MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

Tutorial 11

Monday, May 3 and Tuesday, May 4, 2004

Announcements

- Problem set 10 is due this Friday. This is the last graded problem set of the semester.
- The final exam will be held on **Tuesday, May 18**, 9 a.m.–12 p.m. in the Johnson ice rink. Coverage will be comprehensive.
- The TAs will jointly hold office hours in the week of May 10. A schedule will be posted on the 6.003 website.
- A final exam review package will be available on the 6.003 website this Thursday. TAs will hold two identical optional exam review sessions on Tuesday, May 11, **7:00–10:00 p.m.**, and Friday, May 14, **1:30–4:30 p.m.**, both in 34-101. Please note that the times of the two reviews are different from each other and from those of the reviews for the quizzes.

Today's Agenda

- The Routh and Hurwitz Stability Criteria for CT System Functions
- Introduction to Feedback
  - Block pushing and unity feedback
  - Types of controllers
  - Tracking systems
  - Stabilization of systems
- Root Locus

1 The Routh and Hurwitz Stability Criteria for CT System Functions

A causal CT system with a rational system function is stable if and only if all its poles are to the left of the $j\omega$-axis, *i.e.* they all have negative real parts. So, if $P(s)$ is the denominator of the system function, stability is equivalent to all the roots of $P(s)$ having negative real parts. If $P(s)$ is first- or second-order, it is easy to find the roots and check this. If $P(s)$ is third- or fourth-order, analytic forms of the roots still exist, but it is a much more tedious task to calculate them. For fifth-order $P(s)$ and up, no general analytic expression for the roots exists and we are resigned to using numerical methods to find the roots. But we are not always interested in the roots themselves, just whether their real parts are negative. In 1877, Edward Routh published a solution for determining the necessary and sufficient conditions for all the roots being the left-half plane using the coefficients of $P(s)$. In 1895, Adolf Hurwitz produced an independent solution. These are known as the *Routh and Hurwitz stability criteria*.

We won't go into the Routh and Hurwitz stability criteria in detail, but here are the results for polynomials up to fourth-order:

The Routh and Hurwitz Stability Criteria for CT System Functions up to Fourth-Order:

Consider an n^{th}-order polynomial of the form:

$$P(s) = s^n + \sum_{k=0}^{n-1} a_k s^k = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0,$$

where the coefficients a_k are real. Then:

- For first-order $P(s) = s + a_0$, the single real root is negative if and only if:
$$a_0 > 0.$$
- For second-order $P(s) = s^2 + a_1 s + a_0$, both roots have negative real parts if and only if:
$$a_0,\, a_1 > 0.$$
- For third-order $P(s) = s^3 + a_2 s^2 + a_1 s + a_0$, all three roots have negative real parts if and only if:
$$\begin{aligned} a_0,\, a_1,\, a_2 &> 0 \quad \text{and} \\ a_2 a_1 &> a_0. \end{aligned}$$
- For fourth-order $P(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$, all four roots have negative real parts if and only if:
$$\begin{aligned} a_0,\, a_1,\, a_2,\, a_3 &> 0, \\ a_3 a_2 &> a_1, \quad \text{and} \\ a_3 a_2 a_1 &> a_1^2 + a_3^2 a_0. \end{aligned}$$

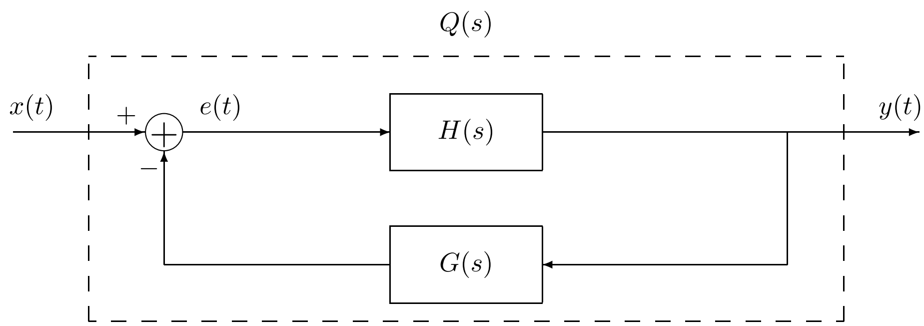
2 Introduction to Feedback

Feedback systems are used for a variety of reasons, including:

- Inverse system design.
- Compensation for nonideal elements.
- Compensation for uncertainties.
- Disturbance and noise rejection.
- Improvement of dynamic behavior characteristics.
- Stabilization of unstable systems.
- Tracking systems.

