MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

TUTORIAL 4

Monday, March 1 and Tuesday, March 2, 2004

Announcements

- Problem set 4 is due this Friday.
- Quiz 1 will be held on **Thursday, March 11**, 7:30–9:30 p.m. in Walker Memorial. The quiz will cover material in Chapters 1–3 of O&W, Lectures and Recitations through Feburary 27, Problem Sets #1–3, and that part of Problem Set #4 involving problems from Chapter 3.
- The TAs will jointly hold office hours from 2–8 p.m. on Wednesday, March 10 and again from 10 a.m.–3 p.m. on Thursday, March 11. A schedule will be posted on the 6.003 website.
- A quiz review package will be available on the 6.003 website this Thursday. TAs will hold two identical optional quiz review sessions on Monday, March 8 and Tuesday, March 9, 7:30–9:30 p.m. in 34-101.

Today's Agenda

- Frequency Response of LTI Systems
  - Differential and difference equations
  - Filtering
  - Real systems
  - Frequency response of cascaded systems
- CT Fourier Transform
  - Synthesis and analysis equations
  - Variations of the synthesis and analysis equations
  - Rectangular pulse and sinc pair
  - The multiplication and convolution properties

1 Frequency Response of LTI Systems

In our Fourier series representation of periodic signals, we set the CT variable $s = j\omega$, so that e^{st} becomes $e^{j\omega t}$. Likewise, in DT, we set $z = e^{j\omega}$, so that z^n becomes $e^{j\omega n}$. Then, the eigenvalue of the LTI system corresponding to the eigenfunction $\mathrm{e}^{j\omega t}$ (CT) and $\mathrm{e}^{j\omega n}$ (DT) is the *frequency response* of a system H. It is defined through the impulse response $h(t)$ (CT) and $h[n]$ (DT) as:

The Frequency Response of LTI Systems in Terms of the Impulse Response:

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,\mathrm{d}t \qquad \text{(CT)}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} \qquad \text{(DT)}$$

From the eigenfunction property, when these exponentials are the inputs of an LTI system, the outputs are the same exponentials scaled by the frequency response of the system. Now that we know how to write periodic input signals as the linear combination of complex exponentials by determining the Fourier series coefficients, we can scale the coefficients appropriately according to the frequency response of the system to get the Fourier series coefficients of the output. So, if the inputs are periodic signals with Fourier series coefficients a_k:

$$x(t) = \sum_k a_k \mathrm{e}^{jk\omega_0 t} \qquad \text{(CT)}$$

$$x[n] = \sum_k a_k \mathrm{e}^{jk\omega_0 n} \qquad \text{(DT)}$$

then the outputs are periodic signals with Fourier series coefficients $b_k = H(jk\omega_0)a_k$ for CT and $b_k = H(\mathrm{e}^{jk\omega_0})a_k$ for DT:

$$y(t) = \sum_k a_k H(jk\omega_0)\mathrm{e}^{jk\omega_0 t} \qquad \text{(CT)}$$

$$y[n] = \sum_k a_k H(\mathrm{e}^{jk\omega_0})\mathrm{e}^{jk\omega_0 n} \qquad \text{(DT)}$$

There is a caveat when speaking about the frequency response of LTI systems. All stable systems have well-defined frequency responses for all frequencies. However, unstable systems generally do not have a frequency response.

1.1 Differential and difference equations

A large number of LTI systems that we study in real life are described by linear constant-coefficient ordinary differential (CT) and difference (DT) equations (LCCODEs), so it would be helpful to develop techniques to analyze such systems. As we found in problem set 2, finding the impulse response of such systems (*time-domain* analysis) is a rather tedious procedure. However, it turns out that a *frequency-domain* analysis is much more straightforward:

Finding the Frequency Response of Differential and Difference Equations:

Suppose we are given a *stable* CT or DT system described by a differential or difference equation. To find the frequency response, we do the following:

1. Let $x(t) = \mathrm{e}^{j\omega t}$ for CT or $x[n] = \mathrm{e}^{j\omega n}$ for DT.
2. Let $y(t) = H(j\omega)\mathrm{e}^{j\omega t}$ for CT or $y[n] = H(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}$ for DT.
3. Plug $x(t)$ and $y(t)$ for CT or $x[n]$ and $y[n]$ for DT into the differential or difference equation.
4. Solve for $H(j\omega)$ for CT or $H(\mathrm{e}^{j\omega})$ for DT.

