

6.003: Signals and Systems — Spring 2004

TUTORIAL 10

Monday, April 26 and Tuesday, April 27, 2004

Announcements

- Problem set 9 is due this Friday.
- The final exam is **THREE WEEKS** away. The clock is ticking as you read this....

Today's Agenda

- The Laplace Transform
 - Motivation for the Laplace transforms
 - The relationship between Laplace and Fourier transforms
- Region of Convergence and its properties
- Poles and Zeros
 - Multiplicity in poles and zeros
 - Poles and zeros at infinity and zero
- Properties of the Region of Convergence
- “Pathological” Signals
- Partial-Fraction Expansion and the Inverse Laplace Transform
- The Laplace Transform and LTI Systems
 - The eigenfunction property
 - Causality
 - Stability
 - Systems described by linear constant-coefficient differential equations
- Geometric Evaluation of Rational Laplace Transforms and their Relation to Bode plots
- The Impulse and Step Responses
- The Initial- and Final-Value Theorems

1 The Laplace Transform

1.1 Motivation for the Laplace transforms

Notice the trend... we began by expressing periodic CT signals as the sum of exponentials e^{st} , where $s = j\omega_0 k$ takes on discrete values on the $j\omega$ axis. We then extended this to express aperiodic CT signals using a continuum of values $s = j\omega$. Now why restrict ourselves to using only the $j\omega$ axis? Why not utilize the entire s -plane? Can we represent even more signals if we do so? We saw that many signals, such as unstable signals, cannot be represented using the Fourier transform. However, we will find that *by using the entire s -plane, we can represent an even broader scope of signals as the linear combination of complex exponentials of the form e^{st} .*

As we have seen earlier with the general form of the eigenfunction property of LTI system, we define the Laplace transform as follows:

The Laplace Transform:

The Laplace transform of a CT signal is:

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st} dt$$

This is also written as:

$$\begin{aligned} X(s) &= \mathcal{L}\{x(t)\} \\ x(t) &\stackrel{\mathcal{L}}{\longleftrightarrow} X(s) \end{aligned}$$

These also suggest an inverse Laplace transform, but we'll see that knowledge about the region of convergence is an essential component in this operation:

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

1.2 The relationship between the Laplace and Fourier transforms

Let's look at how the Laplace and Fourier transforms are related to each other. We write the variable s as $s = \sigma + j\omega$, where σ and ω are the real and imaginary parts of s , respectively. If we set σ to zero, then we get the Fourier transform of $x(t)$:

$$\begin{aligned} X(s)|_{s=j\omega} &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt \\ &= \mathcal{F}\{x(t)\} \end{aligned}$$

Now if we break s into its real and imaginary parts and recombine terms, we get the Fourier transform of $x(t)e^{-\sigma t}$:

$$\begin{aligned} X(s)|_{s=\sigma+j\omega} &= \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t} dt \\ &= \int_{-\infty}^{+\infty} (x(t)e^{-\sigma t}) e^{-j\omega t} dt \\ &= \mathcal{F}\{x(t)e^{-\sigma t}\} \end{aligned}$$

To summarize:

The Fourier and Laplace Transforms:

The Laplace transform $X(s)$ of a CT signal $x(t)$ is the Fourier transform of $x(t)e^{-\sigma t}$, where σ is the real part of s , and the Fourier transform $X(j\omega)$ is the Laplace transform evaluated at $s = j\omega$.

2 Region of Convergence

Of course, the Laplace transform does not always converge for all values of s . We call range of s for which the transform converges the *region of convergence*, or the ROC, which we can depict as a shaded region in the s -plane. With the interpretation of the Laplace transform $X(s)$ as the Fourier transform of $x(t)e^{-\sigma t}$, we see that $X(s)$ converges when $x(t)e^{-\sigma t}$ is absolutely integrable. The classic cases to see this is in the calculation of the Laplace transform of $x_1(t) = e^{-at}u(t)$ (see Example 9.1 on page 656 of the textbook). Likewise, we can work out the Laplace transform of $x_2(t) = -e^{-at}u(-t)$ (see Example 9.2 on page 657 of the textbook). We have two signals, $x_1(t)$ and $x_2(t)$, that have the same Laplace transform, but converge for different values of s . Thus:

Specifying the Region of Convergence:

When writing the Laplace transform of a signal, both the algebraic expression and the region of convergence are required.

