MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

Tutorial 11 Solutions

Tuesday, May 4, 2004

Problem 11.1

(a)

$$\begin{aligned} V_o(s) &= A[V_+(s) - V_-(s)] \\ V_+(s) &= \frac{1/(sC)}{R + 1/(sC)} V_i(s) \\ V_-(s) &= \frac{1}{2}[V_i(s) + V_o(s)] \end{aligned}$$

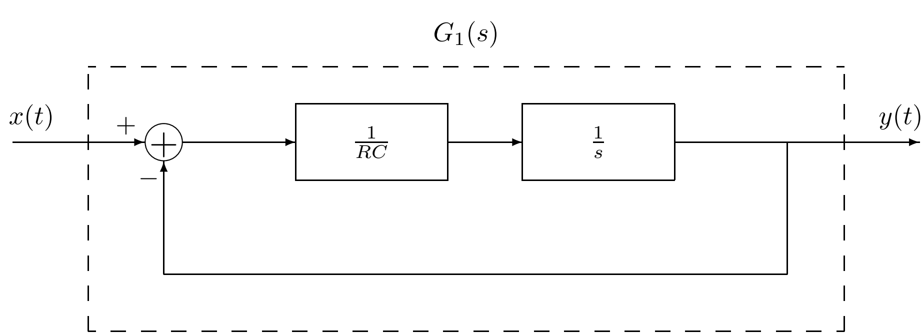
(b)

$$\begin{aligned} G_1(s) &= \frac{1/(sC)}{R + 1/(sC)} \\ G_2(s) &= A \\ G_3(s) &= \frac{1}{2} \end{aligned}$$

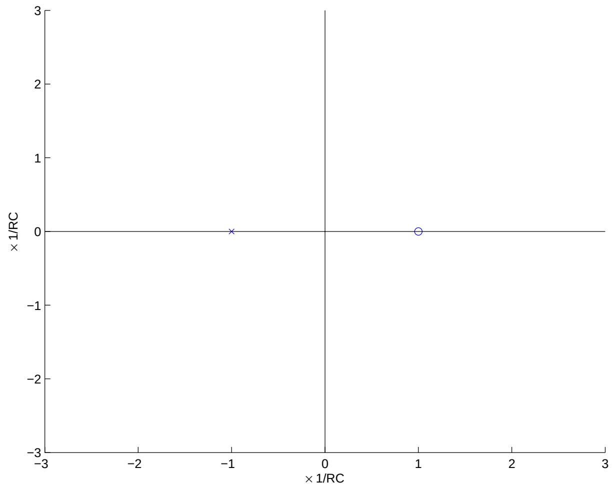
(c) To get $G_1(s)$ into the form of Black's formula $\frac{H(s)}{1+G(s)H(s)}$, we divide the numerator and denominator by R to get:

$$G_1(s) = \frac{1/(sRC)}{1 + 1/(sRC)}.$$

We see that the forward path system function is $1/(sRC)$, and the feedback path system function is 1 (unity feedback). $1/(sRC)$ is the cascade of an integrator $1/s$ and a gain $1/(RC)$, so we get the following block diagram for $G_1(s)$:



(d) We plug the second two equations from part (a) for $V_+(s)$ and $V_-(s)$ into the first equation for $V_o(s)$, and solve for $H(s)$. We then take the limit as $A \to \infty$. There is a pole at $\frac{-1}{RC}$ and a zero at $\frac{1}{RC}$. The pole-zero diagram is shown below:

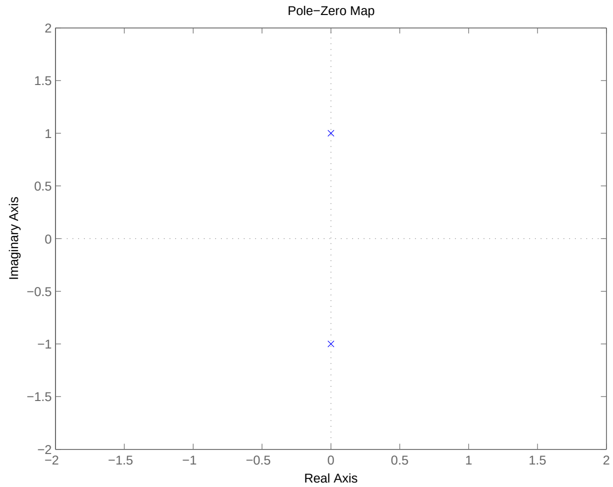


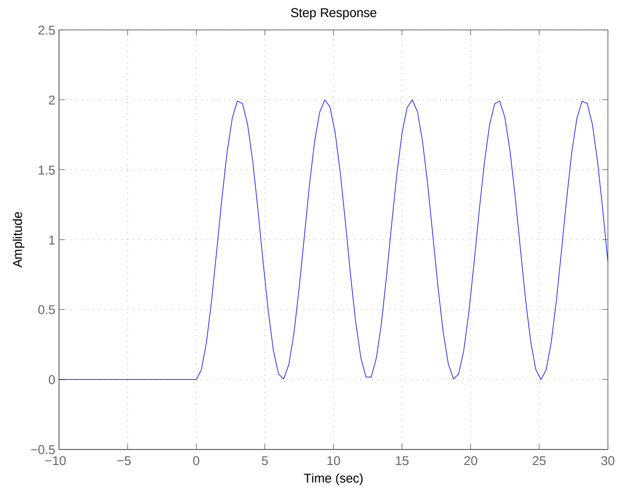
(e)

$$
\begin{aligned}
|H(j\omega)| &= \left| -\frac{j\omega - \frac{1}{RC}}{j\omega + \frac{1}{RC}} \right| \\
&= \frac{\left|j\omega - \frac{1}{RC}\right|}{\left|j\omega + \frac{1}{RC}\right|} \\
&= \frac{\omega^2 + \left(-\frac{1}{RC}\right)^2}{\omega^2 + \left(\frac{1}{RC}\right)^2} \\
\longrightarrow \quad |H(j\omega)| &= 1, \quad \text{for all } \omega.
\end{aligned}
$$

Problem 11.2

(a) $H_p(s)$ has two poles at $\pm j$. The pole-zero diagram and step response are shown below. Since the real part of j is not negative, $H_p(s)$ is unstable. This is an undamped second-order system, so its step response is a right-sided everlasting sinusoid:





(b) (i) The closed-loop transfer function is:

$$Q(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} = \frac{K}{s^2 + K + 1}.$$

(ii) The poles are at $\pm j\sqrt{K+1}$. If $K \geq -1$, both poles are purely imaginary and lie on the $j\omega$ axis, and the system is unstable. If $K \leq -1$, both poles are real, but one is positive and the other is negative, so the system is unstable. Thus, we cannot stabilize the system with a proportional controller.

(c) (i) The closed-loop transfer function is:

$$Q(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} = \frac{As + B}{s^2 + As + (B+1)}.$$

(ii) According to Routh and Hurwitz, we need $A > 0$ and $B+1 > 0$. Thus, we need $A > 0$ and $B > -1$.

(iii) The transfer function from the input $x(t)$ to the error $e(t)$ is:

$$\frac{E}{X}(s) = \frac{1}{1 + H_c(s)H_p(s)} = \frac{s^2 + 1}{s^2 + As + (B+1)}.$$

The steady-state error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sX(s)\frac{E}{X}(s) = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{s^2+1}{s^2 + As + (B+1)} = \frac{1}{B+1}.$$

(d) (i) The closed-loop transfer function is:

$$Q(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} = \frac{As^2 + Bs + C}{s^3 + As^2 + (B+1)s + C}.$$

(ii) According to Routh and Hurwitz, we need $A > 0$, $B + 1 > 0$, $C > 0$ and $A(B+1) > C$. Thus, we need $A > 0$, $B > -1$, $C > 0$ and $A(B+1) > C$.

