

6.003: Signals and Systems — Spring 2004

TUTORIAL 2

Tuesday, February 17, 2004

Announcements

- Problem set 2 is due this Friday.
- There are no tutorials this week, but we will provide this handout, and TAs will hold extra office hours during the week.

Today's Agenda

- DT Convolution
 - Calculating the DT convolution
- CT Convolution
 - Calculating the CT convolution
- LTI System Properties
 - Impulse response
 - Commutativity, distributivity, associativity, and time shift
 - Effect of stability on commutativity
 - Causality and stability
- Signal Properties vs. System Properties
- Singularity Functions

1 DT Convolution

In class, we saw how to write any DT signal as the superposition sum of scaled, shifted impulses using the sifting property of the DT unit impulse:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

We define the *unit impulse response*, $h[n]$, of a DT system H as the output when the input is the unit impulse $\delta[n]$. If the system H is LTI, then the output $y[n]$ for any arbitrary input $x[n]$ is

$$\begin{aligned} y[n] &= H\{x[n]\} \\ &= H\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{+\infty} x[k]H\{\delta[n-k]\} \quad (\text{by linearity}) \\ &= \sum_{k=-\infty}^{+\infty} x[k]\left(H\{\delta[n]\}_{\text{shifted by } k}\right) \quad (\text{by time-invariance}) \\ &= \sum_{k=-\infty}^{+\infty} x[k]\left(h[n]_{\text{shifted by } k}\right) \quad (\text{apply system } H) \\ y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad (\text{simplify}) \end{aligned}$$

So, the output $y[n]$ is the *convolution* of the input $x[n]$ and the unit impulse response $h[n]$ of the LTI system. We write this as

$$y[n] = x[n] * h[n].$$

One useful sanity check when performing DT convolutions is that a signal of length m convolved with a signal of length n produces a signal with no more than $m + n - 1$ non-zero terms.

1.1 Calculating the DT convolution

Calculating the convolution of two DT signals can be a rather tedious process. Consider the following example.

Given $x[n] = \delta[n+1] - 2\delta[n] + 3\delta[n-1]$ and $h[n] = \delta[n] + 4\delta[n-1] - 2\delta[n-2] + \delta[n-3]$, calculate $y[n] = x[n] * h[n]$. We can calculate the convolution by applying the convolution sum directly

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Since $x[n]$ is nonzero for $n \in \{-1, 0, 1\}$,

$$\begin{aligned}
 y[n] &= \sum_{k=-1}^1 x[k]h[n-k] \\
 y[-1] &= x[-1]h[0] = 1 \\
 y[0] &= x[-1]h[1] + x[0]h[0] = 4 + -2 = 2 \\
 y[1] &= x[-1]h[2] + x[0]h[1] + x[1]h[0] = -2 + -8 + 3 = -7 \\
 y[2] &= x[-1]h[3] + x[0]h[2] + x[1]h[1] = 1 + 4 + 12 = 17 \\
 y[3] &= x[0]h[3] + x[1]h[2] = -2 + -6 = -8 \\
 y[4] &= x[1]h[3] = 3
 \end{aligned}$$

Thus, $y[n] = \delta[n+1] + 2\delta[n] - 7\delta[n-1] + 17\delta[n-2] - 8\delta[n-3] + 3\delta[n-4]$.

It is sometimes quicker, and less error prone, to use the following trick when calculating the convolution of two *finite length DT signals*. First we form the following table:

$x[n] \setminus h[n]$	<u>1</u>	4	-2	1
1	1	4	-2	1
<u>-2</u>	<u>-2</u>	-8	4	-2
3	<u>3</u>	12	-6	3

The table consists of the values of $x[n]$ in the first column and the values of $h[n]$ in the top row. The elements in the interior of the table are formed by multiplying the corresponding elements in the first column and first row. Note that the *underlined* entries correspond to the element at zero (i.e. $x[0]$ and $h[0]$). The *double-underlined* element corresponds to the entry formed by multiplying the two underlined elements (i.e. $x[0]h[0]$). Now we can simply form $y[n]$ by summing diagonally. Using sequence notation:

$$\begin{aligned}
 y[n] &= \left\{ 1, \underline{4 + (-2)}, (-2) + (-8) + 3, 1 + 4 + 12, (-2) + (-6), 3 \right\} \\
 &= \{1, \underline{2}, -7, 17, -8, 3\}
 \end{aligned}$$

Note again that we have underlined the element resulting from a sum containing the double-underlined element, which corresponds to $y[0]$. This result for $y[n]$ is equivalent to the one above.

