same* complex exponential scaled by a complex factor. When we restricted the exponents to be purely imaginary, that scale factor, or the eigenvalue corresponding to that eigenfunction, is defined to be the *frequency response* of the LTI system.[1] Thus, we have a third way of specifying LTI systems: the frequency response. However, this representation can only be used for *stable* LTI systems (and a few exceptional pathological systems). Now, how do we use the frequency response to find outputs from inputs? Well, if we can express an input as the superposition of complex exponentials, then the output would be the superposition of the *same* exponentials, each scaled by the frequency response evaluated at that frequency. In Chapter 3, we restricted ourselves to *periodic* input signals[2], so we developed the *Fourier series* representation of signals. In Chapter 4, we generalized to *aperiodic* signals and used the *Fourier transform*[3] representation.

[1] Later, when we relax this restriction, we will call the eigenvalue the *transfer function* or *system function*.
[2] Subject to the Dirichlet conditions, that is.
[3] Fourier transforms are **not** on Quiz 1.

How is the frequency-domain representation related to the others? We can translate diff eq's into a frequency response by the steps detailed in Tutorial 4. The time and frequency relationship is quite remarkable: *the frequency response is the Fourier transform of the impulse response.*

Three Ways to Characterize LTI Systems So Far:



- Impulse response: complete characterization, all LTI systems have one.
- Frequency response: only for stable systems (with a few exceptions, like an integrator).
- Diff Eq: only certain LTI systems can be described this way, plus the diff eq alone does not specify the system; need additional info like stability and causality.

Depending on what we are trying to analyze, one of the characterizations above may be better suited. If we are modelling physical phenomena, such as a mechanical system or circuit, it may be easy to derive a differential equation to model that system. From that differential/difference equation we can then solve for the impulse response or the frequency response. We saw that working with the impulse response can be difficult (i.e. convolution). Recently, with the FT we have begun to see that using the frequency domain can turn problems involving convolution into ones involving simple multiplication. In other words, *convolution in one domain corresponds to multiplication in the other domain.*

Representation	Applicable to All LTI Systems?	Is It a Unique and Complete Description?	How to Find Output Given Input
Impulse response	Yes	Yes	Convolution
Diff eq	No, but many useful ones	No, auxiliary conditions required	Solve the diff eq (tedious)
Frequency response	No, stable systems only	Yes	Scale each Fourier series coef by frequency response

Table 1: Comparison chart of LTI system representations

2 Complex Number Tricks

2.1 Evaluating the magnitude and phase of a sum of complex exponentials

Suppose we wanted to find the magnitude and phase of the signal:

$$x[n] = \mathrm{e}^{j\left(-\frac{\pi}{6}n-\frac{\pi}{2}\right)} + \frac{1}{2}\mathrm{e}^{j\frac{\pi}{6}n} + 3\mathrm{e}^{j\frac{\pi}{2}n} + \frac{1}{2}\mathrm{e}^{j\frac{5\pi}{6}n} + \mathrm{e}^{j\left(\frac{7\pi}{6}n+\frac{\pi}{2}\right)}.$$

This form of a signal comes up all the time, and we would like a quick and neat method to do this. Let's rearrange and group the terms:

$$x[n] = 3\mathrm{e}^{j\frac{\pi}{2}n} + \frac{1}{2}\left(\mathrm{e}^{j\frac{\pi}{6}n} + \mathrm{e}^{j\frac{5\pi}{6}n}\right) + \left(\mathrm{e}^{j\left(\frac{7\pi}{6}n+\frac{\pi}{2}\right)} + \mathrm{e}^{j\left(-\frac{\pi}{6}n-\frac{\pi}{2}\right)}\right).$$

Note that in each group, the average of the exponents is $\mathrm{e}^{j\frac{\pi}{2}n}$, so let's factor it out:

$$x[n] = \mathrm{e}^{j\frac{\pi}{2}n}\left[3 + \frac{1}{2}\left(\mathrm{e}^{-j\frac{\pi}{3}n} + \mathrm{e}^{j\frac{\pi}{3}n}\right) + \left(\mathrm{e}^{j\left(\frac{2\pi}{3}n+\frac{\pi}{2}\right)} + \mathrm{e}^{j\left(-\frac{2\pi}{3}n-\frac{\pi}{2}\right)}\right)\right].$$

