MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

Tutorial 12

Monday, May 10 and Tuesday, May 11, 2004

Announcements

- Problem set 11 has been assigned for you to study the z-transform and DT feedback systems, but we will not collect it to grade. You will be responsible for this material for the final exam.
- The final exam will be held on **Tuesday, May 18**, 9 a.m.–12 p.m. in the Johnson ice rink. Coverage will be comprehensive.
- The TAs will jointly hold office hours in the week of May 10. A schedule is posted on the 6.003 website.
- A final exam review package is available on the 6.003 website. TAs will hold two identical optional exam review sessions on Tuesday, May 11, **7:00–10:00 p.m.**, and Friday, May 14, **1:30–4:30 p.m.**, both in 34-101. Please note that the times of the two reviews are different from each other and from those of the reviews for the quizzes.

Today's Agenda

- Introduction to the z-Transform
  - Definition
  - The relationship between the z- and Fourier transforms
- Region of Convergence
- Poles and Zeros
  - Caveats about poles at zero and infinity
- Properties of the Region of Convergence
- The Inverse z-Transform
  - z vs. z^{-1}
- The z-Transform, LTI Systems, and the Eigenfunction Property
  - The eigenfunction property
  - Causality
  - Stability
  - Systems described by linear constant-coefficient difference equations
  - Real systems

- Geometric Evaluation of Rational z-Transforms
- System Function Algebra and Block Diagram Representation
- DT Feedback

1 Introduction to the z-Transform

1.1 Definition

In much the same way that the CT Fourier transform was generalized to the Laplace transform, we can generalize the DT Fourier transform to the z-transform, which is defined as follows:

The z-transform:

The z-transform $X(z)$ of a DT signal $x[n]$ is:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}.$$

We write this as:

$$\begin{aligned} X(z) &= \mathcal{Z}\{x[n]\} \\ x[n] &\overset{\mathcal{Z}}{\longleftrightarrow} X(z) \end{aligned}$$

The inverse z-transform is written as:

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}.$$

1.2 The relationship between the z- and Fourier transforms

We discovered a relationship between the Laplace and Fourier transforms in CT; let's do the same in DT for the z- and Fourier transforms. Let's write the variable z in polar form $z = re^{j\omega}$, where r is the magnitude of z and ω is the phase of z. If we set r to 1, then the z-transform reduces to the Fourier transform:

$$\begin{aligned} X(z)|_{z=e^{j\omega}} &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \\ &= \mathcal{F}\{x[n]\} \end{aligned}$$

Now if we plug in the polar form of z into the definition of the z-transform and recombine terms, we get the Fourier transform of $x[n]r^{-n}$:

$$
\begin{aligned}
X(z)|_{z=re^{j\omega}} &= \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n} \\
&= \sum_{-\infty}^{+\infty} \left(x[n]r^{-n}\right) e^{-j\omega n} \\
&= \mathcal{F}\{x[n]r^{-n}\}.
\end{aligned}
$$

To summarize:

> **The Fourier and z-Transforms:**
>
> The z-transform $X(z)$ of a DT signal $x[n]$ is the Fourier transform of $x[n]r^{-n}$, where r is the magnitude of z, and the Fourier transform $X(e^{j\omega})$ is the z-transform evaluated at $z = e^{j\omega}$.

The $j\omega$-axis played a special role in the Laplace transform; it was the location of the CT Fourier transform. We'll see that the unit circle in the z-transform plays an analogous role; it is the location of the DT Fourier transform. Basic properties of those transforms are consistent with their location. For instance, the CT Fourier transform is generally aperiodic in ω, and so it occupies an infinitely long line of the s-plane. The DT Fourier transform is periodic with period 2π in ω, so it occupies a unit circle in the z-plane, which has length 2π to enforce the periodicity.

2 Region of Convergence

Like the Laplace transform, the z-transform converges for only certain values of z, namely those z such that $x[n]r^{-n}$ is absolutely summable, where r is the magnitude of z. These values are in the region of convergence (ROC) of the z-transform.