3 Poles and Zeros

Most of the Laplace transforms we'll see are ratios of polynomials in s :

$$X(s) = \frac{N(s)}{D(s)},$$

where $N(s)$ and $D(s)$ are the numerator and denominator polynomials, respectively. Rational Laplace transforms result from signals that are linear combinations of complex exponentials and from LTI systems described as linear constant-coefficient differential equations. Since the roots of $N(s)$ and $D(s)$, which are called the *zeros* and *poles*, respectively, specify the algebraic expression up to a scale factor, it will be analytically and pictorially convenient for us to view rational Laplace transforms in terms of the poles and zeros. We draw a pole as “ \times ” and a zero as “ \circ ” in a *pole-zero plot* of the Laplace transform.

Problem 10.1

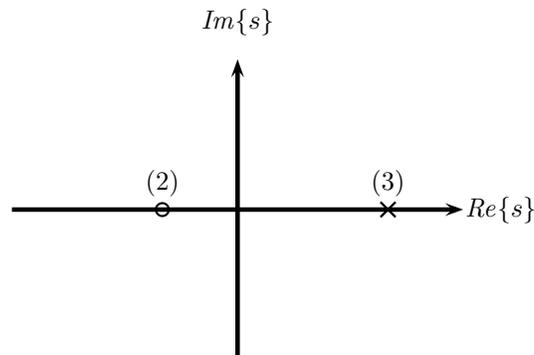
Draw the pole-zero plot of the following Laplace transform:

$$X(s) = \frac{(s+1)(s^2+9)}{(s-1)(s^2-2s+5)}$$

(Work space)

3.1 Multiplicity in poles and zeros

If a rational Laplace transform has a multiplicity in its poles and zeros, it is important to indicate the exact number. Sometimes, this is written as a “power” near the repeated pole and concentric circles for repeated zeros. For instance, the following diagram shows the pole-zero plot of $\frac{(s+1)^2}{(s-2)^3}$:



3.2 Poles and zeros at infinity and zero

If the order of the denominator polynomial is greater than that of the numerator polynomial by some integer k , then the Laplace transform will approach zero as s approaches infinity. Likewise, if the order of the numerator polynomial is greater than that of the denominator polynomial by some integer k , then the Laplace transform will approach infinity as s approaches infinity. These are considered to be k zeros and k poles, respectively, at infinity. For now, this concept is not terribly crucial, but they will become useful bookkeeping tools when we get to feedback and root locus.

4 Properties of the Region of Convergence

The textbook outlines eight basic properties of the region of convergence:

Properties of the Region of Convergence:

1. The ROC consists of strips parallel to the $j\omega$ -axis in the s -plane.
2. For rational Laplace transforms, the ROC does not contain any poles.
3. If the signal is of finite duration and is absolutely integrable, then the ROC is the entire s -plane.
4. If the signal is right-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC.
5. If the signal is left-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC.
6. If the signal is two-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s -plane that includes the line $Re\{s\} = \sigma_0$.
7. If the Laplace transform is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles are contained in the ROC.
8. If the Laplace transform is rational, then if the signal is right-sided, the ROC is the region of the s -plane to the right of the rightmost pole. If the signal is left-sided, the ROC is the region of the s -plane to the left of the leftmost pole.

Keep in mind that a signal either doesn't have a Laplace transform or falls into one of the four categories described in Properties 3 through 6. This chart would be a really handy one to copy onto a quiz reference sheet.

5 “Pathological” Signals

Notice that the ever-lasting pure complex exponential $e^{j\omega t}$, sine, and cosine do not appear in the Laplace transform charts. Why? They don't converge anywhere! But then why do they have Fourier transforms? Well, we relaxed the condition that the signals had to be stable in order to have FT's, and this created impulses in the frequency domain. Although it was kind of sketchy to allow impulses in the frequency domain, the math works and we can solve lots of problems with them. However, while the Fourier allows for impulses, Laplace does not. Where does the delta go in Laplace? We force *strict* inequality in defining the ROC, which makes impulses go away.