If we apply this method, we get the following result.

The Frequency Response of Differential and Difference Equations:

Suppose we are given a *stable* CT system described by the following differential equation:

$$\begin{aligned} &a_N \frac{\mathrm{d}^N}{\mathrm{d}t^N} y(t) + a_{N-1} \frac{\mathrm{d}^{N-1}}{\mathrm{d}t^{N-1}} y(t) + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}t} y(t) + a_0 y(t) \\ &= b_M \frac{\mathrm{d}^M}{\mathrm{d}t^M} x(t) + b_{M-1} \frac{\mathrm{d}^{M-1}}{\mathrm{d}t^{M-1}} x(t) + \cdots + b_1 \frac{\mathrm{d}}{\mathrm{d}t} x(t) + b_0 x(t). \end{aligned}$$

Its frequency response is:

$$H(j\omega) = \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_1(j\omega) + a_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0}$$

Similarly, a *stable* DT system described by the following difference equation:

$$\begin{aligned} &a_N y[n-N] + a_{N-1} y[n-(N-1)] + \cdots + a_1 y[n-1] + a_0 y[n] \\ &= b_M x[n-M] + b_{M-1} x[n-(M-1)] + \cdots + b_1 x[n-1] + b_0 x[n], \end{aligned}$$

has frequency response:

$$H(\mathrm{e}^{j\omega}) = \frac{b_M \mathrm{e}^{-jM\omega} + b_{M-1}\mathrm{e}^{-j(M-1)\omega} + \cdots + b_1 \mathrm{e}^{-j\omega} + b_0}{a_N \mathrm{e}^{-jN\omega} + a_{N-1}\mathrm{e}^{-j(N-1)\omega} + \cdots + a_1 \mathrm{e}^{-j\omega} + a_0}$$

1.2 Real systems

Frequency Response of Real Systems:

For a real system, namely, a systems where the impulse response $h(t)$ in CT (and $h[n]$ in DT) is real, the frequency response H is *conjugate even* (magnitude and real part are even signals, angle and imaginary parts are odd signals):

CT:

$$\begin{aligned} |H(j\omega)| &= |H(-j\omega)|, \\ \angle H(j\omega) &= -\angle H(-j\omega). \end{aligned}$$

DT:

$$\begin{aligned} |H(\mathrm{e}^{j\omega})| &= |H(\mathrm{e}^{-j\omega})|, \\ \angle H(\mathrm{e}^{j\omega}) &= -\angle H(\mathrm{e}^{-j\omega}). \end{aligned}$$

Thus, when the input of the system is:

$$\begin{aligned} x(t) &= \cos(\omega_0 t) \qquad \text{(CT)}, \\ x[n] &= \cos(\omega_0 n) \qquad \text{(DT)}, \end{aligned}$$

the output is:

$$\begin{aligned} y(t) &= |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0)) \qquad \text{(CT)}, \\ y[n] &= |H(\mathrm{e}^{j\omega_0})| \cos(\omega_0 n + \angle H(\mathrm{e}^{j\omega_0})) \qquad \text{(DT)}. \end{aligned}$$

Problem 4.1

Consider the stable CT LTI system described by (see problem set 2):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + \frac{5}{2}\frac{\mathrm{d}}{\mathrm{d}t}y(t) - \frac{3}{2}y(t) = x(t).$$

(a) Find the frequency response of the system.

(b) Find the output when the inputs are:

  (i) $x_{\mathrm{i}}(t) = \cos(3\pi t)$.

  (ii) $x_{\mathrm{ii}}(t) = \sin(\frac{3\pi}{4}t) + \cos(\pi t + \frac{\pi}{3})$.

(c) Let $x_c(t)$ be an input signal with fundamental period T and Fourier series coefficients a_k. Write the Fourier series coefficients b_k of the corresponding output signal $y_c(t)$ in terms of a_k and the frequency response. Assume that the fundamental period of $y_c(t)$ is also T.

Compare this method to finding the impulse response $h(t)$ and convolving $h(t)$ with the inputs to find the outputs.

Problem 4.2

Consider the stable DT LTI system described by (see problem set 2):

$$y[n] + \frac{5}{2}y[n-1] - \frac{3}{2}y[n-2] = x[n].$$

(a) Find the frequency response of the system.