For example, the right-sided exponential:

$$x[n] = a^n u[n]$$

has the following z-transform with the associated ROC:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|.$$

The left-sided exponential:

$$x[n] = -a^n u[-n-1]$$

has the same algebraic z-transform with a different ROC:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| < |a|.$$

These signals are analogous to the CT single-sided exponentials with the same differences: Flipping the ROC is equivalent to time reversal and a minus sign. One annoying asymmetry is that although the CT case goes from $u(t)$ to $u(-t)$ when the ROC is flipped, the DT case goes from $u[n]$ to $u[-n-1]$, rather than $u[-n]$.

Like Laplace, we have:

Specifying the Region of Convergence:
When writing the z-transform of a DT signal, both the algebraic expression and the region of convergence are required.

3 Poles and zeros

Many of the z-transforms we come across are ratios of polynomials:

$$X(z) = \frac{N(z)}{D(z)},$$

where $N(z)$ and $D(z)$ are the numerator and denominator polynomials, respectively. Rational z-transforms result from signals that are linear combinations of complex exponentials and from LTI systems described as linear constant-cofficient difference equations. Once again, we have the concept of poles and zeros of rational z-transforms, including multiple poles and zeros and poles and zeros at infinity and zero. However, these behave a bit differently than they do in the Laplace transform, as we'll see later.

3.1 Caveats about poles at zero and infinity

We have to be careful about poles at zero and infinity for z-transforms, even more so than we did for Laplace transforms. For example, property 3 above states that a finite-duration signal might have poles at zero or infinity. From the definition of the z-transform, we see that if such a signal $x[n]$ has non-zero values for some $n < 0$, then its transform has terms with positive powers of z. Thus, it has a pole at infinity, and the ROC does not include $z = \infty$. Likewise, the transforms of a signal that has non-zero values for some $n > 0$ has negative powers of z, so it has a pole at zero and the ROC does not include $z = 0$. Thus, the only way for the ROC of a transform to be the entire z-plane, including at $z = 0$ and $z = \infty$ is for the corresponding signal $x[n]$ to be zero for all non-zero time, so $x[n]$ is of the form:

$$x[n] = A\delta[n],$$

where A is a constant. The same reasoning applies for the last parts of propoerties 8 and 9.

We also saw with the Laplace transform properties that ROCs sometimes have the words "at least" attached, which comes about when there is pole-zero cancellation. z-transform properties have the same problem, but they are even worse because now we also need to watch out for the "except for the possible addition or deletion of the origin" for the ROC of a time-shifted signal. The next problem illustrates these issues.

Problem 12.1

Find the z-transforms and the associated ROCs of the following finite-length DT signals. Be careful about poles at zero and infinity.

(a) $x_a[n] = 3\delta[n+4] - \delta[n+3] + 2\delta[n+2]$.

(b) $x_b[n] = x_a[n-3] = 3\delta[n+1] - \delta[n] + 2\delta[n-1]$.

(c) $x_c[n] = x_a[n-5] = 3\delta[n-1] - \delta[n-2] + 2\delta[n-3]$.

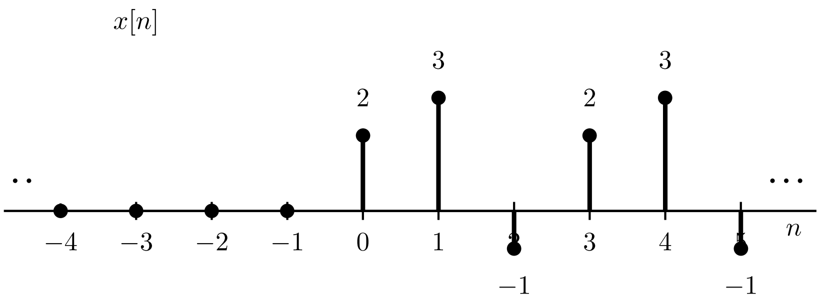
(d) $x_d[n] = 3\delta[n]$.

(Work space)

Problem 12.2

Find the z-transform $X(z)$ and the associated ROC of the signal $x[n]$, where $x[n]$ is a semi-periodic signal that satisfies the following conditions:

- $x[n] = 0$ for $n < 0$.
- $x[0] = 2$.
- $x[1] = 3$.
- $x[2] = -1$.
- $x[n] = x[n-3]$ for $n > 2$.



There are several ways to solve this problem!