Problem 10.2

Several of the ROC properties say “at least...” How are we to know when to include more regions to the ROC? This can be tricky, and it comes up when we have *pole-zero cancellation*. The following example will illustrate this.

Find the Laplace transform and the associated ROC of the following signal, where $*$ indicates convolution:

$$x(t) = e^{-2t}u(t) * [\delta(t) - e^{-3t}u(t)].$$

(Work space)

6 Partial-Fraction Expansion and the Inverse Laplace Transform

How do we find the inverse Laplace transforms? Formally, we would use *contour integration*, which we will not cover in 6.003. Instead, we will rely on partial-fraction expansion and our table of properties, just like we did for the Fourier transform. We have to infer information about the components of the signal based on information about the ROC. Let's look at an example problem.

Problem 10.3

(a) The Laplace transform $X_a(s)$ and the associated ROC of a CT signal $x_a(t)$ is:

$$X_a(s) = \ln(1 + s), \quad \text{Re}\{s\} > -1.$$

Find $x_a(t)$ using properties.

(b) Suppose we are given the Laplace transform:

$$X_b(s) = \frac{2s + 1}{(s + 3)(s + 2)}$$

For each of the following ROC's, determine the corresponding signal $x_b(t)$

- (i) $\text{Re}\{s\} < -3$.
- (ii) $-3 < \text{Re}\{s\} < -2$.
- (iii) $\text{Re}\{s\} > -2$.

7 The Laplace Transform and LTI Systems

Just like we used the Fourier transform to analyze LTI systems, we can do the same with the Laplace transform. Recall that:

$$y(t) = h(t) * x(t),$$

where $x(t)$ is the input of an LTI system, $h(t)$ is the impulse response, and $y(t)$ is the output. Then, by the convolution property of the Laplace transform, we have:

$$Y(s) = X(s)H(s),$$

where the uppercase functions in s are the Laplace transforms of the corresponding lowercase functions in t . We call $H(s)$ the *transfer function*, or *system function*, of the LTI system. When the real part of s is zero, and the ROC of the transfer function includes the $j\omega$ -axis, then the transfer function reduces to the frequency response $H(j\omega)$ of the LTI system. We now have three ways of characterizing LTI systems:

Ways of Characterizing LTI Systems:

1. The impulse response $h(t)$.
2. The frequency response $H(j\omega)$, which is the Fourier transform of the impulse response $h(t)$ and the transfer function $H(s)$ evaluated on the $j\omega$ -axis.
3. The transfer function $H(s)$, which is the Laplace transform of the impulse response $h(t)$.

The utility and choice of each method depends on the system and the problem we want to solve. Although all LTI systems have impulse responses, only stable LTI systems (and special ones that have impulses in the frequency domain) have frequency responses. Likewise, not all LTI systems have transfer functions, though the set that do is broader than the set of systems that have frequency responses.

7.1 The eigenfunction property

The main motivation for our development of the Fourier and Laplace transforms in the first place is the eigenfunction property of LTI system:

The Eigenfunction Property of LTI Systems:

If the input $x(t)$ to an LTI system with transfer function $H(s)$ is a complex exponential of the form:

$$x(t) = e^{st},$$

where s is in the ROC of $H(s)$, then the output $y(t)$ is:

$$y(t) = H(s)e^{st}.$$

This is one of the most important concepts we learn in 6.003.

7.2 Causality

Some of the properties of transfer functions and their ROCs can be directly related to properties of the LTI system itself. For instance, we know that right-sided signals have ROCs that extend to infinity to the right. We also know that a causal system has a right-sided impulse response that is zero for negative time. Therefore:

The ROC associated with the transfer function for a causal system is a right-half plane.

As the textbook cautions, the converse of this statement is not necessarily true. It is possible that a right-sided signal starts in negative time; it would then have the right-half plane in its ROC yet be the impulse response of a non-causal system. However, this can't happen in *rational* transfer functions. So:

Causal Rational Systems and the ROC:

A system with a rational transfer function is causal if and only if the ROC is to the right of the rightmost pole.

Similarly:

Anti-Causal Rational Systems and the ROC:

A system with a rational transfer function is anti-causal if and only if the ROC is to the left of the leftmost pole.

