MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

TUTORIAL 12 SOLUTIONS

Tuesday, May 11, 2004

Problem 12.1

(a) $X_a(z) = 3z^4 - z^3 + 2z^2$ ROC: All z except ∞

(b) $X_b(z) = 3z - 1 + 2z^{-1}$ ROC: All z except 0 and ∞

(c) $X_c(z) = 3z^{-1} - z^{-2} + 2z^{-3}$ ROC: All z except 0

(d) $X_d(z) = 3$ ROC: All z

Problem 12.2 $X(z) = \frac{2+3z^{-1}-z^{-2}}{1-z^{-3}}$

Problem 12.3

(a) (i) $x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] - \left(-\frac{1}{3}\right)^n u[-n-1]$

(ii) $x[n] = -3\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$

(iii) $x[n] = 3\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

(b) $\frac{-(-1)^n}{n} u[n-1]$

Problem 12.4

(a) $x_a[n] = \delta[n] + 2\delta[n-2] - 5\delta[n-3]$

(b) $x_b[n] = \delta[n+1] - \frac{1}{3!}\delta[n+3] + \frac{1}{5!}\delta[n+5] - \frac{1}{7!}\delta[n+7] + \frac{1}{9!}\delta[n+9] - ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{n!}\delta[n+2k+1]$

Problem 12.5 We are tempted to use the convolution property of z-transforms, *i.e.*, we multiply the z-transform $X(z)$ of the input by the transfer function $H(z)$ to obtain the z-transform $Y(z)$ of the output, then we take the inverse z-transform of $Y(z)$ to produce the output in time $y[n]$. The problem here is that $x[n]$ does not have a z-transform, so we write $x[n]$ in a more suggestive form:

$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{3}n\right) \\ &= \left(\frac{1}{2}\right)^n \frac{1}{2j}\left(e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}\right) \\ \Longrightarrow \quad x[n] &= \frac{1}{2j}\left[\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - \left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right]. \end{aligned}$$

Note that $x[n]$ is a linear combination of complex exponentials of the form z_0^n, where:

$$z_0 = \frac{1}{2}e^{\pm j\frac{\pi}{3}}$$

Thus, we can use the eigenfunction property of LTI systems. If the input is:

$$x_0[n] = z_0^n,$$

then the output is the input scaled by the transfer function evaluated at z_0:

$$y_0[n] = H(z_0)z_0^n.$$

The output $y[n]$ is therefore:

$$y[n] = \frac{1}{2j}\left[H\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - H\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right].$$

So, we get:

$$\begin{aligned}
H\left(\frac{1}{2}e^{\pm j\frac{\pi}{3}}\right) &= \frac{1}{z^3 + 2\sqrt{2}z^{3/2} + \frac{1}{8}}\Big|_{z=\frac{1}{2}e^{\pm j\frac{\pi}{3}}} \\
&= \frac{1}{\left(\frac{1}{2}e^{\pm j\frac{\pi}{3}}\right)^3 + 2\sqrt{2}\left(\frac{1}{2}e^{\pm j\frac{\pi}{3}}\right)^{3/2} + \frac{1}{8}} \\
&= \frac{1}{\frac{1}{8}e^{\pm j\pi} + 2\sqrt{2}\left(\frac{1}{2\sqrt{2}}e^{\pm j\frac{\pi}{2}}\right) + \frac{1}{8}} \\
&= \frac{1}{-\frac{1}{8} \pm j + \frac{1}{8}} \\
\Longrightarrow \quad H\left(\frac{1}{2}e^{\pm j\frac{\pi}{3}}\right) &= \mp j
\end{aligned}$$

Therefore, the output $y[n]$ is:

$$\begin{aligned}
y[n] &= \frac{1}{2j}\left[-j\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n - j\left(\frac{1}{2}e^{-j\frac{\pi}{3}}\right)^n\right] \\
&= \left(\frac{1}{2}\right)^n \frac{-j}{2j}\left[\left(e^{j\frac{\pi}{3}}\right)^n + \left(e^{-j\frac{\pi}{3}}\right)^n\right] \\
\Longrightarrow \quad y[n] &= -\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{3}n\right).
\end{aligned}$$

Problem 12.6 $h[n] = u[n+1]$
This is a right-sided signal, but since it becomes non-zero at time n=-1 it is non-causal.

Problem 12.7

1. $H(z) = \frac{1}{1+\frac{5}{2}z^{-1}-\frac{3}{2}z^{-2}}$

2. $H(z) = \frac{\frac{6}{7}}{1+3z^{-1}} + \frac{\frac{1}{7}}{1-\frac{1}{2}z^{-1}}$

3. Poles at -3 and $-\frac{1}{2}$.

4. If the system is causal, the ROC is $|z| > 3$.
   If the system is stable, the ROC is $3 > |z| > \frac{1}{2}$.

5. If the system is causal, then $h[n] = \left[\frac{6}{7}(-3)^n + \frac{1}{7}\left(\frac{1}{2}\right)^n\right] u[n]$.
   If the system is stable, then $h[n] = \left(-\frac{6}{7}\right)(-3)^n u[-n-1] + \left(\frac{1}{7}\right)\left(\frac{1}{2}\right)^n u[n]$.

Problem 12.8

Diagram	**Impulse Response**	$\|H(e^{j\omega})\|$
I	D	6
II	F	2
III	E	5
IV	B	4
V	C	1
VI	A	3