(Work space)

4 Properties of the Region of Convergence

The textbook outlines nine basic properties of the region of convergence of the z-transform:

Properties of the Region of Convergence of the z-Transform:

1. The ROC consists of a ring in the z-plane centered about the origin.
2. For rational z-transforms, the ROC does not contain any poles.
3. If the signal is of finite duration, then the ROC is the entire z-plane, except possibly at $z = 0$ and/or $z = \infty$.
4. If the signal is right-sided, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $|z| > r_0$ will also be in the ROC.
5. If the signal is left-sided, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.
6. If the signal is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.
7. If the z-transform is rational, then its ROC is bounded by poles or extends to infinity.
8. If the z-transform is rational and the signal is right-sided, then the ROC is the region of the z-plane outside the of the outermost pole. Furthermore, if the signal $x[n]$ is zero for $n < 0$, then the ROC also includes $z = \infty$.
9. If the z-transform is rational and the signal is left-sided, then the ROC is the region of the z-plane inside the of the innermost pole. Furthermore, if the signal $x[n]$ is zero for $n > 0$, then the ROC also includes $z = 0$.

5 The Inverse z-Transform

As with Laplace, taking the inverse z-transform generally requires contour integration, but we can use partial fraction expansion and the table of properties. We will often do partial fraction expansion in z^{-1} instead of z.

The partial fraction expansion for z-transform is calculated in the same way as Laplace (the partial fraction expansion is just an algebraic manipulation). Thus, we can treat the variable z or z^{-1} in the same way we treated s.

Of course, just like with Laplace, pattern-matching and using tables of properties is another favorite way of find the inverse z-transforms.

5.1 z vs. z^{-1}

We can use two different forms of rational z-transforms:

(a) Polynomials in z, or the products of factors of the form $(z - a)$.

(b) Polynomials in z^{-1}, or the products of factors of the form $(1 - az^{-1})$.

In both forms, the constant a indicates the poles and zeros. It doesn't really matter which form you use, just be consistent! For example, suppose we would like to find the inverse z-transform of the following:

$$X(z) = \frac{z^3}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}.$$

Three possible ways of doing this are:

- Find the inverse transform of:

  $$X_1(z) = z^{-3}X(z) = \frac{1}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})},$$

  then use the time-shift property. This is generally the easiest method.

- Rewrite the transform as:

  $$X(z) = \frac{z^5}{(z + \frac{1}{4})(z - \frac{1}{2})},$$

  and do long division in z, followed by partial fraction expansion in z.

- Rewrite the transform as:

  $$X(z) = \frac{1}{z^{-3}(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})},$$

  and do partial fraction expansion in z^{-1}. This is rather tedious because there are multiple roots.

Keep in mind when doing partial fraction expansion to use long division for top-heavy rationals (when the numerator is of the same or higher order than the denominator).

Problem 12.3

(a) Suppose we are given the z-transform:

$$X(z) = \frac{4z^2 + \frac{1}{2}z}{z^2 - \frac{1}{6}z - \frac{1}{6}} = \frac{4 + \frac{1}{2}z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{4 + \frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{3}z^{-1})}.$$

For each of the following ROC's, determine the corresponding signal $x[n]$

(i) $|z| < \frac{1}{3}$.

(ii) $\frac{1}{3} < |z| < \frac{1}{2}$.

(iii) $|z| > \frac{1}{2}$.

(b) The z-transform $X(z)$ and the associated ROC of a DT signal $x[n]$ is:

$$X(z) = \ln(1 + z^{-1}), \quad |z| > 1.$$

Find $x[n]$.

(Work space)

Problem 12.4

As this problem illustrates, in addition to applying properties and reading tables to find inverse z-transforms, we can also try a power series.