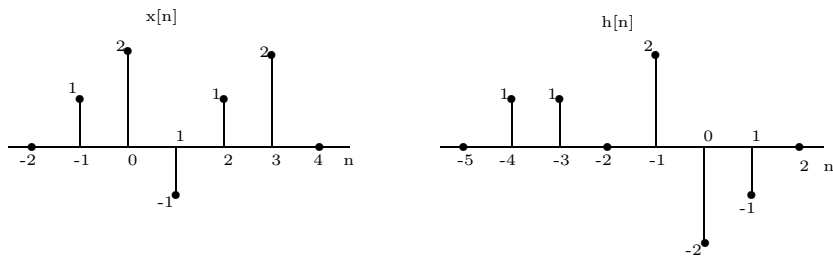
Problem 2.1

Evaluate the following discrete-time convolution sums given below:

(a) $y[n] = \cos\left(\frac{1}{2}\pi n\right) * \left(\frac{1}{2}\right)^n u[n - 2]$

(b) $y[n] = u[n] * \sum_{p=0}^{\infty} \delta[n - 2p]$

(c) $y[n] = x[n] * h[n]$, where $x[n]$ and $h[n]$ are given below:



(Workspace)

2 CT Convolution

Similarly, we can express CT signals as the superposition sum of scaled, shifted impulses using the sifting property of the CT unit impulse:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau$$

We likewise define the *unit impulse response*, $h(t)$, of a CT system H as the output when the input is the unit impulse $\delta(t)$. If the system H is LTI, then the output $y[n]$ for any arbitrary input $x[n]$ is

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(\tau - t)d\tau$$

or

$$y(t) = x(t) * h(t)$$

One useful sanity check when performing CT convolutions is that a signal of length t_1 convolved with a signal of length t_2 produces a signal of length no more than $t_1 + t_2$.

2.1 Calculating the CT convolution

To graphically compute the convolution integral $x(t) * h(t)$, use the following steps:

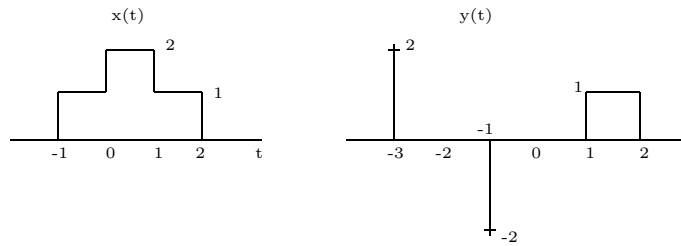
1. Plot $x(\tau)$ and $h(\tau)$.
2. Select which signal to flip, say $h(\tau)$. In general, the signal of shorter duration is the one you should flip to make the computation easier.
3. Flip $h(\tau)$ about the vertical axis, which gives us $h(-\tau)$.
4. Plot $h(-\tau + t)$ on the τ axis. This is just $h(-\tau)$ shifted to the RIGHT by t . Note that in order to shift $h(-\tau)$ to the right by t , we replace τ by $\tau - t$ which gives us $h(-(\tau - t)) = h(-\tau + t)$.
5. Identify different regions of overlap as t is varied from $-\infty$ to $+\infty$ where there are “breaks” in either function. These correspond to discontinuities and regions where the mathematical expressions describing the functions change.
6. Multiply both functions and integrate them for each of the regions identified above. Make sure to use the correct limits for the integrals.