Because feedback systems are almost always real-life systems, we will assume they are causal. Such systems are stable if and only if the poles are to the left of the $j\omega$-axis.

The standard figure for a feedback system is:



We define the following quantities (Laplace domain):

- The (reference) input: $X(s)$ (or $R(s)$).
- The (controlled) output: $Y(s)$ (or $C(s)$).
- The error signal: $E(s) = X(s) - G(s)Y(s)$.
- The forward path system function: $H(s)$.
- The feedback path system function: $G(s)$.
- The open-loop transfer function: $L(s) = G(s)H(s)$.
- The closed-loop transfer function: $Q(s) = \frac{Y}{X}(s) = \frac{H(s)}{1+G(s)H(s)} = \frac{H(s)}{1+L(s)}$.
- The characteristic equation: $P(s) = 1 + G(s)H(s) = 1 + L(s) = 0$.

The formula for the closed-loop transfer function is known as Black's formula. $P(s) = 1+G(s)H(s)$ is the denominator of the closed-loop transfer function. Setting it to zero gives the characteristic equation, which yields the closed-loop poles.

Caution: In some sources, the notation for the forward path system function $H(s)$ and the feedback path system function $G(s)$ are reversed. In these notes, they are consistent with Oppenheim and Willsky.

2.1 Block pushing and unity feedback

Sometimes we'll find it convenient to have the feedback path system function be $G(s) = 1$ so that the system block diagram is in *unity feedback form.* For such systems, the closed-loop transfer function is:

$$Q(s) = \frac{H(s)}{1 + H(s)}.$$

How can we turn any feedback system into this form, perhaps cascaded with non-feedback systems? For a general closed-loop transfer function:

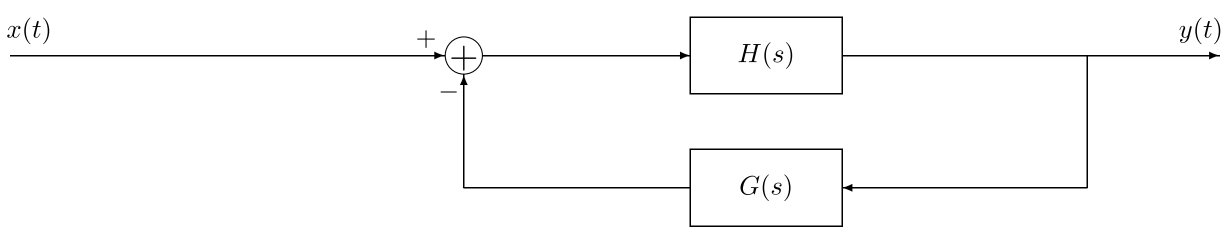
$$Q(s) = \frac{H(s)}{1 + G(s)H(s)},$$

we can multiply both the numerator and denominator by $G(s)$ to obtain:

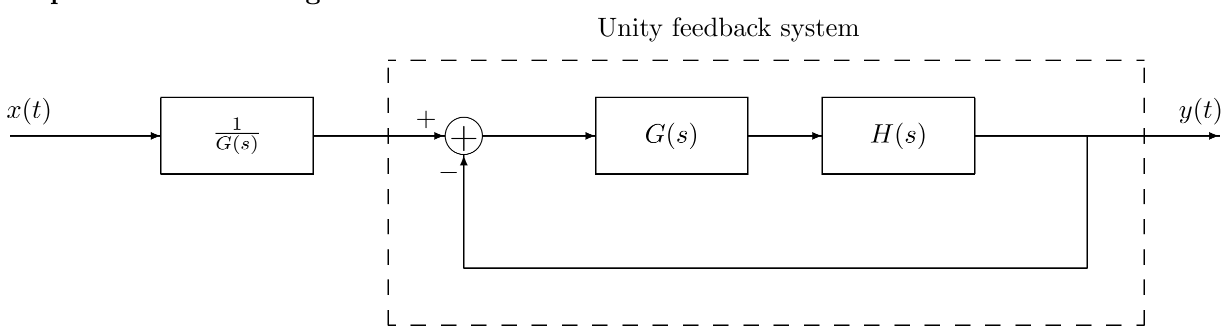
$$Q(s) = \frac{1}{G(s)} \cdot \frac{G(s)H(s)}{1 + G(s)H(s)}.$$