Find the signals that correspond to the following z-transforms and their associated ROCs:

(a) $X_a(z) = 1 + 2z^{-2} - 5z^{-3}, \quad |z| > 0.$

(b) $X_b(z) = \sin z, \quad |z| \geq 0.$

(Work space)

6 The z-Transform, LTI Systems, and the Eigenfunction Property

Just like we used the Fourier transform to analyze LTI systems, we can do the same with the Laplace transform. Recall that:

$$y[n] = h[n] * x[n],$$

where $x[n]$ is the input of a DT LTI system, $h[n]$ is the impulse response, and $y[n]$ is the output. Then, by the convolution property of the z-transform, we have:

$$Y(z) = X(z)H(z),$$

where the the uppercase functions in z are the z-transforms of the corresponding lowercase functions in n. We call $H(z)$ the *transfer function*, or *system function*, of the DT LTI system. When the magnitude of z is one, and the ROC of the transfer function includes the unit circle, then the transfer function reduces to the frequency response $H(e^{j\omega})$ of the LTI system. We now have three ways of characterizing DT LTI systems:

Ways of Characterizing DT LTI Systems:

1. The impulse response $h[n]$.
2. The frequency response $H(e^{j\omega})$, which is the Fourier transform of the impulse response $h[n]$ and the transfer function $H(z)$ evaluated on the unit circle.
3. The transfer function $H(z)$, which is the z-transform of the impulse response $h[n]$.

6.1 The eigenfunction property

Like continuous-time, there's also the eigenfunction property in discrete-time:

The Eigenfunction Property of DT LTI Systems:

If the input $x[n]$ to a DT LTI system with transfer function $H(z)$ is a complex exponential of the form:

$$x[n] = z^n,$$

where z is in the ROC of $H(z)$, then the output $y[n]$ is:

$$y[n] = H(z)z^n.$$

Problem 12.5

Suppose a DT LTI system has the following transfer function:

$$H(z) = \frac{1}{z^3 + 2\sqrt{2}z^{3/2} + \frac{1}{8}}.$$

It is also known that the circle

$$|z| = \frac{1}{2}$$

is in the ROC of $H(z)$. Find the output $y[n]$ of the system when the input $x[n]$ is:

$$x[n] = \left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{3}n\right).$$

(Caution: Does $x[n]$ have a z-transform?)

(Work space)

6.2 Causality

Analogously with Laplace, we have:

> A DT LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.

The "including infinity" condition is important. In order for this to be met for rational system functions, we need an additional condition:

> **Causal Rational Systems and the ROC:**
>
> A DT LTI system with a rational transfer function is causal if and only if: (a) the ROC is the exterior of a circle outside the outermost pole; and (b) the order of the numerator of the transfer function does not exceed that of the denominator.

The second condition is rather subtle, for this does not occur in the Laplace transform. Why do we need it? If the order of the numerator exceeds that of the denominator, then we can use long division to create terms with positive powers of z, which correspond to DT impulses in $n < 0$. The following problem illustrates this point.

6.3 Stability

Recall that a DT system is stable if and only if the impulse response is absolutely summable. In that case, the frequency response, which is the transfer function evaluated on the unit circle, exists. Thus:

> **Stable Systems and the ROC:**
>
> An DT LTI system is stable if and only if the ROC of its transfer function $H(z)$ includes the unit circle, $|z| = 1$.

It follows that:

> **Rational Causal and Stable Systems and the ROC:**
>
> An DT LTI system described by a rational transfer function $H(z)$ is causal and stable if and only if all of the poles of $H(z)$ lie inside the unit circle.

Problem 12.6

Find the impulse response $h[n]$ of the system with the following transfer function and verify that the system is non-causal, even though $h[n]$ is right-sided:

$$H(z) = \frac{z^2}{z-1}, |z| > 1.$$

(Work space)

6.4 Systems described by linear constant-coefficent difference equations

We saw how the Fourier transform could be used to find the frequency response of systems described by difference equations. We can do the same with the z-transform. A general linear constant-coefficent difference equation has the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

Delay in time by n_0 corresponds to multiplication by z^{-n_0} in the z-transform domain. So, taking the z-transform of both sides results in:

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z).$$

Thus, the transfer function $H(z)$ is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left\{\sum_{k=0}^{M} b_k z^{-k}\right\}}{\left\{\sum_{k=0}^{N} a_k z^{-k}\right\}}.$$

Recall that a difference equation alone does not specify the system. In the z-transform domain, this is translated to the fact that a z-transform alone does not specify a signal. Knowing the ROC does completely specify the system, and this may come in the form of knowing whether a system is causal or stable. The next problem shows how this works.

6.5 Real systems

Like in CT, **real** DT LTI systems with rational transfer functions have the poles and zeros in conjugate pairs.