(b) Find the output when the inputs are:

(i) $x_{\text{i}}[n] = \cos(\frac{\pi}{3}n)$.

(ii) $x_{\text{i}}[n] = \sin(\frac{3\pi}{4}n) + \cos(\frac{\pi}{4}n + \frac{\pi}{3})$.

(c) Let $x_c[n]$ be an input signal with fundamental period N and Fourier series coefficients a_k. Write the Fourier series coefficients b_k of the corresponding output signal $y_c[n]$ in terms of a_k and the frequency response. Assume that the fundamental period of $y_c[n]$ is also N.

Compare this method to finding the impulse response $h[n]$ and convolving $h[n]$ with the inputs to find the outputs.

Problem 4.3

(From 6.003 Quiz 1, Fall 2000) A CT signal $x(t)$ is periodic with period $T = 6$. Over one period, the signal is given by

$$x(t) = \begin{cases} 2, & |t| \leq 1 \\ 0, & 1 < |t| < 3 \end{cases}$$

The corresponding Fourier series coefficients for $x(t)$ are given by

$$a_k = \begin{cases} \frac{2}{3}, & k = 0 \\ \frac{2\sin(k\pi/3)}{k\pi}, & k \neq 0 \end{cases}$$

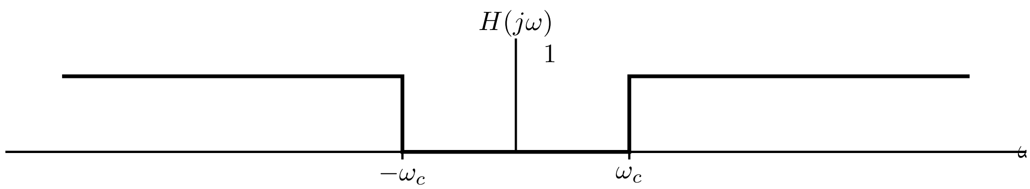
We are given the following facts about an LTI system into which $x(t)$ will be the input:

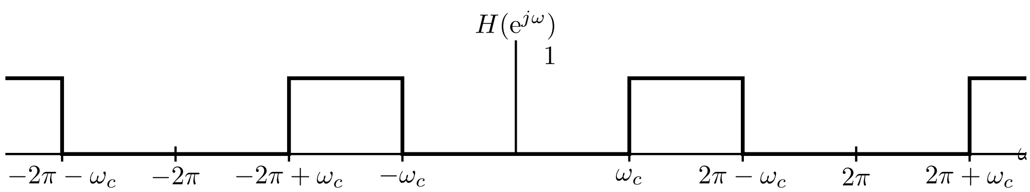
- The frequency response of the system is zero for $|\omega| > \pi$.
- The response to an input signal $1 + \cos[\frac{2\pi}{3}(t-5)]$ is a constant signal of 4.
- The response to an input signal $\cos(\frac{\pi}{3}t)$ is $\pi\cos(\frac{\pi}{3}t)$.

Find the output $y(t)$ when $x(t)$ is the input.