7.3 Stability

Recall that a system is stable if and only if the impulse response is absolutely integrable. In that case, the frequency response, which is the transfer function evaluated on the $j\omega$ -axis, exists. Thus:

Stable Systems and the ROC:

An LTI system is stable if and only if the ROC of its transfer function $H(s)$ includes the entire $j\omega$ -axis.

A reminder on terminology that is recapped from tutorial 2: The terms “causal” and “stable” are properties of systems, not signals. You may sometimes hear people casually throw around terms like “causal signal” or “stable signal.” This usage can be misleading. These people are actually describing (LTI) *systems* with those properties and the “signals” refer to the impulse response signals of those systems. We are describing LTI *systems* using *signals*, namely the impulse response, so its important to keep this in mind, especially when those signals don't even correspond to the impulse response of an LTI system.

7.4 Systems described by linear constant-coefficient differential equations

A few weeks ago, we saw how the Fourier transform could be used to find the frequency response of systems described by differential equations. We can do the same with the Laplace transform; in fact, we can do more. A general linear constant-coefficient differential equations has the form:

$$\sum_{k=0}^N a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^M b_k \frac{d^k}{dt^k} x(t).$$

Recall that differentiation in time corresponds to multiplication by s in the Laplace transform domain. So, taking the Laplace transform of both sides results in:

$$\left(\sum_{k=0}^N a_k s^k \right) Y(s) = \left(\sum_{k=0}^M b_k s^k \right) X(s).$$

Thus, the transfer function $H(s)$ is:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\left\{ \sum_{k=0}^M b_k s^k \right\}}{\left\{ \sum_{k=0}^N a_k s^k \right\}}.$$

We just transformed a *differential* equation, which is hard to solve, into an *algebraic* equation, which is generally much easier to solve. We can find the poles and zeros of the transfer function by finding the roots of the polynomials in the denominator and numerator, respectively. Recall also that a differential equation alone does not specify the system. In the Laplace transform domain, this is translated to the fact that a Laplace transform alone does not specify a signal. Knowing the ROC does completely specify the system, and this may come in the form of knowing whether a system is causal or stable. The next problem shows how this works.

Problem 10.4 (Extended from the 6.003 Spring 2004 Problem Set 2)

In problem set 2, problem 7, we considered the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{5}{2} \frac{dy(t)}{dt} - \frac{3}{2}y(t) = x(t)$$

We showed that if $x(t) = 0$ for all t , then

$$y(t) = Ae^{-3t} + Be^{\frac{1}{2}t}$$

satisfies the above differential equation for any constant values of A and B . This information led us to the impulse response of the system described by the differential equation by matching boundary conditions. We found:

(a) If the system is causal, then

$$h(t) = \left(-\frac{2}{7}e^{-3t} + \frac{2}{7}e^{\frac{t}{2}} \right) u(t).$$

(b) If the system is BIBO stable, then

$$h(t) = \left(-\frac{2}{7} \right) e^{-3t}u(t) + \left(-\frac{2}{7} \right) e^{\frac{t}{2}}u(-t).$$

Let us derive the same results using the Laplace transform through the following steps:

1. Find the transfer function of the system.
2. Use partial-fraction expansion.
3. Find the poles of the transfer function.
4. Identify the region of convergence.
5. Take the inverse Laplace transform.

(Work space)

(Work space)

8 Geometric Evaluation of Rational Laplace Transforms and their Relation to Bode plots

Suppose we have a rational transfer function $H(s)$:

$$H(s) = M \frac{(s - z_1)(s - z_2) \cdots (s - z_Q)}{(s - p_1)(s - p_2) \cdots (s - p_P)},$$

where z_i and p_i are the zeros and poles, respectively. Each factor can be viewed as a complex number or vector in the complex plane that starts at z_i or p_i and ends at s .

We can then express each factor in polar (magnitude-phase form), where the magnitude is the length of the vector and the phase is the angle that the vector forms with the real axis. Multiplying complex numbers multiplies their magnitudes and adds their phases, so:

The Geometric Evaluation of Rational Laplace Transforms:

A rational Laplace transform $H(s)$ can be evaluated at some point s_0 in the ROC in the following manner. The magnitude of $H(s_0)$ is the magnitude of the scale factor M (see above equation) time the product of the lengths of the vectors from the zeros to s_0 divided by the product of the lengths of the vectors from the poles to s_0 . The phase of $H(s_0)$ is the phase of M plus the sum of the angles of the zero vectors minus the sum of the angles of the pole vectors.

