6.003: Signals and Systems — Spring 2004

Tutorial 3

Monday, February 23 and Tuesday, February 24, 2004

Announcements

- Problem set 3 is due this Friday.

Today's Agenda

- Choosing a Basis

- Eigenfunctions of LTI Systems

- CT Fourier Series

  - Dirichlet conditions
  - Properties of Fourier series

- DT Fourier Series

- Parseval's Relation

1 Choosing a Basis

So far, we have thought of signals as functions of time: $x(t)$ and $x[n]$. However, it is sometimes more convenient to express signals in terms of another *basis*. Let's define a (discrete) basis to be a set of signals indexed by k called $\{\phi_k(t)\}$ in CT and $\{\phi_k[n]\}$ in DT so that there exists coefficients a_k such that we can express a large class of signals in terms of the basis signals as superposition sums of the basis signals:

$$x(t) = \sum_k a_k \phi_k(t)$$

$$x[n] = \sum_k a_k \phi_k[n].$$

Then, given a basis, the coefficients a_k are *just as good* at specifying the signal as the original $x(t)$ and $x[n]$. In DT, it is easy to identify a "default" basis. Since:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k],$$

we see that we can choose the basis signals to be $\phi_k[n] = \delta[n-k]$, so that $a_k = x[k]$.

We do this all the time in other areas. The coefficients a_k are **boldfaced** for emphasis.

- **Whole numbers** - Represent a general number by their decimal coefficients so that the basis is $\{10^k\}$:

$$32618 = (\mathbf{3} \times 10^4) + (\mathbf{2} \times 10^3) + (\mathbf{6} \times 10^2) + (\mathbf{1} \times 10^1) + (\mathbf{8} \times 10^0).$$

  Now, let's change to the $\{16^k\}$ basis and write 32618 in base 16:

$$32618 = (\mathbf{7} \times 16^3) + (\mathbf{15} \times 16^2) + (\mathbf{6} \times 16^1) + (\mathbf{10} \times 16^0).$$

  We see that if we know the basis, the set of numbers $\{3, 2, 6, 1, 8\}$ and $\{7, 15, 6, 10\}$ both identify the number 32618.

- **Vectors** - Represent any vector by orthogonal components:

$$\vec{v} = (3, -1, 2) = \mathbf{3} \times (1, 0, 0) + -\mathbf{1} \times (0, 1, 0) + \mathbf{2} \times (0, 0, 1).$$

  Let's rotate our coordinate system so that we can express $\vec{v}$ in terms of new basis vectors:

$$\vec{v} = (3, -1, 2) = \mathbf{3} \times (1, 0, 0) + \left(-\frac{\sqrt{\mathbf{3}}}{\mathbf{2}} - \mathbf{1}\right) \times \left(0, \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) + \left(\sqrt{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{2}}\right) \times \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

- **Currency** - Represent an amount by more elementary denominations:

$$
\begin{aligned}
\$34.92 \quad = \quad & \mathbf{1} \times (20 \text{ dollar bill}) + \mathbf{1} \times (10 \text{ dollar bill}) + \mathbf{4} \times (1 \text{ dollar bill}) \\
& + \mathbf{3} \times (\text{quarter}) + \mathbf{1} \times (\text{dime}) + \mathbf{1} \times (\text{nickel}) + \mathbf{2} \times (\text{penny})
\end{aligned}
$$

Of course, we can break down \$32.41 several ways, and can even write it terms of foreign currency.

- **Taylor series** (more specifically, the MacLaurin series) - Represent a CT signal $f(t)$ with continuous derivatives as a polynomial; the basis is $\{t^k\}$:

$$f(t) = \left(\mathbf{f(0)} \times t^0\right) + \left(\mathbf{f'(0)} \times t\right) + \left(\frac{\mathbf{f''(0)}}{\mathbf{2}} \times t^2\right) + \left(\frac{\mathbf{f'''(0)}}{\mathbf{3!}} \times t^3\right) + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$$

What's the most appropriate basis for representing signals that are the inputs and outputs of LTI systems? To answer this question, we need to determine the eigenfunctions of LTI systems.