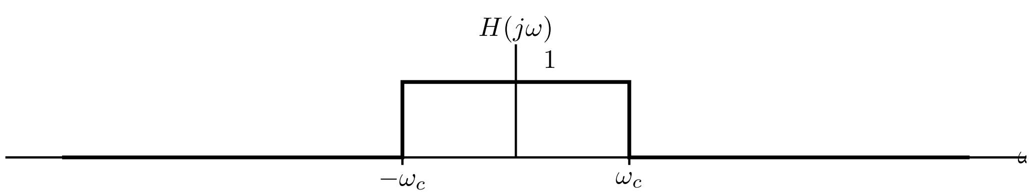
1.3 Filtering

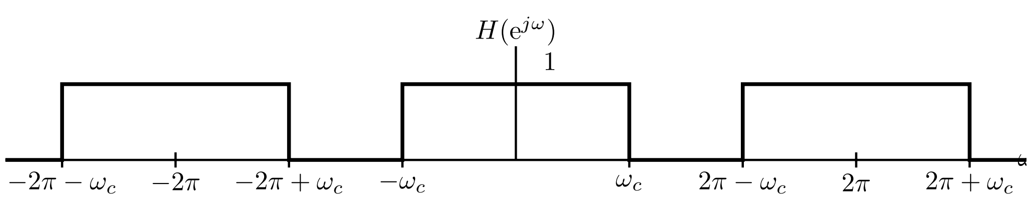
- Systems that act on input signals to produce a desired outcome.
- For LTI systems, they can be characterized by the impulse response or the frequency response, $H(j\omega)$ for CT and $H(e^{j\omega})$ for DT which tells you the scaling that is applied to each frequency component. (Keep in mind that the DT frequency response is periodic with period 2π; thus DT filters are usually defined between $-\pi$ and π)
- Often described by their magnitude characteristics, e.g. lowpass, highpass, bandpass etc.
  - **Highpass filters** have a high magnitude for the frequency response at high frequencies and low magnitude for lower frequencies. Below are examples of CT and DT **ideal** highpass filters with cutoff frequency ω_c:



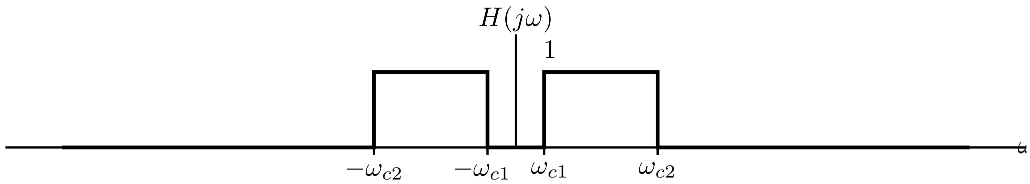


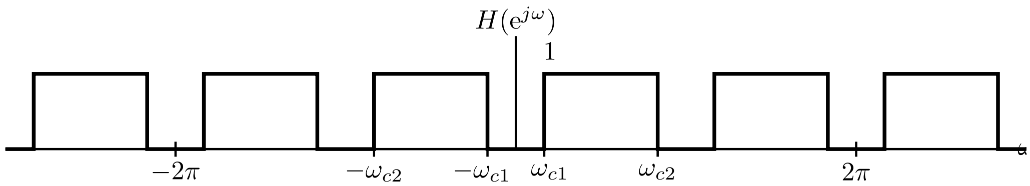
  - **Lowpass filters** have a high magnitude for the frequency response at low frequencies and low magnitude for higher frequencies.Below are examples of CT and DT **ideal** lowpass filters with cutoff frequency ω_c:





– **Bandpass filters** have a high magnitude for the frequency response of a specific band (or range) of frequencies and low magnitude for frequencies outside of that band. Below are examples of CT and DT **ideal** bandpass filters with cutoff frequencies ω_{c1} and ω_{c2}:





Problem 4.4

Consider a DT LTI system with frequency response:

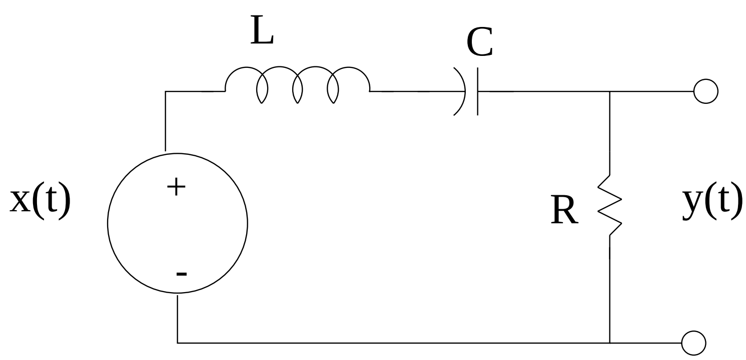
$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \geq \frac{\pi}{4}, \\ 0, & \text{otherwise} \end{cases}$$

For input $x[n] = \sum_{<N>} a_k e^{jk\frac{2\pi}{N}n}$ and $N = 10$, find a range of k for which the output $y[n]$ has Fourier coefficients b_k that are equal to zero.

(Work space)

Problem 4.5

Let's solve 6.002 problems the 6.003 way. Consider the CT causal and stable system implemented by the following RCL circuit, where the voltage source is the input signal $x(t)$ and the voltage across the resistor is the output signal $y(t)$:



(a) Derive a differential equation that relates $x(t)$ and $y(t)$.

(b) Find the frequency response $H(j\omega)$ of the system.

(c) Use the impedance method for a voltage divider to verify your solution to the previous part.

(d) Is this a lowpass, bandpass or highpass filter?