Problem 2.2 Evaluate the following continuous-time convolution integrals given below:

(a) $y(t) = u(t + 1) * u(t - 2)$

(b) $y(t) = (u(t + 2) - u(t - 1)) * u(-t + 2)$

(c) $y(t) = x(t) * h(t)$, where $x(t)$ and $h(t)$ are given below:



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3 LTI System Properties

3.1 Impulse response

The impulse response *completely characterizes* an LTI system, *i.e.* there is a one-to-one correspondence between the set of impulse responses and the set of LTI systems. In fact, we can even *define* or *identify* an LTI systems by its impulse response, $h(t)$. If we want to find $h(t)$, we can set the input $x(t) = \delta(t)$:

$$\begin{aligned}y(t) &= h(t) * \delta(t) \\ &= h(t)\end{aligned}$$

For example, to find the impulse response, $h(t)$, of the system

$$y(t) = -x(t - t_2) + 6 \int_{-\infty}^t x(\tau - t_1) d\tau,$$

we substitute $x(t) = \delta(t)$:

$$\begin{aligned}h(t) &= -\delta(t - t_2) + 6 \int_{-\infty}^t \delta(\tau - t_1) d\tau \\ &= 6u(t - t_1) - \delta(t - t_2)\end{aligned}$$

Problem 2.3 Find the impulse response of the following systems:

(a) $y[n] = 2x[n] + 3x[n - 1] - 4x[n - 2]$

(b) $y[n] = \sum_{k=-\infty}^n x[k]$

(c) $y(t) = \int_{-\infty}^t x(\tau - 7) d\tau$

(Workspace)

3.2 Commutativity, distributivity, associativity, and time shift

The convolution operation satisfies the following properties (in DT notation):

- Commutativity: $x[n] * h[n] = h[n] * x[n]$ (see section below)
- Distributivity: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
- Associativity: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- Time shift: $x[n] * \delta[n - n_0] = x[n - n_0]$

Analogous properties hold in CT. These convolution properties will simplify our study of LTI systems. We don't need to worry about the order in which convolution is done and we know how to combine LTI systems together. From a practical standpoint, a system implemented in a straightforward manner might suffer tremendously from various constraints, such as noise or signal size. However, we may find that building the same system (in the mathematical sense) but with its components rearranged might alleviate those problems.

3.3 Effect of stability on commutativity

Two LTI systems may not commute for all inputs if one of them is unstable. When the unstable system is first in the cascade, then there may be inputs where the output of the first system is infinity for all time. The following example illustrates this issue.

Example 2.4

Let systems 1 and 2 be DT LTI systems with the following impulses responses, respectively:

$$\begin{aligned}h_1[n] &= \delta[n] - \delta[n - 1], \\h_2[n] &= u[n].\end{aligned}$$

Let $x[n]$ be the signal:

$$x[n] = -u[-n - 1].$$

Now, let configuration A be the following: $x[n]$ is the input to system 1; call the output $y_A[n]$. $y_A[n]$ is then the input to system 2; call the output $z_A[n]$. Let configuration B be the same as A but with the two systems swapped: $x[n]$ is the input to system 2; call the output $y_B[n]$. $y_B[n]$ is then the input to system 1; call the output $z_B[n]$. If systems 1 and 2 commute, then $z_A[n]$ and $z_B[n]$ equal each other. We will show that this is not the case.

We see that configuration A produces:

$$x[n] = -u[-n-1] \quad \longrightarrow \quad y_A[n] = -u[-n-1] * (\delta[n] - \delta[n-1]) = \delta[n] \quad \longrightarrow \quad z_A[n] = \delta[n] * u[n] = u[n].$$

However, configuration B produces:

$$x[n] = -u[-n - 1] \quad \longrightarrow \quad y_B[n] = -u[-n - 1] * u[n] = \infty \text{ for all } n.$$

Since the intermediate signal $y_B[n]$ blows up for all time, there does not exist an output $z_B[n]$. Thus, systems 1 and 2 do not commute.

3.4 Causality and stability

It would be nice if we could determine whether a particular LTI system is causal or stable by analyzing its impulse response alone and without dragging the system through the formal definitions of those properties. Fortunately, because of the one-to-one correspondence between LTI systems and their impulse responses, we can. Recall that a system is *causal* if the output at some time t_0 does not depend on the values of the input after time t_0 . This implies that the impulse response $h(t)$ (in CT) must be zero for all negative time (same in DT). A system is *stable* if every bounded input produces a bounded output.¹ It can be shown that an LTI system is stable if and only if its impulse response is absolutely summable (DT) or absolutely integrable (CT):

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$$

It is important to emphasize that the rules mentioned above for determining causality and stability directly from the impulse response *are only true for LTI systems*. We will deal almost exclusively with LTI systems in 6.003 to the point where it is easy to forget that the tools we develop may only work for that class of systems. For general systems, we will have to go back to original definitions of properties.