Now, if we let the open-loop transfer function $L(s) = G(s)H(s)$ be the new forward path system function, we see that the original feedback system is equivalent to a system that consists of a block cascaded with a unity feedback system. Note that the original block diagram and the unity feedback system have the same loop gain $L(s) = G(s)H(s)$. Thus, when we study things that depend on the loop gain $L(s)$, such as root locus, analyzing the unity feedback system is (usually) sufficient.

Original Non-Unity Block Diagram:

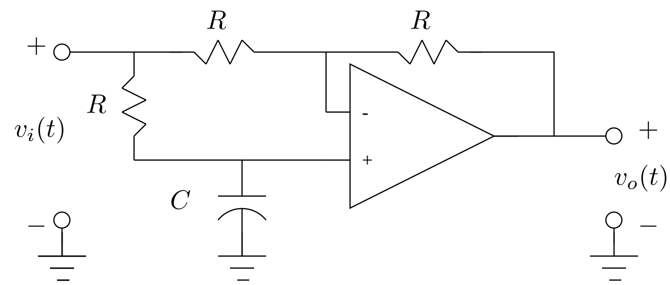


Equivalent Block Diagram:



Problem 11.1

Consider the following op-amp circuit, which has voltage $v_i(t)$ as the input signal and voltage $v_o(t)$ as the output signal.



Assume that the op-amp has some gain $A \gg 1$ and that:

$$V_o(s) = A\left[V_+(s) - V_-(s)\right],$$

where $V_+(s)$, $V_-(s)$ and $V_o(s)$ are the Laplace transforms of $v_+(t)$, $v_-(t)$ and $v_o(t)$, the voltages at the plus, minus and output terminals of the op-amp, respectively. Recall that an ideal op-amp draws no current at its inputs.

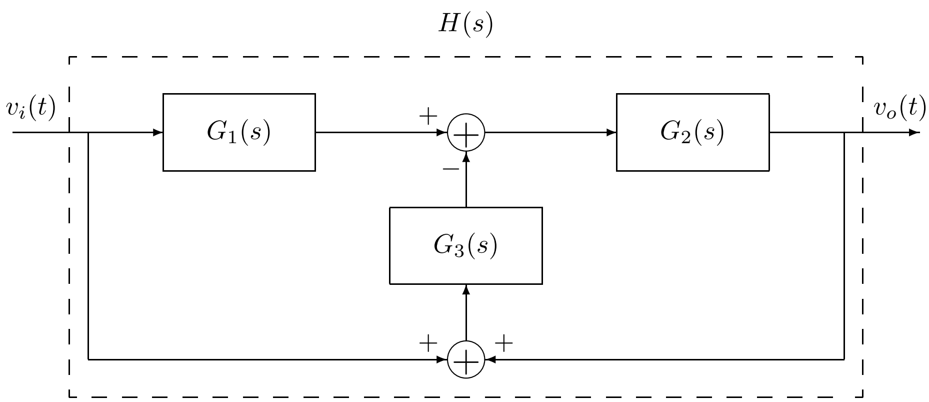
(a) Use the op-amp gain equation above and voltage dividers to set up three equations that relate $V_+(s)$, $V_-(s)$, $V_i(s)$ and $V_o(s)$ in terms of A, R and C:

$V_o(s) =$ ____________________________

$V_+(s) =$ ____________________________

$V_-(s) =$ ____________________________

(b) We can express the operation of the circuit using the following feedback diagram, where $G_1(s)$, $G_2(s)$ and $G_3(s)$ are some transfer functions.