Problem 12.7

In problem set 2, problem 6, we considered the difference equation

$$y[n] + \frac{5}{2}y[n-1] - \frac{3}{2}y[n-2] = x[n].$$

We showed that if $x[n] = 0$ for all n, then

$$y[n] = A(-3)^n \; + \; B\left(\frac{1}{2}\right)^n$$

satisfies the above difference equation for any constant values of A and B. This information led us to the impulse response of the system described by the difference equation by matching boundary conditions. We found:

(a) If the system is causal, then

$$h[n] = \left[\frac{6}{7}\,(-3)^n + \frac{1}{7}\left(\frac{1}{2}\right)^n\right] u[n].$$

(b) If the system is BIBO stable, then

$$h[n] = \left(-\frac{6}{7}\right)(-3)^n\,u[-n-1] + \left(\frac{1}{7}\right)\left(\frac{1}{2}\right)^n u[n].$$

Let us derive the same results using the z-transform through the following steps:

1. Find the transfer function of the system.
2. Use partial-fraction expansion (in z^{-1}, not z).
3. Find the poles of the transfer function.
4. Identify the region of convergence.
5. Take the inverse z-transform.

(Work space)

(Work space)

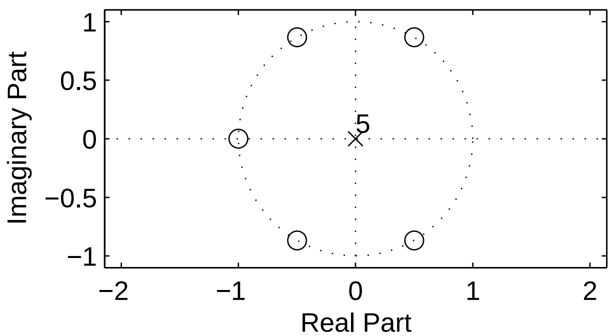
7 Geometric Evaluation of Rational z-Transforms

Like in the Laplace transform, we can use vectors in the complex plane to evaluate geometrically rational z-transforms. You can use this interpretation to verify that for an all-pass system, the existence of a pole at $z = p$ implies the existence of a zero at $z = p^{-1}$. Recall that for CT all-pass systems, the existence of a pole at $s = p$ implies the existence of a zero at $s = -p$.

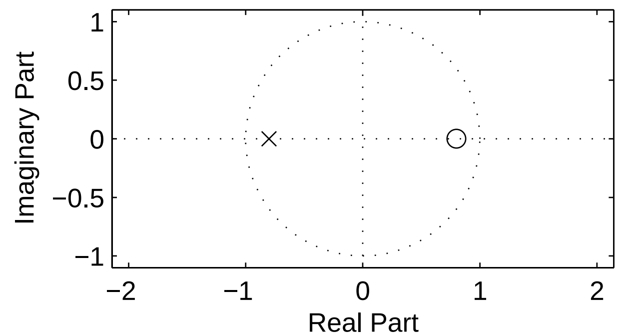
Problem 12.8

Consider 6 causal real DT LTI systems with the corresponding pole-zero diagrams, impulse responses, and frequency response magnitudes (on 3 seperate pages). Find the one-to-one-to-one correspondence between the three types of diagrams.