We can factor out the $\frac{\pi}{2}$ phase, which becomes j:

$$\begin{aligned}
x[n] &= \mathrm{e}^{j\frac{\pi}{2}n}\left[3 + \frac{1}{2}\left(\mathrm{e}^{-j\frac{\pi}{3}n} + \mathrm{e}^{j\frac{\pi}{3}n}\right) + \left(\mathrm{e}^{j\frac{\pi}{2}}\mathrm{e}^{j\left(\frac{2\pi}{3}n\right)} + \mathrm{e}^{-j\frac{\pi}{2}}\mathrm{e}^{j\left(-\frac{2\pi}{3}n\right)}\right)\right] \\
&= \mathrm{e}^{j\frac{\pi}{2}n}\left[3 + \frac{1}{2}\left(\mathrm{e}^{-j\frac{\pi}{3}n} + \mathrm{e}^{j\frac{\pi}{3}n}\right) + \left(j\mathrm{e}^{j\left(\frac{2\pi}{3}n\right)} - j\mathrm{e}^{j\left(-\frac{2\pi}{3}n\right)}\right)\right] \\
&= \mathrm{e}^{j\frac{\pi}{2}n}\left[3 + \frac{1}{2}\left(\mathrm{e}^{-j\frac{\pi}{3}n} + \mathrm{e}^{j\frac{\pi}{3}n}\right) - 2\frac{1}{2j}\left(\mathrm{e}^{j\left(\frac{2\pi}{3}n\right)} - \mathrm{e}^{j\left(-\frac{2\pi}{3}n\right)}\right)\right].
\end{aligned}$$

We apply the Euler relations for cosine and sine:

$$x[n] = \mathrm{e}^{j\frac{\pi}{2}n}\left[3 + \frac{1}{2}\cos\left(\frac{\pi}{3}n\right) - 2\sin\left(\frac{2\pi}{3}n\right)\right].$$

$x[n]$ is now in the form $x[n] = r[n]e^{j\theta[n]}$, where $r[n]$ and $\theta[n]$ are the magnitude and phase, respectively:

$$\boxed{\begin{aligned} r[n] &= 3 + \frac{1}{2}\cos\left(\frac{\pi}{3}n\right) - 2\sin\left(\frac{2\pi}{3}n\right), \\ \theta[n] &= \frac{\pi}{2}n. \end{aligned}}$$

We always have to double-check that $r[n]$ is in fact always nonnegative, since a magnitude cannot be negative. If $r[n]$ takes on negative values, then we need to add π to $\theta[n]$.

2.2 Evaluating the magnitude and phase of a frequency response

As we saw in Tutorial 4, we sometimes want to find the magnitude and phase of CT frequency responses. In Problem 4.1, we have the following frequency response:

$$H(j\omega) = \frac{1}{-\omega^2 - \frac{3}{2} + \frac{5}{2}j\omega}.$$

In part (b-i), we need to evaluate the magnitude and phase when $\omega = 3\pi$:

$$\begin{aligned} H(j3\pi) &= \frac{1}{-(3\pi)^2 - \frac{3}{2} + \frac{5}{2}j(3\pi)} \\ &= \frac{2}{-18\pi^2 - 3 + j15\pi}. \end{aligned}$$

In general we can evaluate the magnitude and phase of a complex number, c, as follows

$$\begin{aligned} |c| &= \sqrt{c\,c^*} = \sqrt{Re\left\{c\right\}^2 + Im\left\{c\right\}^2} \\ \angle c &= \tan^{-1}\left(\frac{Im\left\{c\right\}}{Re\left\{c\right\}}\right). \end{aligned}$$

Oftentimes, we need to find the magnitude and phase of a ratio of complex numbers, a and b,

$$\begin{aligned} \left|\frac{a}{b}\right| &= \frac{|a|}{|b|} \\ \angle\left(\frac{a}{b}\right) &= \angle a - \angle b \end{aligned}$$

Applying these results to the example above, we have

$$\begin{aligned} |H(j3\pi)| &= \frac{2}{\sqrt{(18\pi^2+3)^2 + (15\pi)^2}} \\ \angle H(j3\pi) &= -\tan^{-1}\left(\frac{15\pi}{-18\pi^2 - 3}\right) = \pi + \tan^{-1}\left(\frac{5\pi}{6\pi^2+1}\right) \end{aligned}$$