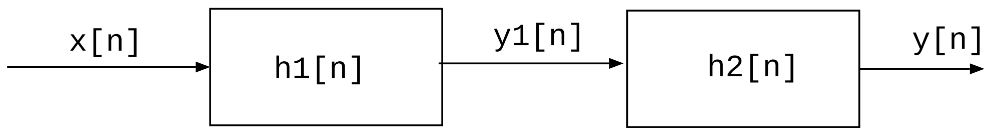
(e) Find the $y(t)$ when the input is an oscillating voltage $x(t) = (2V)\cos(3\pi t + \frac{\pi}{3})$.

(Work space)

(Work space)

1.4 Frequency response of cascaded systems

Consider a cascaded LTI system with input $x(t)$ and output $y(t)$ and impluse response $h(t) = h_1(t) * h_2(t)$. How do we determine the frequency response of the composite system given the frequency responses of the individual systems, $H_1(j\omega)$ and $H_2(j\omega)$?



Recall that to determine the frequency response of an LTI system, we set the input to the complex exponential. $x(t) = \mathrm{e}^{j\omega t}$. Now, $y_1(t)$ is a scaled complex exponential. So the ouput of the second system,

$$y_1(t) = H_1(j\omega)\mathrm{e}^{j\omega t}$$

$y_1(t)$ is a scaled complex exponential.

$$\begin{aligned} y(t) &= H_2(j\omega)y_1(t) \\ &= H_2(j\omega)H_1(j\omega)\mathrm{e}^{j\omega t} \end{aligned}$$

Looking at the composite system, we have a complex exponential as an input, so the output must be $y(t) = H(j\omega)\mathrm{e}^{j\omega t}$. Comparing with the expression above,

$$H(j\omega) = H_2(j\omega)H_1(j\omega)$$

So, instead of convolving the impulse responses, we can multiply the frequency responses.
Since the frequency responses are complex, we can write their product as follows:

$$\begin{aligned} H(j\omega) &= |H_2(j\omega)|.\mathrm{e}^{j\angle H_2(j\omega)}|H_1(j\omega)|.\mathrm{e}^{j\angle H_1(j\omega)} \\ \longrightarrow \quad H(j\omega) &= |H_1(j\omega)||H_2(j\omega)|.\mathrm{e}^{j(\angle H_2(j\omega)+\angle H_1(j\omega))}. \end{aligned}$$

Thus:

$$\begin{aligned} |H(j\omega)| &= |H_1(j\omega)||H_2(j\omega)|, \quad \text{and} \\ \angle H(j\omega) &= \angle H_2(j\omega) + \angle H_1(j\omega). \end{aligned}$$

Problem 4.6

Suppose we have a causal, stable DT LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

Find the output $y[n]$ of the system when the input $x[n]$ is:

$$x[n] = \sin\left(\frac{2\pi}{3}n\right) + u[n-2].$$

The four key points to solving this problem are:

- Finding the impulse response $h[n]$ of a LTI system described by a difference equation given the proper conditions,
- Finding the frequency response $H(e^{j\omega})$ of the system from the difference equation,
- Using the eigenfunction property of LTI systems, and
- Using the commutative property of cascaded LTI systems.

(Work space)

(Work space)

2 CT Fourier Transform

Up to problem set 4, we represented only *periodic* signals as linear combinations of complex exponentials; this analysis was called the Fourier series representation of a signal, and the frequencies of the series were harmonically related. Last week and the week before, we saw the development of a similar analysis for both CT and DT *aperiodic* signals by thinking of such signals as being periodic with the period approaching infinity. Increasing the period (or decreasing the fundamental frequency) had the effect of requiring closer and closer frequencies to represent the signal. When the period becomes infinite and the signal is aperiodic, we see that the signal is the superposition of a continuum of frequencies. Thus, we need to use an integral, rather than a sum, to represent this combination.