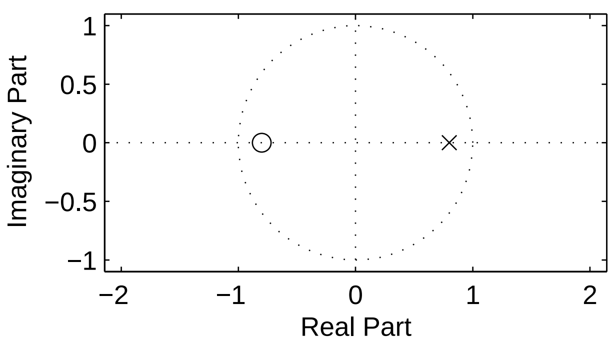
System I



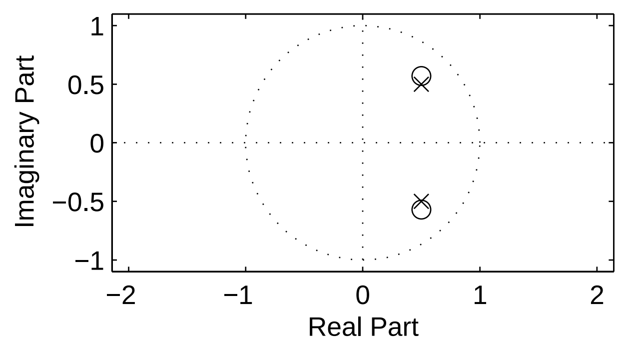
System II



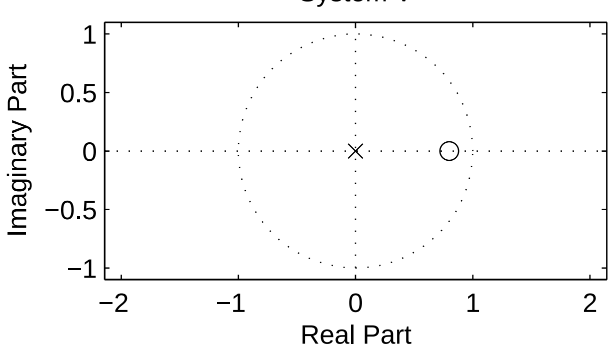
System III



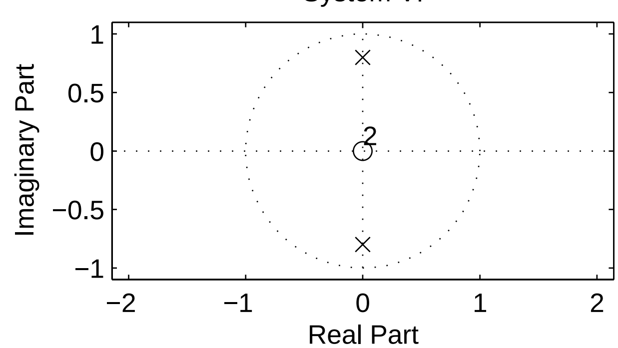
System IV

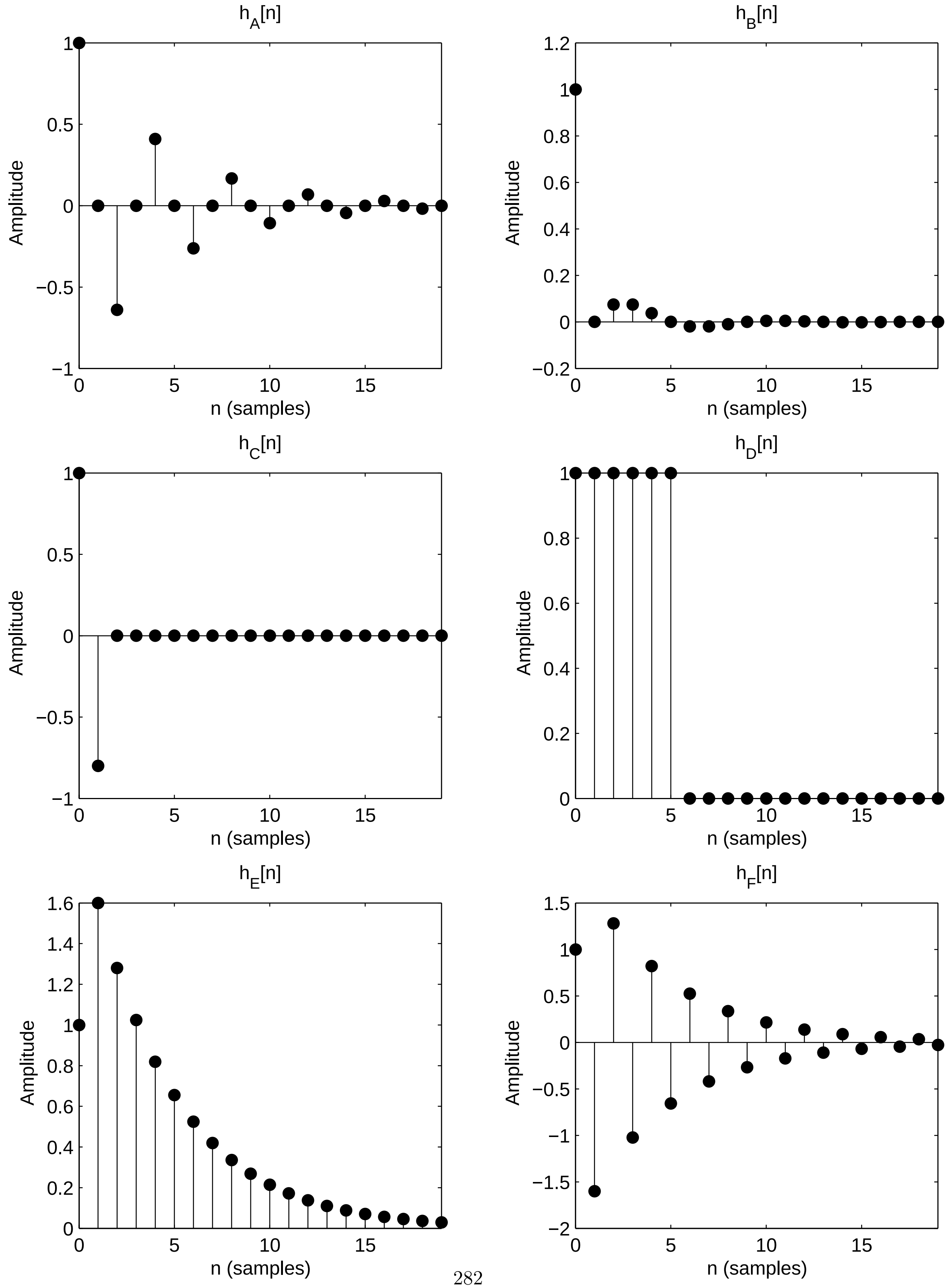


System V

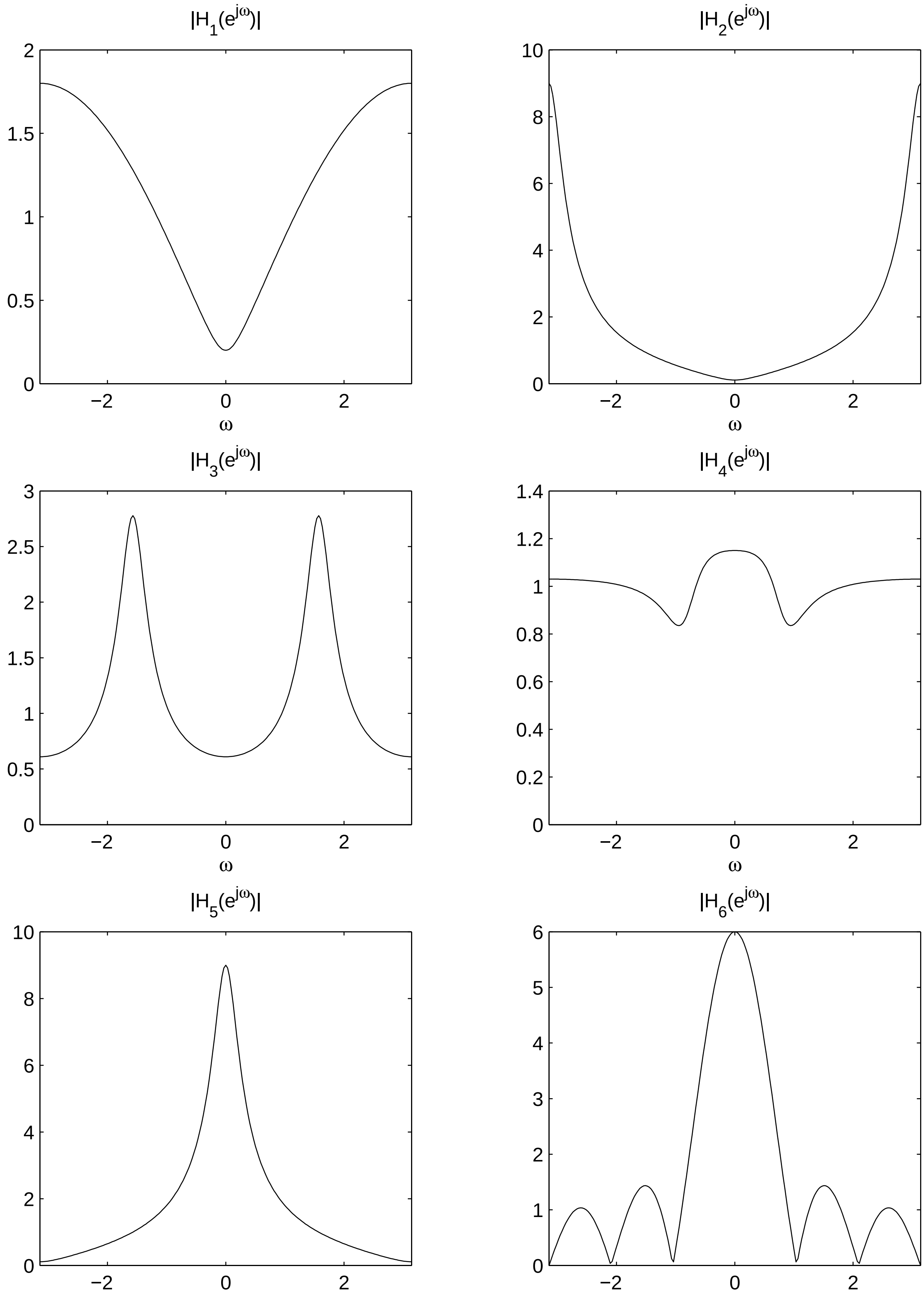


System VI





h_A[n]
h_B[n]
h_C[n]
h_D[n]
h_E[n]
h_F[n]
Amplitude
n (samples)


|H1(ejω)|
2
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1
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−2
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2