2   Eigenfunctions of LTI Systems

We saw how to determine the output $y(t)$ $(y[n])$ of any CT (DT) LTI system given the input $x(t)$ $(x[n])$ and the impulse response $h(t)$ $(h[n])$ through the use of the convolution integral (sum). In general, the output looked nothing like the input and the calculation was rather tedious. Convolution came from expressing signals as superpositions of shifted scaled impulses. Perhaps another set of basis signals would provide more insight on LTI system behavior.

We showed in lecture that a certain set of input signals, namely complex exponentials of the form $x(t) = \mathrm{e}^{st}$ $(x[n] = z^n)$, are *eigenfunctions* of LTI systems, *i.e.* the corresponding outputs are simply scaled versions of inputs of this form, and this scaling factor is the *eigenvalue*. We showed that the outputs of CT and DT LTI systems in response to $x(t) = \mathrm{e}^{st}$ and $x[n] = z^n$ are $y(t) = H(s)\mathrm{e}^{st} = H(s)x(t)$ and $y[n] = H(z)z^n = H(z)x[n]$, respectively, where the eigenvalues $H(s)$ and $H(z)$ associated with the given eigenfunctions are:

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)\mathrm{e}^{-s\tau}\,d\tau$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

where $h(t)$ and $h[n]$ are the impulse responses of the systems.
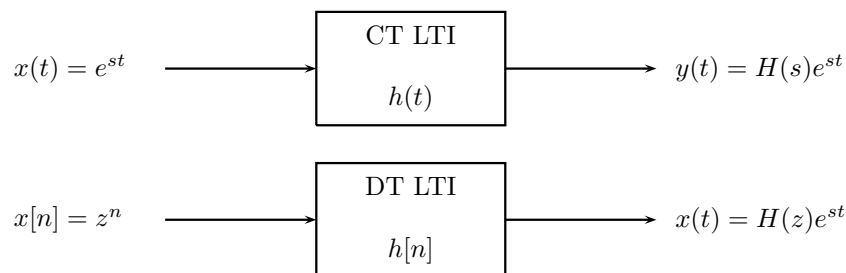
Because of the superposition property of LTI systems, this suggests another way of writing signals. Instead of expressing the input signal as the linear combination of scaled and shifted impulses, we can also express the input as the linear combination of complex exponentials. Then, the output is the same linear combination of the exponentials scaled by the appropriate eigenvalue. So, if the inputs are:

$$x(t) = \sum_k a_k e^{s_k t}$$

$$x[n] = \sum_k a_k z_k^n$$

then the outputs are:

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

$$y[n] = \sum_k a_k H(z_k) z_k^n$$

If you are familiar with linear algebra from 18.03 or 18.06, you've seen the concepts of eigenvectors and eigenvalues of matrices. This is exactly the same idea. We think of a matrix $\mathbf{A}$ as a linear transformation (system) from one (input) vector $\vec{x}$ into another (output) vector $\vec{y}$:

$$\vec{y} = \mathbf{A}\vec{x}.$$

Let $\vec{v}_k$ be the eigenvectors of $\mathbf{A}$ with corresponding eigenvalues λ_k so that when $\mathbf{A}$ is applied to an eigenvector, the result is the same eigenvector scaled by the eigenvalue:

$$\lambda_k \vec{v}_k = \mathbf{A}\vec{v}_k$$

This is powerful, because if we express an arbitrary input vector $\vec{x}$ as the superposition sum of eigenvectors:

$$\vec{x} = \sum_k a_k \vec{v}_k,$$

then we can express the output vector as the *same* linear combination with each component scaled by the corresponding eigenvalue:

$$
\begin{aligned}
\vec{y} &= \mathbf{A}\vec{x} \\
&= \mathbf{A}\left(\sum_k a_k \vec{v}_k\right) \\
&= \sum_k a_k \mathbf{A}\vec{v}_k \\
\longrightarrow \quad \vec{y} &= \sum_k a_k \lambda_k \vec{v}_k,
\end{aligned}
$$

Thus, for the linear transformation $\mathbf{A}$, $\{\vec{v}_k\}$ is the most natural basis for representing vectors.

We are now see a need to express signals in terms of complex exponentials. But how do we do so? This question motivates us to spend this tutorial studying the *Fourier series*, which expresses periodic signals in that form.