For DT LTI systems described by difference equations, we should use the conjugate to find the magnitude of the frequency response. For example, the stable DT LTI system described by the difference equation:

$$y[n] + 2y[n-1] = 3x[n]$$

has the frequency response:

$$H(\mathrm{e}^{j\omega}) = \frac{3}{1 + 2\mathrm{e}^{-j\omega}}.$$

The magnitude is:

$$\begin{aligned}
|H(\mathrm{e}^{j\omega})| &= \sqrt{H^*(\mathrm{e}^{j\omega})H(\mathrm{e}^{j\omega})} \\
&= \sqrt{\frac{3}{1+2\mathrm{e}^{j\omega}} \cdot \frac{3}{1+2\mathrm{e}^{-j\omega}}} \\
&= \frac{3}{\sqrt{1 + 2\mathrm{e}^{-j\omega} + 2\mathrm{e}^{j\omega} + 4}} \\
&= \frac{3}{\sqrt{5 + 4\cos(\omega)}}
\end{aligned}$$

2.3 Real/imaginary parts and even/odd parts

All complex numbers can be written as the sum of a real part and j times an imaginary part:

$$z = Re\{z\} + jIm\{z\}.$$

We can find the real part and imaginary part of a complex number by making use of the complex conjugate:

$$\begin{aligned}
Re\{z\} &= \frac{z + z^*}{2}, \\
Im\{z\} &= \frac{z - z^*}{2j}.
\end{aligned}$$

These identities are often useful in proving some of the properties of Fourier series coefficients and the Fourier transform.

Note that all this real and imaginary parts stuff for complex numbers bear a strong resemblance to the even and odd parts of a signal. Any signal can be written as the sum of an even signal and an odd signal:

$$x(t) = Ev\{x(t)\} + Od\{x(t)\}.$$

$$\begin{aligned}
Ev\{x(t)\} &= \frac{x(t) + x(-t)}{2}, \\
Od\{x(t)\} &= \frac{x(t) - x(-t)}{2}.
\end{aligned}$$

3 Caveats About the Unit Impulse

3.1 Time scaling

What happens if we time scale a unit impulse? Watch out! The result is different for DT and CT. For an integer k, there is no change in DT:

$$\delta[kn] \quad = \quad \delta[n].$$

However, in CT and real scale factor a:

$$\delta(at) \quad = \quad \frac{1}{|a|}\delta(t).$$

In DT, the is no additional scale factor in front of the delta function, but it appears in CT because this factor refers to the area of the impulse when integrated.

3.2 Differentiation and the product rule

We know that if the CT unit impulse is multiplied by a signal, we can replace the signal with its value at the time that the delta function is nonzero.

For example:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

Similarly we can compute derivatives of expressions involving delta functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t)\delta(t) = f'(t)\delta(t) + f(t)u_1(t) = f'(0)\delta(t) + f(0)u_1(t)$$

is incorrect, but

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t)\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}f(0)\delta(t) = f(0)u_1(t)$$

is correct. We can avoid errors by simplifying all expressions at each stage of manipulation involving delta functions.

4 Caveats About the CT and DT Fourier Series

4.1 Differentiation rule in CT

Note that the differentiation property for CT FS states that:

$$\begin{aligned} x(t) &\quad\longleftrightarrow\quad a_k \\ \frac{\mathrm{d}}{\mathrm{d}t}x(t) &\quad\longleftrightarrow\quad b_k = jk\omega_0 a_k \end{aligned}$$

Therefore, $b_0 = 0$.

The derivative of a periodic CT signal has no DC value, *i.e.* its Fourier series coefficient a_0 is zero.

When going in the reverse direction (i.e. integrating) we must take care to calculate the a_0 term separately.

4.2 Time scaling

Time scaling a CT signal by a doesn't change its FS representation. However, in DT, time expansion by an integer m gives a height scaling of $\frac{1}{m}$. Consider a_0 before the time expansion

$$a_0 = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]$$

Now, the time scaled version, $x'[n]$, has period mN and

$$a_0' = \frac{1}{mN} \sum_{n=\langle mN \rangle} x'[n] = \frac{1}{mN} \sum_{n=\langle N \rangle} x[n] = \frac{1}{m} a_0$$

4.3 Frequency scaling

Let $x[n]$ be a periodic DT signal with Fourier series coefficients a_k. Let $y[n]$ be the signal whose Fourier series coefficients b_k is related to those of a_k for some integer m, as:

$$b_k = \begin{cases} a_{k/m}, & \text{if } k \text{ is a multiple of } m \\ 0, & \text{if } k \text{ is not a multiple of } m \end{cases}$$

What is relationship between $x[n]$ and $y[n]$?