ω
|H2(ejω)|
10
8
6
4
2
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−2
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ω
|H3(ejω)|
3
2.5
2
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0.5
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ω
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1.4
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−2
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|H5(ejω)|
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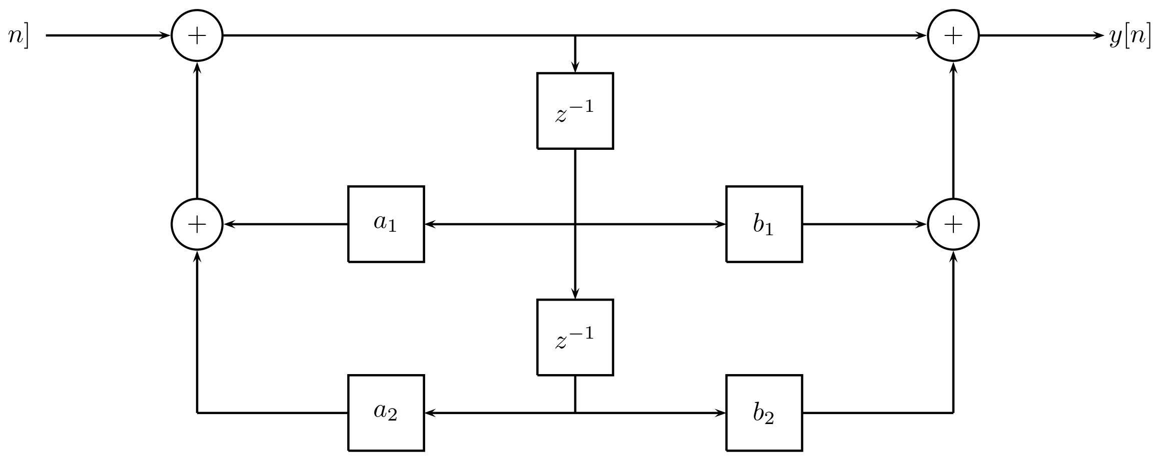
8 System Function Algebra and Block Diagram Representation

The Laplace and z-transforms enable us to translate systems described differential and difference equations into algebraic equations. In CT, we could represent such systems by integrator adder gain block diagrams. Likewise, in DT, we can use *delay adder gain block diagrams*.



$$x[n] + b_1 x[n-1] + b_2 x[n-2] = y[n] - a_1 y[n-1] - a_2 y[n-2]$$

$$H(z) = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

9 DT Feedback

There's also a DT version of feedback. It works the same way.