9 The Impulse and Step Responses

For most of this term, we characterized an LTI system by looking at its *impulse* response. However, in problem set 9, we were asked to find the *step* responses of LTI systems. Why are we now interested in the step response and not the impulse response? The impulse response is the output of an LTI system when the input is an impulse. For physical CT systems, it may be difficult to measure the impulse response; we would have to put in a very high input for a very short period of time. Such an input might even damage the system. On the other hand, the step response, or the output when the input is the unit step $u(t)$, is usually very easy to obtain. We can then differentiate the step response to produce the impulse response. Also, for many systems that we design, inputs are usually very simple, such as a step. Thus, our interest in the step response of LTI systems is motivated by the ease by which it can be measured and by its relevance to the specifications of the system.

The impulse response of a CT LTI system with transfer function $H(s)$ is $h(t)$, or the inverse Laplace transform of $H(s)$. The step response $s(t)$ is then the input, the unit step $u(t)$, with the impulse response:

$$s(t) = h(t) * u(t).$$

Taking the Laplace transform of both sides produces:

$$\begin{aligned} S(s) &= H(s) \cdot \mathcal{L}\{u(t)\} \\ &= H(s) \cdot \frac{1}{s} \\ \implies S(s) &= \frac{H(s)}{s}. \end{aligned}$$

Taking the inverse Laplace transform of both sides produces:

$$s(t) = \mathcal{L}^{-1} \left\{ \frac{H(s)}{s} \right\}.$$

Thus, the step response $s(t)$ is the inverse Laplace transform of the transfer function $H(s)$ divided by s . Usually, the ROC is such that the system is causal, but this is not always the case.

10 The Initial- and Final-Value Theorems

The initial- and final-value theorems are useful when analyzing the initial and steady-state behavior of systems to many inputs (usually the unit step). The basic idea is to find the Laplace transform of the output by multiplying the Laplace transforms of the input and the impulse response. Then, use the final-value theorem to check whether the system has a steady-state output value for that input.

In the Laplace transform table, we have the *initial-* and *final-value theorems*:

Initial-Value Theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then:

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s).$$

Final-Value Theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \rightarrow \infty$, then:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s).$$

It's not immediately obvious why the above theorems are true. We saw examples in recitation where the steady-state value of the output ($\lim_{t \rightarrow \infty} y(t)$) of a system when the input is a step was determined without explicitly using the final-value theorem. These used the eigenfunction property; let's see how we can understand why the final-value theorem is true from this analysis.

According to the eigenfunction property of LTI systems, when the input $x(t)$ to an LTI system with transfer function $H(s)$ is:

$$x(t) = e^{s_0 t},$$

the output is:

$$y(t) = H(s_0)e^{s_0 t},$$

provided that s_0 is in the ROC of the system. However, what if the input were not "everlasting?" Suppose the input were:

$$x(t) = e^{s_0 t}u(t).$$

Then the output would be:

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{+\infty} h(\tau)e^{s_0(t-\tau)}u(t-\tau)d\tau \\
\Rightarrow y(t) &= e^{s_0t} \int_{-\infty}^t h(\tau)e^{-s_0\tau}d\tau
\end{aligned}$$

When $t \gg 0$, the integral on the right-hand side approaches $H(s_0)$, or the Laplace transform of $h(t)$, provided that s_0 is in the ROC. Thus, when we put a complex exponential into an LTI system beginning at time $t = 0$, the output approaches what have been the output had the input been an everlasting exponential starting from $t = -\infty$:

Steady-State Approximation:

When:

$$x(t) = e^{s_0t}u(t)$$

is the input to an LTI system with system function $H(s)$, the output for large times after transients have died away is:

$$y(t) \approx e^{s_0t}H(s_0).$$

We can think of a unit step input $u(t)$ as the input e^{s_0t} where s_0 approaches zero. So, $y(t)$ for large times goes to $H(0)$. We know that $H(s) = Y(s)/X(s)$, where $Y(s)$ and $X(s)$ are the Laplace transforms of the output and input, respectively. Since $x(t) = u(t)$, $X(s) = 1/s$. So, $H(s) = sY(s)$. So, $y(t)$ as $t \rightarrow \infty$ is $H(s)$ as $s \rightarrow zero$, or $sY(s)$ as $s \rightarrow 0$, which is the final-value theorem.