Using our previous example:

$$h(t) = 6u(t - t_1) - \delta(t - t_2)$$

Is this system causal?

Well, if *both* $t_1, t_2 \geq 0$, then this system is causal. If either $t_1 < 0$, $t_2 < 0$, or $t_1, t_2 < 0$, this system would not be causal since $h(t) \neq 0$ for some $t < 0$.

Is this system stable?

Again, we apply the result from before:

$$\int_{-\infty}^{+\infty} |6u(t - t_1) - \delta(t - t_2)| dt \geq \int_{-\infty}^{+\infty} |6u(t - t_1)| dt - \int_{-\infty}^{+\infty} |\delta(t - t_2)| dt$$

$$\geq \infty - 1$$

$$= \infty$$

So, this system is not stable.

¹This is more precisely known as *BIBO stability*. In 6.003, this is the only definition of stability that we'll use.

Problem 2.5

Are the following impulse responses indicative of causal systems? Are the following impulse responses indicative of stable systems?

(a) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[n - 1]$

(b) $h[n] = (\frac{1}{2})^n u[-n]$

(c) $h[n] = 5^n u[3 - n]$

(d) $h(t) = e^{-t} u(3 - t)$

(e) $h(t) = e^{-2t} u(t + 50)$

(Workspace)

4 Signal Properties vs. System Properties

Distinguish *signal* properties from *system* properties. For instance, signals can be even/odd or periodic, and systems can be linear, time-invariant, causal, or stable. After today's discussion, this become even more confusing because we now have a way of describing LTI *systems* using *signals*, namely the impulse response. You may sometimes hear people casually throw around terms like “causal signal” or “stable signal.” This terminology can be misleading. These people are actually describing (LTI) *systems* with those properties and the “signals” refer to the impulse response signals of those systems.

5 Singularity Functions

With the system's impulse response, we can determine the output of the system for any given input using:

$$\begin{aligned}y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{+\infty} x(\tau)h(t - \tau) d\tau\end{aligned}$$

We can use certain inputs to learn more about a system. For example, we know that if we want to find the system's impulse response, we set $x(t) = \delta(t)$, then $y(t) = h(t)$. If we want to find the system's response to a step input $u(t)$, we can set

$$x(t) = u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

then $y(t) = s(t)$. Notice that $s(t) = \int_{-\infty}^t h(\tau) d\tau$. Likewise, if we want to find the system's response to a ramp input $tu(t)$, we can set

$$x(t) = tu(t) = \int_{-\infty}^t u(s) ds = \int_{-\infty}^t \left[\int_{-\infty}^s \delta(\tau) d\tau \right] ds$$

So, using $\delta(t)$ we can create inputs called *singularity functions* that have the form:

$$u_n(t) = \frac{d^n}{dt^n} \delta(t)$$

For example:

$$\begin{aligned}u_0(t) &= \frac{d^0}{dt^0} \delta(t) = \delta(t) \\ u_{-1}(t) &= \int_{-\infty}^t \delta(\tau) d\tau = u(t) \\ u_1(t) &= \frac{d}{dt} \delta(t)\end{aligned}$$

We can use these singularity functions to help us analyze systems. For example, if we had to convolve $y(t) = x(t) * h(t)$, since

$$\delta(t) = u_{-1}(t) * u_1(t)$$

we can use singularity functions to do the following:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= x(t) * \delta(t) * h(t) \\ &= x(t) * (u_{-1}(t) * u_1(t)) * h(t) \\ &= (x(t) * u_1(t)) * h(t) * u_{-1}(t) \\ &= \left(\frac{dx(t)}{dt} * h(t) \right) * u_{-1}(t). \end{aligned}$$

That is, after convolving $\left(\frac{dx(t)}{dt} * h(t) \right)$, we then integrate the result to find $y(t)$.