First we note that b_k consists of the coefficients a_k spread out by a factor of m, this means that we must sum over a period of $N' = mN$ to include all the coefficients. Thus,

$$\begin{aligned} y[n] &= \sum_{k=\langle mN \rangle} b_k e^{-jk\frac{2\pi}{mN}n} \\ \text{setting } k = lm \text{ gives} & \\ &= \sum_{l=\langle N \rangle} b_{lm} e^{-jlm\frac{2\pi}{mN}n} \\ &= \sum_{l=\langle N \rangle} a_l e^{-jl\frac{2\pi}{N}n} \\ &= x[n] \end{aligned}$$

The same is also true in CT.

4.4 Multiplication by $(-1)^k$ and $(-1)^n$

Let $x[n]$ be a periodic DT signal with period N and Fourier series coefficients a_k. Let $y[n]$ be the signal whose Fourier series coefficients b_k is related to those of a_k as:

$$b_k = (-1)^k a_k.$$

What's the relationship between $x[n]$ and $y[n]$? We rewrite -1 as $\mathrm{e}^{\pm j\pi} = \mathrm{e}^{\pm j(N/2)(2\pi/N)} = \mathrm{e}^{\pm j(N/2)\omega_0}$. From the time-shifting property, this is simply a time shift by half a period:

$$y[n] = x\left[n \pm \frac{N}{2}\right].$$

Note that shifting a periodic signal to the *left* by half a period is the same as shifting it to the *right* by a half a period. The same result holds in CT.

Similarly, multiplying a periodic DT signal $x[n]$ by $(-1)^n$ to form $(-1)^n x[n]$ corresponds to shifting the Fourier series coefficients by half a period. This has no counterpart in CT.

4.5 Matrix view of the DTFS (optional)

We can view the operations of the analysis and synthesis equations for DTFS in terms of matrix operations and transformations in vector space. Consider the $N \times N$ matrix $\mathbf{A}$ defined as:

$$\mathbf{A} \triangleq \begin{pmatrix} e^{-j\omega_0 \cdot 0 \cdot 0} & e^{-j\omega_0 \cdot 0 \cdot 1} & \dots & e^{-j\omega_0 \cdot 0 \cdot (N-1)} \\ e^{-j\omega_0 \cdot 1 \cdot 0} & e^{-j\omega_0 \cdot 1 \cdot 1} & \dots & e^{-j\omega_0 \cdot 1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\omega_0 \cdot (N-1) \cdot 0} & e^{-j\omega_0 \cdot (N-1) \cdot 1} & \dots & e^{-j\omega_0 \cdot (N-1) \cdot (N-1)} \end{pmatrix}$$

Now consider two $N \times 1$ vectors $\mathbf{x}$ and $\mathbf{a}$ representing one period of a DT signal and its corresponding FS coefficients. Then, we can write the synthesis and analysis equations as a linear transformation from one vector to the other:

$$\begin{aligned} \mathbf{a} &= \frac{1}{N}\mathbf{A}\mathbf{x} \\ \mathbf{x} &= \mathbf{A}^\dagger \mathbf{a} \end{aligned}$$

where $(\cdot)^\dagger$ represents conjugate transpose.

Note that $\mathbf{A}^\dagger \mathbf{A} = \mathbf{A}\mathbf{A}^\dagger = N\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Thus, $\frac{1}{\sqrt{N}}\mathbf{A}$ is a unitary matrix and represents an orthonormal transformation. If we remove the $\frac{1}{\sqrt{N}}$ scale factor, then $\mathbf{A}$ is an orthogonal (but not orthonormal) transformation. This reinforces the idea the the Fourier series is nothing more than an *orthogonal change of coordinate system*.