This is a non-rigorous approach (we have to be more careful in how limits are evaluated and we need to use the causality of the system; most proofs of this use the unilateral Laplace transform, which you have seen in 18.03 but we will not use in 6.003), but at least it gives us some idea behind the final-value theorem.

Problem 10.5

Consider eight distinct causal real CT LTI systems with rational transfer functions $H_i(s)$. Figure 1 shows their pole-zero diagrams (labeled 1 through 8). Figure 2 shows their step responses (labeled A through H). Figure 3 shows their Bode plots (labeled I through VIII). Match them (it's a one-to-one-to-one correspondence).

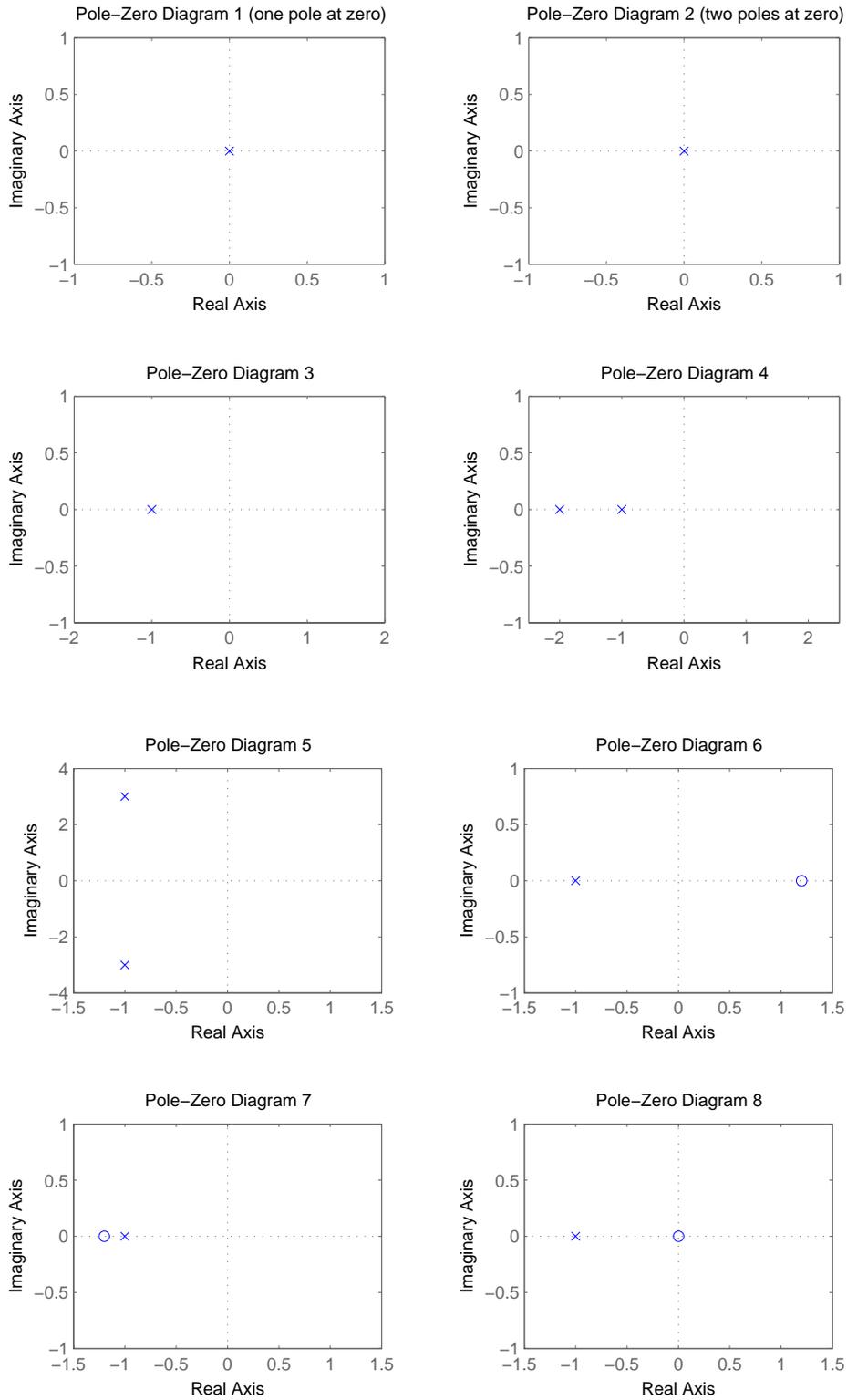


Figure 1: Matching Problem: Pole-Zero Diagrams

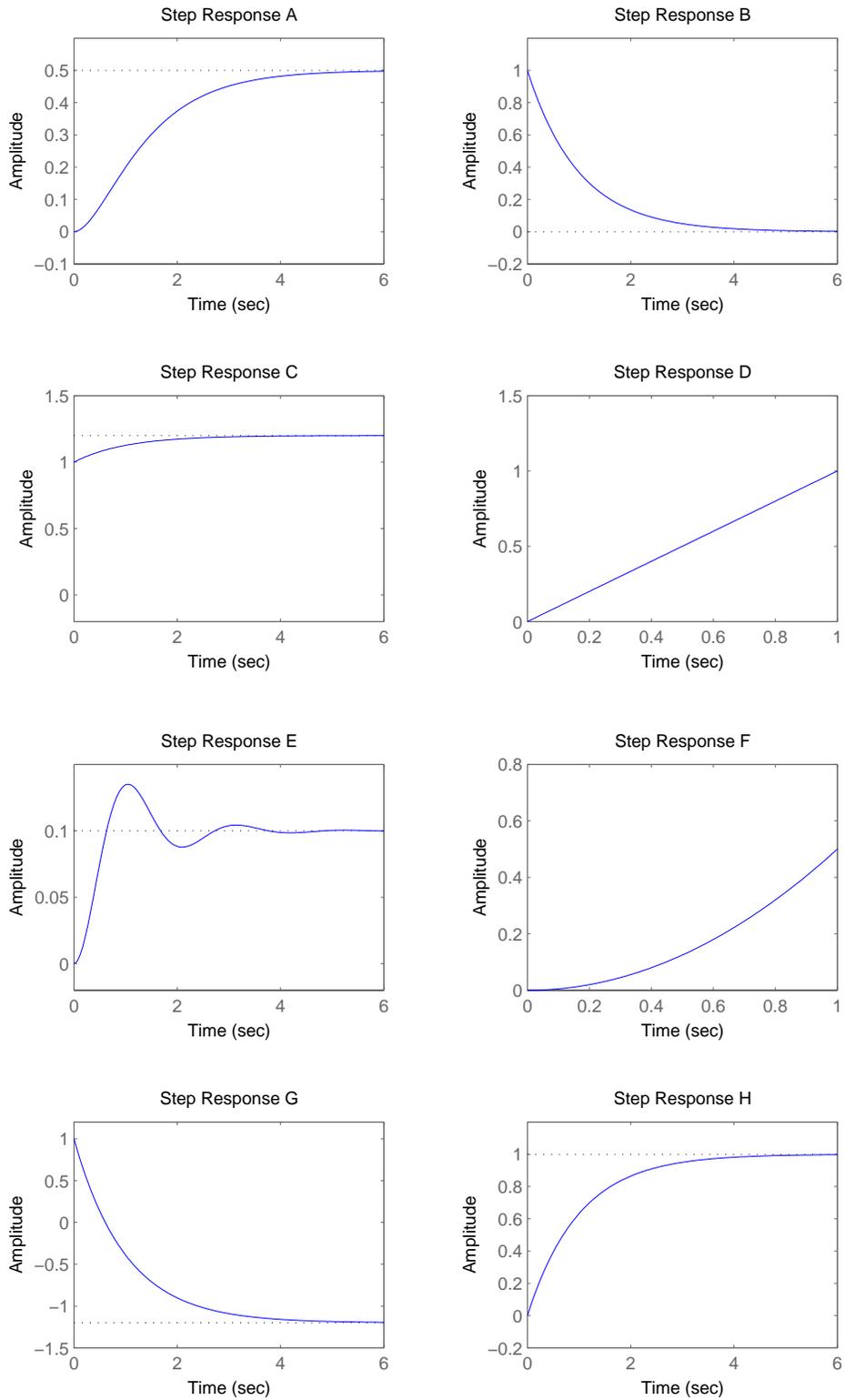


Figure 2: Matching Problem: Step Responses

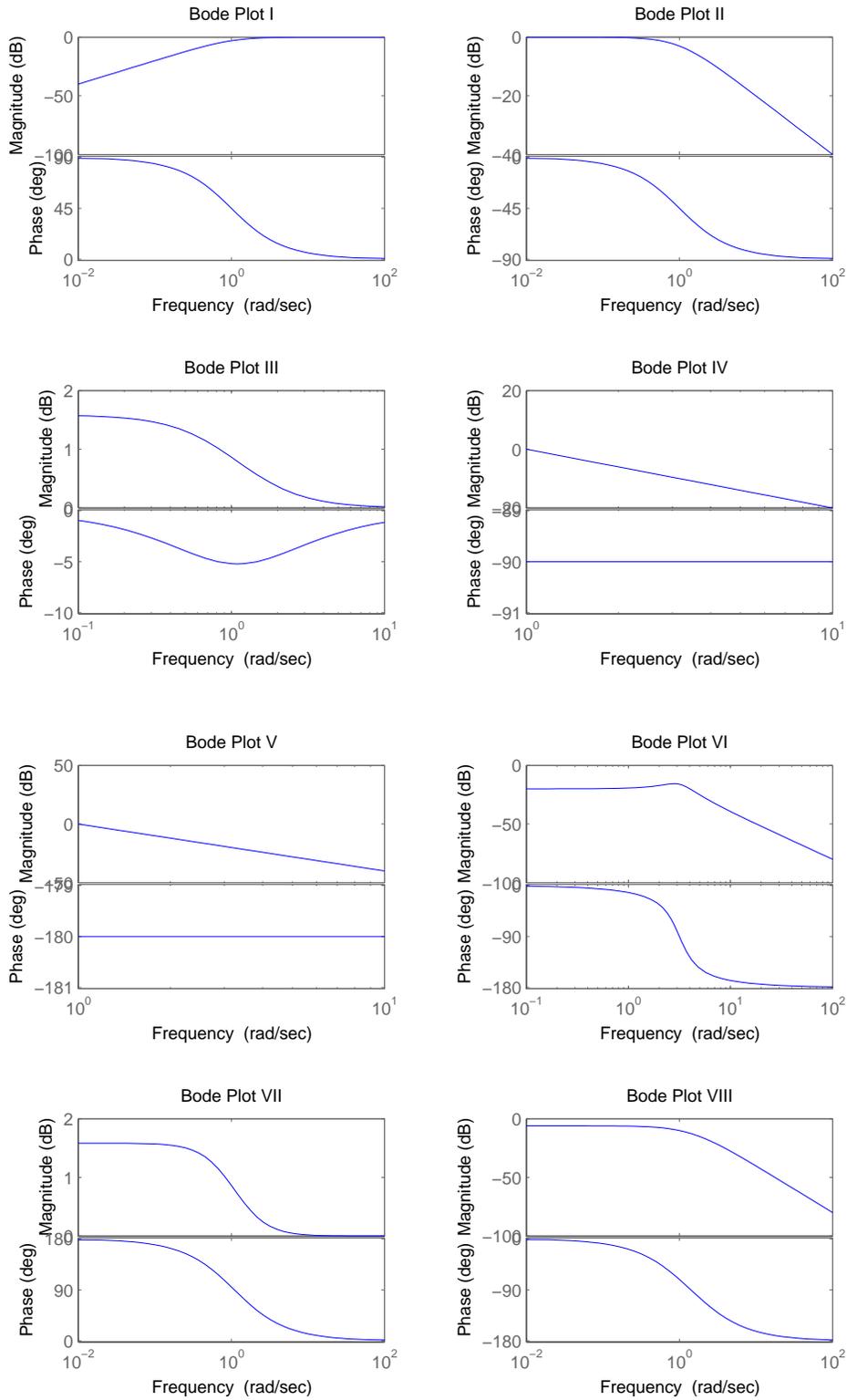


Figure 3: Matching Problem: Bode Plots