(iii) The transfer function from the input $x(t)$ to the error $e(t)$ is:

$$\frac{E}{X}(s) = \frac{1}{1 + H_c(s)H_p(s)} = \frac{s(s^2+1)}{s^3 + As^2 + (B+1)s + C}.$$

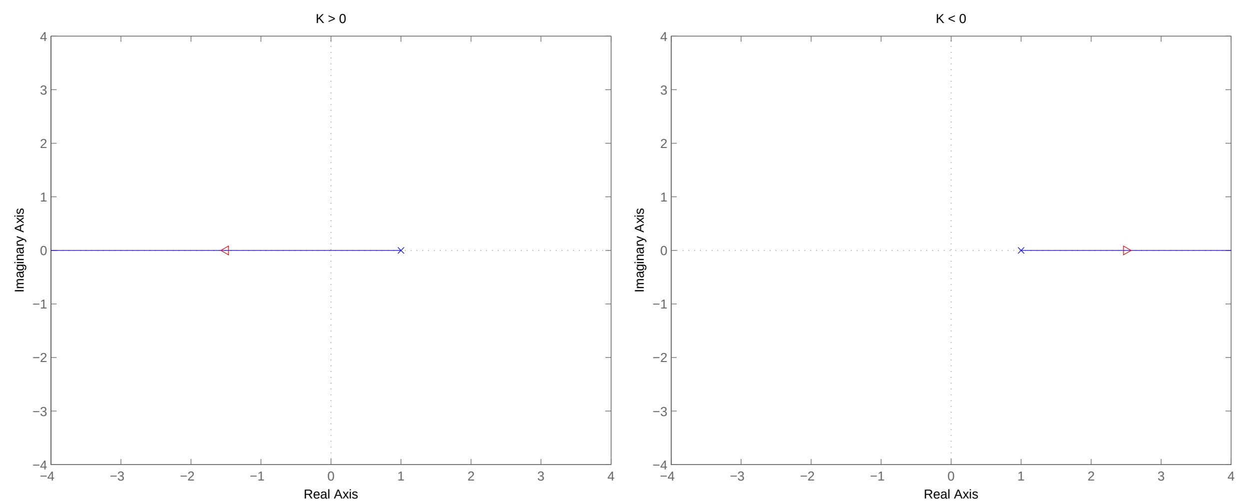
The steady-state error is:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sX(s)\frac{E}{X}(s) = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{s(s^2+1)}{s^3 + As^2 + (B+1)s + C} = 0.$$

A PID controller enables us to stabilize the system *and* have steady-state error.

Problem 11.3

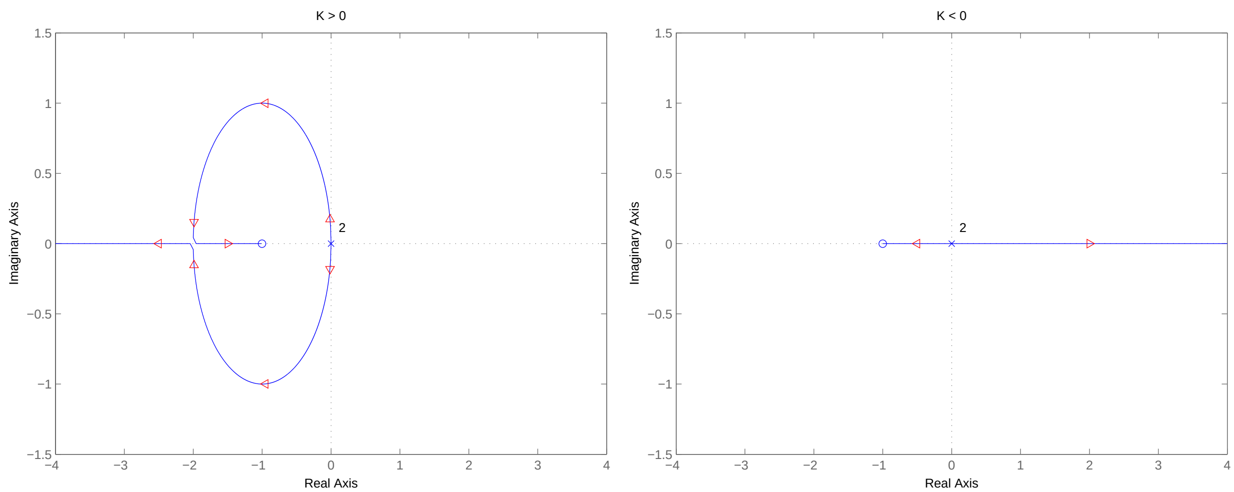
(a) Open-loop pole: 1. Open-loop zero: one at infinity. Characteristic equation: $0 = 1 + KL_0(s) = 1 + \frac{K}{s-1} = s - (1-K)$. Thus, the single closed-loop pole is at $1-K$. We need $K > 1$ to make the closed-loop system stable. The root locus plot is shown below:



(b) Open-loop poles: two at the origin. Open-loop zeros: one at -1, one at infinity. Characteristic equation: $0 = 1 + KL_0(s) = 1 + \frac{K(s+1)}{s^2} = s^2 + Ks + K$. Thus, the closed-loop poles are at:

$$p_\pm = \frac{-K \pm \sqrt{K^2 - 4K}}{2}.$$

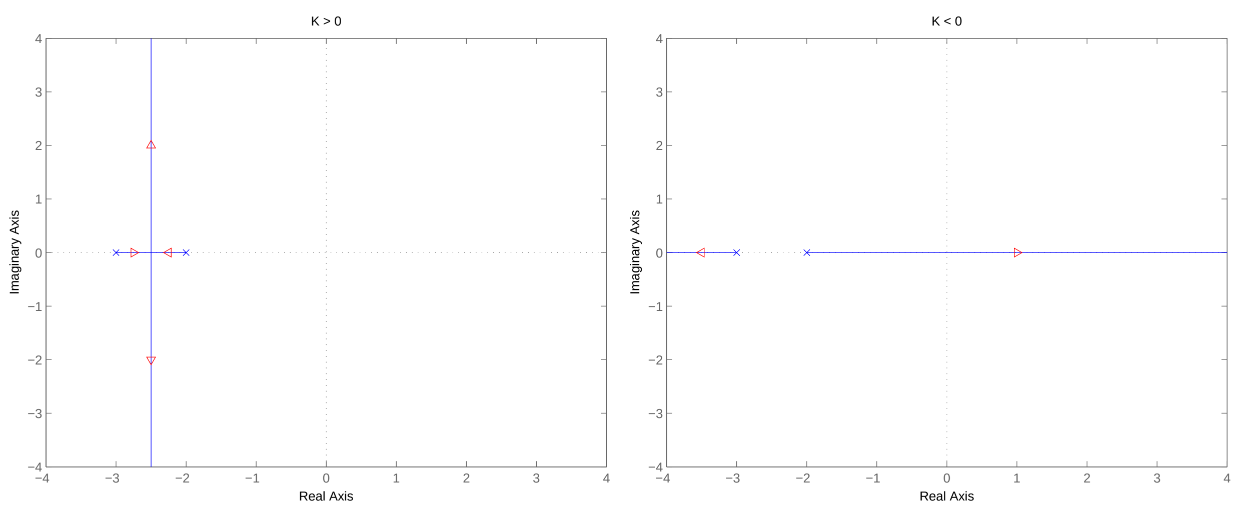
When $0 < K < 4$, $p_\pm$ are both complex. When $K = 4$, $p_+ = p_- = -2$ (both poles meet at -2). When $K > 4$, $p_+ \to -1$ and $p_- \to -\infty$ (both real). When $K < 0$, $p_+ \to +\infty$ and $p_- \to -1$ (both real). According to Routh and Hurwitz, we need $K > 0$ to make the closed-loop system stable. The root locus plot is shown below:



(c) Open-loop poles: -2, -3. Open-loop zeros: two at infinity. Characteristic equation: $0 = 1 + KL_0(s) = 1 + \frac{K}{(s+2)(s+3)} = s^2 + 5s + (6 + K)$. Thus, the closed-loop poles are at:

$$p_{\pm} = \frac{-5 \pm \sqrt{1 - 4K}}{2}.$$

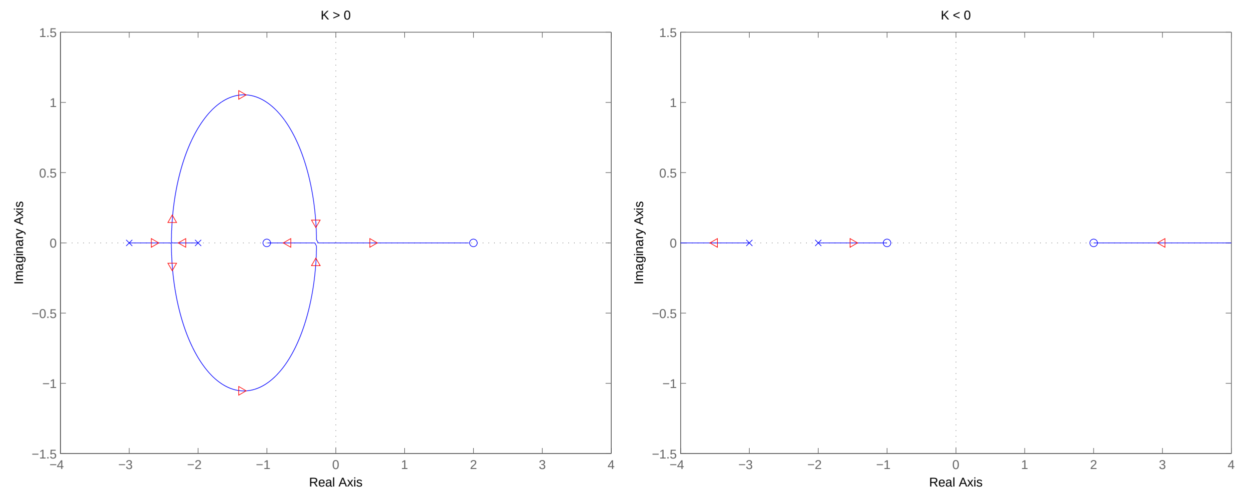
When $0 < K < \frac{1}{4}$, $p_{\pm}$ are beteen -3 and -2. When $K = \frac{1}{4}$, $p_{+} = p_{-} = -2.5$ (both poles meet at -2.5). When $K > \frac{1}{4}$, both poles are complex and have -2.5 as their real part; the imaginary parts go to $\pm\infty$. When $K < 0$, both closed-loop poles are real and grow further apart. According to Routh and Hurwitz, we need $K > -6$ to make the closed-loop system stable. The root locus plot is shown below:



(d) Open-loop poles: -2, -3. Open-loop zeros: -1, 2. Each open-loop pole is paired up with an explicit open-loop zero, so there are no open-loop zeros or poles at infinity. Characteristic equation: $0 = 1 + KL_0(s) = 1 + \frac{K(s+1)(s-2)}{(s+2)(s+3)} = (K+1)s^2 + (5-K)s + (6-2K)$. Thus, the closed-loop poles are at:

$$p_{\pm} = \frac{K - 5 \pm \sqrt{9K^2 - 26K + 1}}{2(K+1)}.$$

When $K > 0$, the closed-loop poles move toward each other. When $K \approx 0.3899$, the poles meet at around -2.387 (on the real axis). As K is increased further, the poles become complex and move in a curved manner back toward the real axis. When $K \approx 2.8499$, they meet on the real axis again at around -0.279. As K is increased, they move apart toward the open-loop zeros. When $K = 3$, one of the poles hits the $j\omega$-axis. When $K < 0$, both closed-loop poles are real. One moves from -2 (open-loop pole) to -1 (open-loop zero). The other one starts at -3 and heads towards negative infinity. When $K = -1$, it hits infinity and starts coming back ("wrap around") on the *positive* real axis toward 2 (open-loop zero). Overall, the closed-loop system is stable when $-1 < K < 3$. The root locus plot is shown below:



The wrap around effect for p_- is interesting, and it can be verified that it actually happens. The following shows the location of p_- as a function of K. We see that as K goes from 0 to -1, p_- goes from -3 down to negative infinity. As K goes from -1 down to negative infinity, p_- goes from positive infinity down to 2.