3 CT Fourier Series

Let us begin by representing periodic CT signals as the linear combination of complex exponentials using the *Fourier series* representation. We will then do the same with periodic DT signals. Later, we will do the same with aperiod CT and DT signals using the *Fourier transform*. Say $x(t)$ is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$. Then, we can write $x(t)$ as the sum of the harmonically related components $\phi_k(t) = e^{jk\omega_0 t}$ (all with period T) and the coefficients are unique. They are related by:

CT Fourier Series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt \qquad \text{(Analysis equation)}$$

The coefficients a_k are known as *Fourier series coefficients*. We can extract a_k by exploiting the orthogonality of the $\phi_k(t)$ components, that is:

$$\frac{1}{T} \int_T \phi_m^*(t) \phi_k(t) dt = \delta[m - k].$$

As discussed in the prevous section on choosing a basis, we are simply using a different set of numbers to represent the original signal. In particular, we are representing a *periodic CT* signal $x(t)$ by an *aperiodic DT* signal a_k.

It is important to note if we plug in $t = 0$ into the synthesis equation and $k = 0$ into the analysis equation, we see that the value of the signal at zero is the sum of all the Fourier series coefficients and that the coefficient a_0 represents the DC or average value of the signal over time; these facts will come up often:

$$x(0) = \sum_{k=-\infty}^{+\infty} a_k$$

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

3.1 Dirichlet conditions

There are three conditions that must be satified in order for the Fourier series to converge to the periodic signal, except at discontinuities. They are:

1. $x(t)$ must be absolutely integrable over any period.

2. There are only a finite number of max and min in any period.

3. There are only a finite number of discontinuities in any finite interval of time, and each discontinuity is finite.

Nearly all the signals we will encounter in 6.003 will satisfy these conditions, so we won't need to worry about testing them each time, but it's important to keep in mind the limitations of using Fourier analysis. For the signals with discontinuities that can be represented, the Fourier series converges to the average of the limiting values of the signal from each side of the discontinuity. Mathematically:

Convergence of the CT Fourier Series at Discontinuities:

Let $x(t)$ be a signal that satisfies the Dirichlet conditions. Let $x(t)$ be discontinuous at $t = t_0$. Let $x_N(t)$ be the signal created from the Fourier series coefficients from $-N$ to N. Then, regardless of whether $x(t_0)$ is defined:

$$\lim_{N \to \infty} x_N(t) = \frac{\lim_{t \to t_0^+} x(t) + \lim_{t \to t_0^-} x(t)}{2}.$$

3.2   Properties of Fourier series

The table on page 206 of Oppenheim and Willsky lists the properties of the CT Fourier series. It will be very helpful to become familiar with applying these rules to solve problems.

Problem 3.1

Find the Fourier coefficients for the following signals.

(a) $x_a(t) = \sin(\pi t) + \cos(2\pi t)$

(b) $x_b(t) = j \sin(\pi t) - \pi \cos(2\pi t)$

(c) $x_c(t) = \cos\left(\frac{\pi}{2}t\right) + \sin(\pi t) + \cos(2\pi t)$

(d) $x_d(t) = \cos^2(2\pi t) + \cos\left(4\pi t + \frac{\pi}{3}\right)$

(e) A periodic square wave with period $T = 10$, defined over one period as
$$x_e(t) = \begin{cases} 1, & |t| < 1 \\ 0, & 1 < |t| < 5 \end{cases}$$

Problem 3.2

Let $x(t)$ be a periodic signal with fundamental period of T and its Fourier coefficients a_k. Derive the Fourier coefficients of each of the following signals in terms of a_k.

(a) $-2x(t) + jx(t)$

(b) $x(t - 1)$

(c) $\frac{d}{dt}x(t)$

(d) $1 + x(-t)$

(e) $x(1 - t)$

(f) $x^2(t)$

Problem 3.3

A continuous-time periodic signal $x(t)$ is real-valued and has a fundamental period of $T = 8$. The non-zero Fourier series coefficients for $x(t)$ are specified as:

$$
\begin{aligned}
a_1 = a_{-1}^* &= j, \\
a_2 = a_{-2}^* &= 1 + j\sqrt{3}, \\
a_5 = a_{-5}^* &= -3.
\end{aligned}
$$

Express $x(t)$ in the form:

$$
x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k)
$$

Problem 3.4

Let $x(t) = \cos^2 \pi t$.

(a) Rewrite $x(t)$ using the cosine double angle formula and find the Fourier series coefficients a_k of $x(t)$.