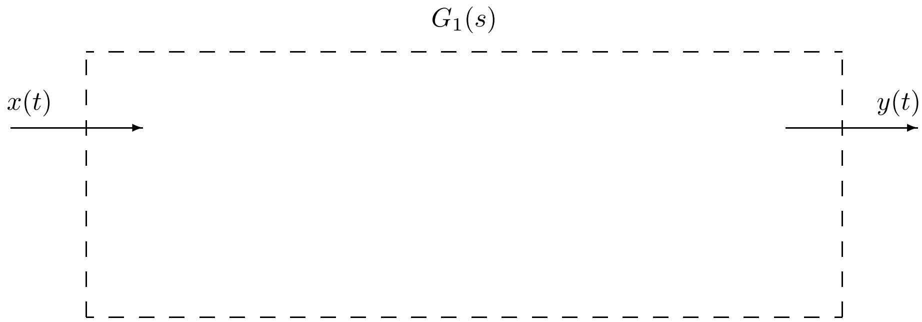
Use your results from the previous part to find $G_1(s)$, $G_2(s)$ and $G_3(s)$ in terms of A, R and C.

$G_1(s)$ = ____________________

$G_2(s)$ = ____________________

$G_3(s)$ = ____________________

(c) In part (b), you found that $G_2(s)$ and $G_3(s)$ were simply constants. $G_1(s)$ is more complicated, but it can be represented as a combination of gains, integrators[1] and adders. Draw such a block diagram for $G_1(s)$. *Hint:* Feedback is necessary, and you may want to write $G_1(s)$ as Black's formula and use pattern-matching to find the forward path and feedback path system functions.



Now, the entire system $H(s)$ is represented by a combination of gains, integrators and adders.

(Work space)

[1] You can, in fact, also construct $G_1(s)$ using differentiators instead of integrators (you may thought of this construction first). However, ideal differentiators cannot be built in the real causal world, and are thus an undesirable component of these block diagrams. More generally, a rational system function where the order of the numerator exceeds that of the denominator is not physically realizable.

(d) Show that the transfer function of the circuit $H(s) = \frac{V_o(s)}{V_i(s)}$ is:

$$H(s) = \frac{V_o(s)}{V_i(s)} = -\frac{s - \frac{1}{RC}}{s + \frac{1}{RC}}.$$

Sketch the pole-zero diagram.

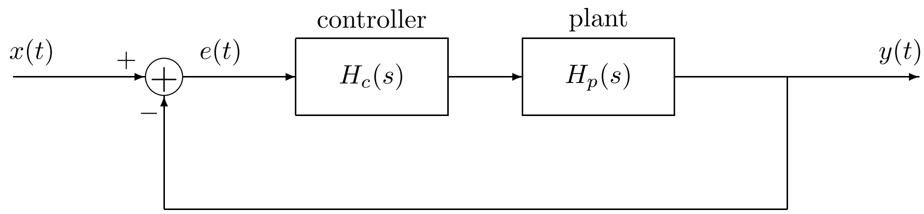
(Work space)

(e) Show that the magnitude $|H(j\omega)|$ of the frequency response is unity for all frequencies so that the circuit implements an all-pass filter.

(Work space)

2.2 Types of controllers

When we apply feedback to building systems, some components (called "plants") are fixed, and others (called "compensators" or "controllers") are free for us to design. We typically see these sort of systems in this form:



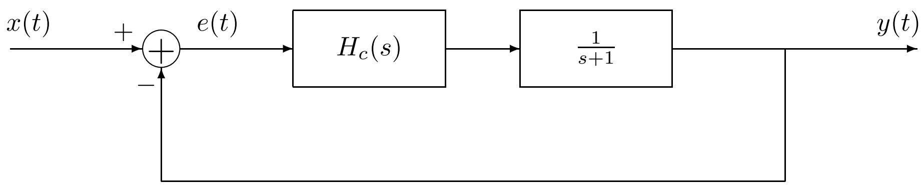
Some types of controllers $H_c(s)$ include:

- Proportional: $H_c(s) = K$.
- Integral: $H_c(s) = \frac{K}{s}$.
- Proportional plus integral (PI) : $H_c(s) = K_1 + \frac{K_2}{s}$.
- Proportional plus derivative (PD) : $H_c(s) = K_1 + K_2 s$.
- Proportional integral derivative (PID) : $H_c(s) = K_1 + \frac{K_2}{s} + K_3 s$.