It is now easy to prove Parseval's relation. The square of the length a vector is the inner product of the vector with itself, or the sum of the squares of each orthogonal component:

$$\begin{aligned} \sum_{n=\langle N \rangle} |x[n]|^2 &= \mathbf{x}^\dagger \mathbf{x} \; = \; \left(\mathbf{A}^\dagger \mathbf{a}\right)^\dagger \left(\mathbf{A}^\dagger \mathbf{a}\right) \; = \; \left(\mathbf{a}^\dagger \mathbf{A}\right)\left(\mathbf{A}^\dagger \mathbf{a}\right) \\ &= \mathbf{a}^\dagger \left(\mathbf{A}\mathbf{A}^\dagger\right) \mathbf{a} \; = \; \mathbf{a}^\dagger \left(N\mathbf{I}\right) \mathbf{a} \; = \; N\mathbf{a}^\dagger \mathbf{a} \\ &= N \sum_{k=\langle N \rangle} |a_k|^2 \end{aligned}$$

Thus:

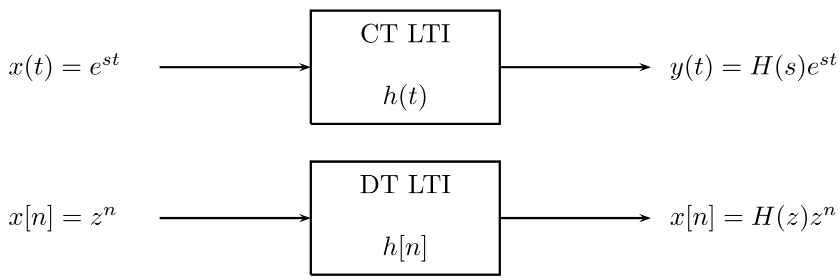
$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 \quad = \quad \sum_{k=\langle N \rangle} |a_k|^2.$$

5 Eigenstuff

We showed in lecture that a certain set of input signals, namely complex exponentials of the form $x(t) = \mathrm{e}^{st}$ ($x[n] = z^n$), are *eigenfunctions* of LTI systems, i.e. the corresponding outputs are simply scaled versions of inputs of this form, and this scaling factor is the *eigenvalue*. We showed that the outputs of CT and DT LTI systems in response to $x(t) = \mathrm{e}^{st}$ and $x[n] = z^n$ are $y(t) = H(s)\mathrm{e}^{st} = H(s)x(t)$ and $y[n] = H(z)z^n = H(z)x[n]$, respectively, where the eigenvalues $H(s)$ and $H(z)$ associated with the given eigenfunctions are:

$$\begin{aligned} H(s) &= \int_{-\infty}^{+\infty} h(\tau)\mathrm{e}^{-s\tau}\, d\tau \\ H(z) &= \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \end{aligned}$$

where $h(t)$ and $h[n]$ are the impulse responses of the systems.



The superposition property of LTI systems suggests another way of writing signals. We can express an input signal as a linear combination of complex exponentials. Then, the output of an LTI system is the same linear combination of the exponentials scaled by the appropriate eigenvalue. So, if the inputs are:

$$\begin{aligned} x(t) &= \sum_k a_k \mathrm{e}^{s_k t} \\ x[n] &= \sum_k a_k z_k^n \end{aligned}$$

then the outputs are:

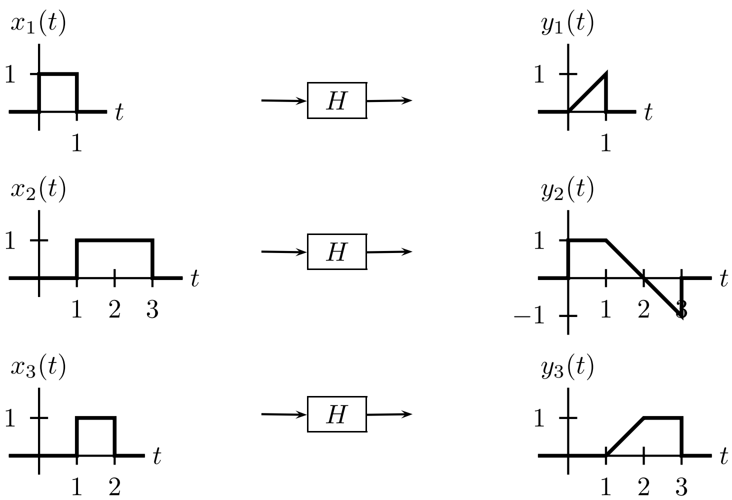
$$\begin{aligned} y(t) &= \sum_k a_k H(s_k)\mathrm{e}^{s_k t} \\ y[n] &= \sum_k a_k H(z_k) z_k^n \end{aligned}$$

Problem 5.1

Consider a *linear* system H that has input-output pairs depicted in the figure below. Determine the following and explain your answers:

(a) Is this system causal?