(b) Rewrite $x(t)$ as $\cos \pi t \cdot \cos \pi t$. Using the multiplication property, find the Fourier series coefficients as a convolution of two cosines.

(c) Note that an additional step was necessary in part (b) to find the Fourier series coefficients. Explain the circumstances when this step is needed.

Problem 3.5

Direct application of the analysis equation tells us that the Fourier series coefficients of $x(t) = |\sin t|$ is:

$$a_k = \frac{2}{\pi(1 - 4k^2)}.$$

Let's try to find it by recognizing $x(t)$ as the product of two periodic signals and applying Fourier series properties.

The key points to solving this problem are:

- Knowing the Fourier series coefficients of a canonical square wave,

- Manipulating the canonical square wave to become the square wave that we want and understanding how this affects the Fourier series coefficients,

- Convolving Fourier coefficients as DT signals, and

- Dealing with the change in fundamental period.

(Work space)

4 DT Fourier Series

Like in CT, we can represent periodic DT signals as linear combinations of complex exponentials. So, the DT Fourier series of a DT signal $x[n]$ with fundamental period N and fundamental frequency $\omega_0 = 2\pi/N$ and the coefficients are written as

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DT Fourier Series:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)},$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}.$$

However, DT frequencies "wrap around" every 2π, that is, the set of unique frequencies is restricted to an interval of 2π. In other words, a_k is periodic with period N, just like $x[n]$. The brackets in the summation mean we sum over *any* set of N consecutive values of n or k. DT signals do not have the difficulties that CT signals had with the Dirichlet conditions. All periodic DT signals can be represented by the DT Fourier series. Note that since we need N complex numbers to completely specify $x[n]$ with period N, we also need exactly N Fourier coefficients to completely specify. We can see that there is a perfect one-to-one correspondence between the set of all possible periodic DT signals and the set of all possible DT Fourier coefficients. We are representing a periodic DT signal $x[n]$ by another periodic DT signal a_k.

Like CT, we have

$$x[0] = \sum_{k=\langle N \rangle} a_k,$$

$$a_0 = \frac{1}{N} \sum_{n=\langle N \rangle} x[n].$$

Those familiar with linear algebra from 18.06 may recognize that the DT Fourier series analysis and synthesis equations can be expressed in matrix notation, $\mathbf{a_k} = F\mathbf{x}$ and $\mathbf{x} = F^{-1}\mathbf{a_k}$, where F is called the Fourier matrix. Since F is orthonormal, the DT Fourier series representation can be seen as a change of basis for a N-dimensional vector.

Problem 3.6

Find the Fourier coefficients for the following signals.

(a) $x_a[n] = |e^{j\frac{\pi}{4}n} + 3e^{j\frac{\pi}{2}n} + e^{j\frac{3\pi}{4}n}|$

(b) $x_b[n] = 1 + \sin\left(\frac{\pi}{5}n\right) + (-1)^n$

(c) $x_c[n] = 1 + e^{j\pi n} - \cos(2\pi n)$

5 Parseval's Relation

Parseval's Relation for Periodic Signals

For a periodic CT signal $x(t)$ with period T and Fourier series coefficients a_k and a periodic DT signal $x[n]$ with period N and Fourier series coefficients a_k,

$$\frac{1}{T}\int_T |x(t)|^2\, \mathrm{d}t = \sum_{k=-\infty}^{+\infty} |a_k|^2,$$

$$\frac{1}{N}\sum_{N=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2.$$

One may interpret Parseval's relations as a statement that the average power in a periodic signal is equal to the average power in all of its harmonics harmonic components. Those familiar with linear algebra from 18.06 may recognize Parseval's relations as an outcome of changing to an orthonormal basis; in an orthonormal basis, the square of the length of a vector is the sum of the squares of each component.

Problem 3.7

(From 6.003 Quiz 1, Fall 2000) Consider the periodic CT signal $x(t)$ with fundamental period $2T_0$. $x(t)$ is defined over one period as

$$x(t) = \begin{cases} At^2, & 0 \le t < T_0 \\ 0, & T_0 \le t < 2T_0 \end{cases}$$

Let a_k be the Fourier series cofficients of $x(t)$. Suppose we have the following additional information about a_k:

- $a_0 = 1/6$

- $\sum_{k=-\infty}^{+\infty} |a_k|^2 = \frac{1}{T_0}$

Determine the numerical values of the parameters A and T_0.