2.3 Tracking systems

Sometimes, we want an output to track an input. The beginning of chapter 11 has an example where we want to adjust the position of a telescope. In this case, we would like the output (the actual position of the telescope) to be as close to the input (our signal that indicates what we would like the position to be) as possible. For such systems, we want to look at the error signal $e(t)$ (transform: $E(s)$) that emerges from the summing junction. Let's try an example. (This example is from Keith Santarelli's 6.003 fall 2000 tutorial notes.)

We have a plant $H_p(s) = \frac{1}{s+1}$. We would like to design a controller $H_c(s)$ so that output tracks the input in the following feedback system:



For a unit step input $x(t) = u(t)$ $(X(s) = \frac{1}{s})$, we want the error signal $e(t)$ to approach zero:

$$\lim_{t\to\infty} e(t) = 0.$$

Using the final-value theorem, we get:

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{E(s)}{X(s)}X(s) \\
&= \lim_{s\to 0} s\frac{E(s)}{X(s)}\frac{1}{s} \\
&= \lim_{s\to 0} \frac{E(s)}{X(s)} \\
&= \lim_{s\to 0} \frac{1}{1 + H_c(s)H_p(s)} \\
\Longrightarrow \quad \lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{1}{1 + \frac{H_c(s)}{s+1}}.
\end{aligned}$$

Note that the s in the final-value theorem and the $\frac{1}{s}$ in $X(s)$ conveniently cancel. We see that the right-hand side as s goes to zero only goes to zero as needed for tracking when $H_c(s)$ goes to infinity. This happens exactly when $H_c(s)$ has a pole at zero. We also need stability so that the signals behave properly. So, let's consider a proportional plus integral (PI) controller $H_c(s)$:

$$H_c(s) = K_1 + \frac{K_2}{s}.$$

We see that it has a pole at zero, so now we need to choose the constants so that the system is stable. Overall system stability is equivalent to the stability of the $E(s)/X(s)$ system (they have the same characteristic equation):

$$\begin{aligned}\frac{E(s)}{X(s)} &= \frac{1}{1+\frac{K_1+K_2/s}{s+1}} \\ &= \frac{s(s+1)}{s^2+(K_1+1)s+K_2}.\end{aligned}$$

From Routh and Hurwitz, this is stable when $K_1 > -1$ and $K_2 > 0$. Here's a subtle point: what if $K_2 = 0$? Then, we have:

$$\begin{aligned}\frac{E(s)}{X(s)} &= \frac{s(s+1)}{s^2+(K_1+1)s} \\ &= \frac{s+1}{s+(K_1+1)},\end{aligned}$$

which is stable as long as $K_1 > -1$. Because of the pole-zero cancellation (which will plague us in all of these problems...), we need to adjust the stability conditions to $K_1 > -1$ and $K_2 \geq 0$.

So, the steady-state error is:

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \frac{s(s+1)}{s^2+(K_1+1)s+K_2} \\ &= \frac{s(s+1)}{s^2+(K_1+1)s+K_2} \\ &= \begin{cases} 0, & K_2 > 0 \\ \frac{1}{K_1+1}, & K_2 = 0 \end{cases}\end{aligned}$$

So, we need to choose $K_2 > 0$ to get zero steady-state error. This makes sense; $K_2 = 0$ gets rid of the pole at zero, which we need.

We are not only interested in the steady-state error, but we are also interested in the time-domain behavior. If the step response contains fast oscillations or has too much overshoot, we might damage the telescope.

2.4 Stabilization of systems

Let's look at using feedback to stabilize potentially unstable systems, which will lead us to root locus. The next problem illustrates this.

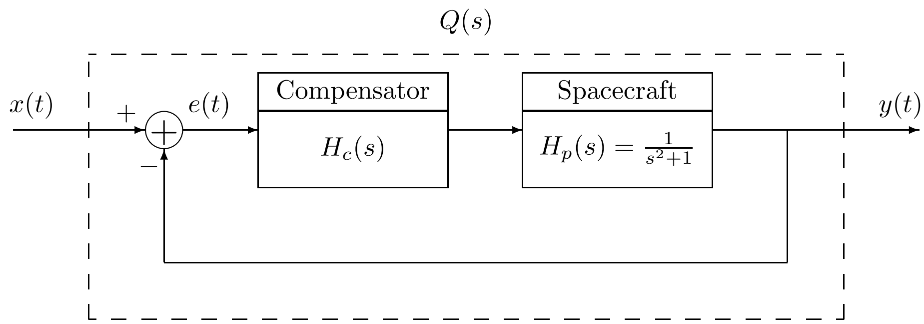
Problem 11.2

Suppose we have a model of the rockets that control a spacecraft's altitude, where the plant system function is:

$$H_p(s) = \frac{1}{s^2+1}.$$

The output of $H_p(s)$ is the deviation of the altitude of the spacecraft from some reference position[2] and the input is the some electrical signal that we can control.