2.1 Synthesis and analysis equations

The Fourier transform (FT) $X(j\omega)$ of the CT signal $x(t)$ and the inverse Fourier transform $x(t)$ of $X(j\omega)$ are related through the *synthesis* and *analysis* formulae:

The Continuous-Time Fourier Transform:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,\mathrm{d}\omega \quad \text{(Synthesis, inverse CTFT)} \\
X(j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,\mathrm{d}t \quad \text{(Analysis, CTFT)}
\end{aligned}
$$

These relationships are also written as

$$
\begin{aligned}
X(j\omega) &= \mathcal{F}\{x(t)\} \\
x(t) &= \mathcal{F}^{-1}\{X(j\omega)\} \\
x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)
\end{aligned}
$$

So, this means that the aperiodic signal $x(t)$ contains exponentials of frequencies between ω and $\omega + \mathrm{d}\omega$ with amplitude $X(j\omega)(\mathrm{d}\omega/2\pi)$. Recall that not all periodic signals have a Fourier series representation. Likewise, not all signals have a Fourier transform. Page 290 of the textbook lists the Dirichlet conditions that the signals must satisfy, but we will not worry too much about them. However, we can “bend” those conditions if delta functions are allowed in the signal or Fourier transform.

2.2 Variations of the synthesis and analysis equations

Unfortunately for us, different people like to use different definitions of the Fourier and inverse Fourier transforms. We can also define the transforms using *cyclic* frequency f instead of *angular* frequency ω:

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}\,\mathrm{d}f \quad \text{(Cyclic synthesis, inverse CTFT)} \\
X(f) &= \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t \quad \text{(Cyclic analysis, CTFT)}
\end{aligned}
$$

One advantage to using f instead of ω is that the annoying $1/2\pi$ term disappears and the synthesis-analysis pair is more symmetric. In fact, we can use this to remember which one of the synthesis-analysis pair contains the factor of $1/2\pi$ in the first definition: since $f = \omega/2\pi$, $\mathrm{d}f = \mathrm{d}\omega/2\pi$; hence the equation that contains $\mathrm{d}\omega$ must have a $1/2\pi$ factor. In quantum mechanics, yet another definition of the Fourier transform is used. It retains the use of angular frequency ω, but it distributes the $1/2\pi$ to *both* equations evenly so we don't have to remember where it is:

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t}\, \mathrm{d}\omega \qquad \text{(QM synthesis, inverse CTFT)}$$

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, \mathrm{d}t \qquad \text{(QM analysis, CTFT)}$$

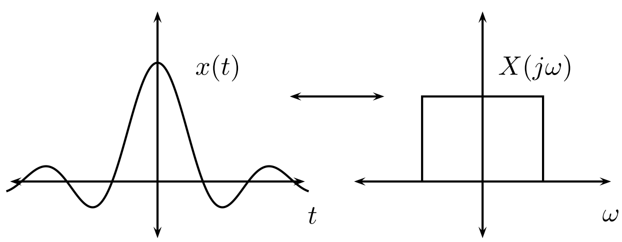
One consequence of doing this is that both sides of Parseval's relation are normalized to one and the integrals of the signal/transform magnitude squared are equal, which is required for a probabilistic interpretation. In 6.003, we will use only the first definition.

2.3 Rectangular pulse and sinc pair

The Fourier transform of the rectangular pulse is known as the *sinc function*, which is defined as:

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}.$$

Graphically, we have:



What's the height and width of each of these functions in terms of the other? If you ever forget the constants that go inside and outside the sinc operation, you can always re-derive them using this following properties that fall directly from the synthesis and analysis equations:

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) d\omega$$

$$X(j0) = \int_{-\infty}^{+\infty} x(t) dt$$

So, the value of $x(t)$ at zero time is the area under its transform divided by 2π. Likewise, the value of $X(j\omega)$ at zero frequency (DC) is the area under the signal. An interesting property of the sinc function is that *the area under the sinc is equal to the area of the triangle formed by the peak and the nearest zeroes on either side of the vertical axis.*

The Fourier Transform of any $x(t)$ of the form $\frac{\sin(At)}{Bt}$ will be a box in frequency. That is, it will be of the form

$$X(j\omega) = \begin{cases} C & \text{for } |\omega| \le D \\ 0 & \text{otherwise} \end{cases}$$

Likewise, a box in time, will have a Fourier Transform which is a sinc in frequency. All you have to really remember is

$$\text{box} \longleftrightarrow \text{sinc}$$

Everything else works out by using CTFT properties

[**Example**]

Find the Fourier Transform of the function $x(t) = \frac{\sin(At)}{Bt}$.

Solution

First, let's characterize this function.