(b) Is this system time invariant?



Problem 5.2

The systems given below have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$, respectively. Determine whether each of them is (i) stable, (ii) causal, (iii) linear, and (iv) time invariant.

(a) $y(t) = \int_{-\infty}^{t/2} x(\tau) d\tau$

(b) $y[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n - 2k]$

(c) $y[n] = \log_{10}(|x[n]|)$

Problem 5.3

Evaluate the following continuous-time convolution integrals given below.

(a) $y_a(t) = [\cos(\pi t)(u(t+1) - u(t-3))] * \left[e^{-2t}u(t)\right]$.

(b) $y_b(t) = \left[(t + 2t^2)(u(t+1) - u(t-1))\right] * 2u(t+2)$.

Problem 5.4

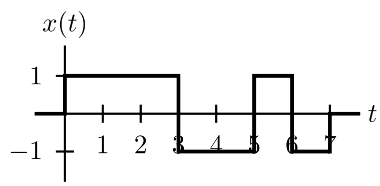
The convolution of a signal with itself turned around in time is called the *autocorrelation function* of that signal. In continuous-time, the autocorrelation function is given by

$$r_x(t) = \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau$$

(a) By comparison with the convolution integral, determine the impulse response of a system which, given a particular $x(t)$ as its input, will yield $r_x(t)$ as its output. Such a system is called a *matched filter.*

(b) Suppose $x(t)$ is given by

$$x(t) = u(t) - 2u(t-3) + 2u(t-5) - 2u(t-6) + u(t-7),$$



sketch the impulse response of the associated matched filter.

(c) Sketch the output of the matched filter with $x(t)$ above as input. That is, sketch the autocorrelation function $r_x(t)$ corresponding to $x(t)$.

Problem 5.5

Suppose we have a causal, stable DT LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

Find the output $y[n]$ of the system when the input $x[n]$ is:

$$x[n] = \sin\left(\frac{2\pi}{3}n\right) + u[n-2].$$

The four key points to solving this problem are:

- Finding the impulse response $h[n]$ of a LTI system described by a difference equation given the proper conditions,
- Finding the frequency response $H(e^{j\omega})$ of the system from the difference equation,
- Using the eigenfunction property of LTI systems, and
- Using the commutative property of cascaded LTI systems.

(Work space)

(Work space)

Problem 5.6

(a) Find the impulse response $h_a[n]$ of a DT LTI system whose output is $y_a[n] = \delta[n]$ when the input is $x_a[n] = (\frac{1}{3})^n u[n]$.

(b) Find the impulse response $h_b(t)$ of a CT LTI system whose output is $y_b(t) = t[u(t) - u(t-3)]$ when the input is $x_b(t) = u(t-1)$.

(c) Find the impulse response $h_c(t)$ of a CT LTI system whose output is $y_c(t) = t[u(t) - u(t-3)] + 3u(t-3)$ when the input is $x_c(t) = u(t-1)$.

Problem 5.7

Evaluate the following sum:

$$S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

You may find the following observation useful. We can also express S as:

$$S = -\frac{\pi^2}{8} + \frac{1}{2} \sum_{k=-\infty}^{+\infty} \left(\pi \frac{\sin(k\pi/2)}{k\pi} \right)^2.$$

Then, use Fourier series properties on the sum over k. You can solve it two different ways using either Parseval's relation or the periodic convolution property. You should find that:

$$S = \frac{\pi^2}{8}.$$

Problem 5.8

Suppose $x[n]$ is a periodic discrete-time signal with Fourier series coefficients a_k. The signal has the following properties:

1. $x[n]$ is real.
2. $x[n]$ is even.
3. The fundamental period of $x[n]$ is $N = 5$.
4. $\sum_{n=-2}^{n=2} x[n] = 0$.
5. $a_6 = 1$.
6. $\frac{1}{5}\sum_{n=0}^{n=4} |x[n]|^2 = 2$.

Find an expression that describes $x[n]$ completely.

(Work space)