Let's use a feedback system to implement an altitude control system. We want a system $Q(s)$ such that the input $x(t)$ is the control signal produced by a human or an automated system and the outupt $y(t)$ is the actual deviation of the altitude of the spacecraft. Ideally, we'd like $y(t)$ to track $x(t)$ as closely as possible. The following figure shows the complete control system:



We would like to design the controller $H_c(s)$.

(a) Suppose we didn't use feeback at all so that $Q(s) = H_p(s)$. Draw the pole-zero diagram of $H_p(s)$. Is $H_p(s)$ stable? Sketch the step response.

(Work space)

[2]A model of a basic rocket system would have the *acceleration* of the ship as the output, but let's say that our plant system is sophisticated enough to include the reference position and velocity of the ship to produce the altitude. We are also assuming that all our systems are linear, which is a major simplication.

(b) Apparently, we need feedback. Suppose $H_c(s) = K$ is a proportional controller, where K is a real number.

(i) Find the transfer function $Q(s) = \frac{Y(s)}{X(s)}$.

(ii) Can we stabilize the system with some value of K?

(Work space)

(c) Okay, so a proportional controller won't stabilize the system. We have enough money to build a more advanced controller, so let's suppose $H_c(s) = As + B$ is a proportional plus derivative (PD) controller, where A and B are real numbers.

(i) Find the transfer function $Q(s) = \frac{Y(s)}{X(s)}$.

(ii) Find the ranges of A and B such that the closed-loop system is stable.

(iii) Find the steady-state error $\lim_{t\to\infty} e(t)$ when the input is a unit step $x(t) = u(t)$.

(Work space)

(d) A PD controller stabilizes the system, but has a non-zero steady-state error. We reach deeper into NASA's pockets and find that we can afford an even more complicated controller. Suppose $H_c(s) = As + B + C/s$ is a proportional integral derivative (PID) controller, where A, B, and C are real numbers.

(i) Find the transfer function $Q(s) = \frac{Y(s)}{X(s)}$.

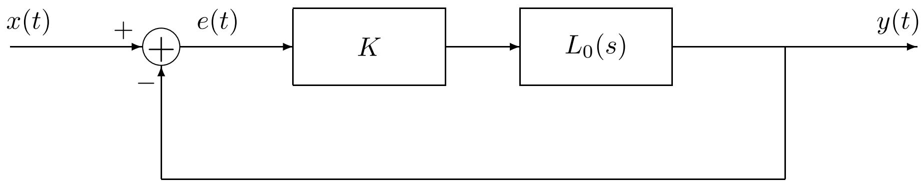
(ii) Find the ranges of A, B, and C such that the closed-loop system is stable.

(iii) Find the steady-state error $\lim_{t\to\infty} e(t)$ when the input is a unit step $x(t) = u(t)$.

(Work space)

3 Root Locus

Suppose we are given the following feedback system with an adjustable gain K and a (rational) plant system function $L_0(s)$[3] so that the open-loop transfer function is $L(s) = KL_0(s)$:



The closed-loop transfer function is:

$$Q(s) = \frac{KL_0(s)}{1 + KL_0(s)}.$$

The closed-loop poles, or the roots of $1 + KL_0(s)$, change as K is adjusted. The path that a pole takes as this gain changes is call the *locus* of that pole. The procedure for determining the shape of these loci is called the *root-locus method*, which was developed by Walter Evans in 1948. The textbook details the development of some root locus rules. The fundamentals of root locus are the following:

Basic Root Locus Rules:

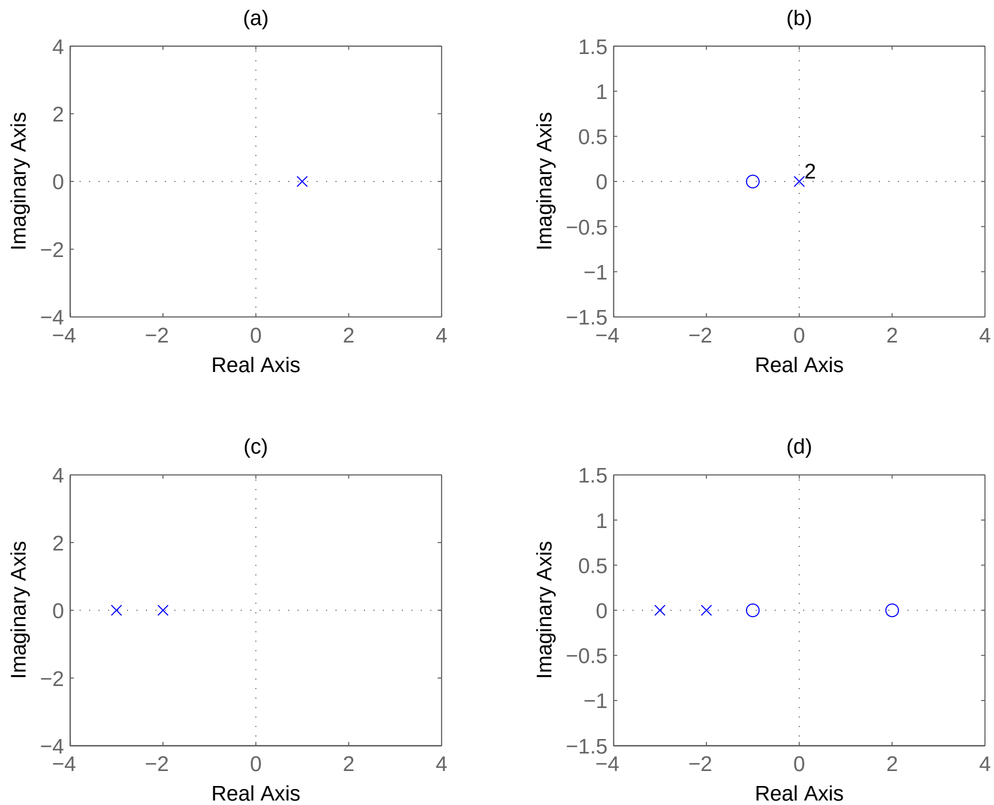
1. The number of branches, which are the paths of the closed-loop poles, is equal to the number of open-loop poles, P.
2. The branches start ($K = 0$) at the open-loop poles and end ($K = \pm\infty$) at the open-loop zeros. In addition to the Z explicit zeros, there also $P - Z$ implicit open-loop zeros at infinity. Thus, the number of open-loop poles is equal to the number of open-loop zeros and they are paired up, one pair for each branch. We'll generally assume that $P \geq Z$.
3. Parts of the real axis that lie to the *left* of an *odd* number of real open-loop poles and zeros are on the root locus for $K > 0$. Similarly, parts of the real axis that lie to the *left* of an *even* number of real open-loop poles and zeros are on the root locus for $K < 0$.
4. Branches of the root locus between two real poles must break off into the complex plane for $|K|$ large enough. Similarly, branches of the root locus between two real zeros must enter the real axis somewhere between the two zeros.

Root locus is where we need to keep track of poles at infinity. There are more rules regarding the angles and asymptotic behavior of the branches, but we won't get into them. Although you won't be responsible for knowing the all the root locus rules, being familiar with them will help you develop some intuition about how the behavior of systems evolves as the compensator gain is changed. Root locus is a very slick tool that system designers use all the time.

[3]When these system functions $L_0(s)$ are written in pole-zero form, we will follow the convention that the multiplicative factor is unity. If this is not the case, all analysis is the same, except that K is scaled by a constant factor.

Problem 11.3

Draw the root locus plots for the following open-loop pole-zero diagrams for both positive and negative gain K. Assume that the multiplicative factor in the rational Laplace transforms are all unity. Determine qualitatively (and quantitatively, if you'd like) when the closed-loop systems are stable.



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