- Zero crossings - We know that $x(t)$ has zero crossings whenever At is a multiple of π. Therefore, the zero crossings occur at $\frac{k\pi}{A}$ for all non-zero integers k.
- $x(0)$ - We can do this by limits:

$$\begin{aligned} \lim_{t\to 0} x(t) &= \frac{\lim_{t\to 0}\sin(At)}{\lim_{t\to 0} Bt} \\ &= \frac{\lim_{t\to 0} A\cos(At)}{\lim_{t\to 0} B} \\ \Rightarrow x(0) &= \frac{A}{B} \end{aligned}$$

Next, let's figure out what it's Fourier Transform is going to be. We're going to make use of the following important properties:

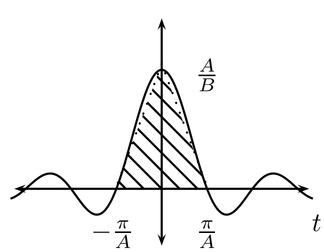
$$\begin{aligned} X(j0) &= \int_{-\infty}^{\infty} x(t)dt \\ \int_{-\infty}^{\infty} X(j\omega)d\omega &= 2\pi x(0) \end{aligned}$$

Since we know that $X(j\omega)$ is of the form

$$X(j\omega) = \begin{cases} C & \text{for } |\omega| \le D \\ 0 & \text{otherwise} \end{cases}$$

we just need to determine C and D.

- C - We know that $C = X(j0)$. This is just the area under $x(t)$. Note that the area under the sinc is just given by the area of the inscribed triangle



$$\int_{-\infty}^{\infty} x(t)dt = \frac{1}{2} \times \left(\frac{A}{B}\right) \times \left(\frac{2\pi}{A}\right)$$

Therefore, $C = X(j0) = \frac{\pi}{B}$.

- D - This is the width of the box. Since we know that the area under $X(j\omega)$ is equal to $C \times (2D)$, we can use

$$\begin{aligned}
\int_{-\infty}^{\infty} X(j\omega)d\omega &= 2\pi x(0) \\
C \times 2D &= 2\pi \times \frac{A}{B} \\
\frac{\pi}{B} \times D &= \pi \times \frac{A}{B} \\
\Rightarrow D &= A
\end{aligned}$$

Therefore,

$$X(j\omega) = \begin{cases} \frac{\pi}{B} & \text{for} \quad |\omega| \leq A \\ 0 & \text{otherwise} \end{cases}$$

Problem 4.7

Draw the Fourier Transform of the following box function in time:

$$x(t) = \begin{cases} \pi, & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(Work space)

2.4 The multiplication and convolution properties

Many properties of the properties of the CT Fourier transform are similar to those of the Fourier series. Two important properties are the multiplication and convolution properties (the uppercase functions are the Fourier transforms of the corresponding lowercase functions), namely, that *convolution in one domain is multiplication in the other*:

- **CT convolution property**

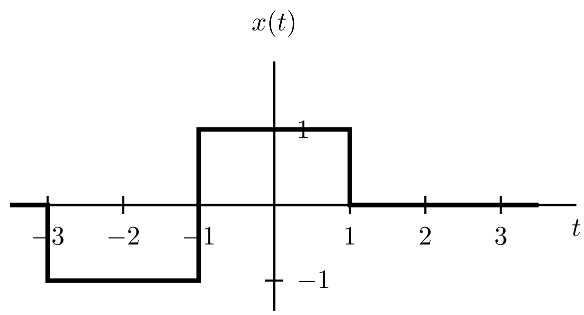
$$y(t) = h(t) * x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(j\omega) = H(j\omega)X(j\omega)$$

- **CT multiplication property**

$$r(t) = s(t)p(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta$$

Problem 4.8

We are given the following CT signal $x(t)$:



Solve the following without explicitly calculating the Fourier transform $X(j\omega)$ of $x(t)$.

(a) $X(j\omega)$ can be expressed as $A(j\omega)\mathrm{e}^{j\theta(j\omega)}$, where $A(j\omega)$ and $\theta(j\omega)$ are both real-valued. These functions are similar to the magnitude and phase of $X(j\omega)$, respectively, but we allow $A(j\omega)$ to take on negative values. Find $\theta(j\omega)$.

(b) Find $X(j0)$.

(c) Find $\int_{-\infty}^{+\infty} X(j\omega)\,\mathrm{d}\omega$.

(d) Find $\int_{-\infty}^{+\infty} |X(j\omega)|^2\,\mathrm{d}\omega$.

(e) Sketch the inverse Fourier transform of $Im\{X(j\omega)\}$.

(Work space)