

6.003: Signals and Systems — Spring 2004

TUTORIAL 1

Monday and Tuesday, February 9 and 10, 2004

Announcements

- Problem set 1 is due this Wednesday. Unless you made previous arrangements with your TA, you must submit it in your assigned recitation section.
- Please check the 6.003 website (<http://web.mit.edu/6.003/www>) and make sure you have been assigned a recitation and a tutorial.

Today's Agenda

- Complex numbers
 - i vs. j
 - Rectangular form
 - Euler's relation and polar form
 - Continuous phase vs. principal value of phase
 - Complex arithmetic
- Signals
 - Linear transformations on the independent variable of signals in CT
 - Transformations of signals in DT
 - Periodic signals
 - DT unit impulse and unit step functions
 - CT unit impulse and unit step functions
- Systems

1 Complex Numbers

1.1 i vs. j

In some fields (and in high school), the imaginary unit is written as i . However, in electrical engineering, the symbol i is also used for electric current, so to prevent confusion, we write j to represent $\sqrt{-1}$. We will use this notation throughout.

1.2 Rectangular form

We can write any complex number z in two forms. The *rectangular* (or *Cartesian*) form has z as a sum of its real and imaginary parts:

$$z = x + jy. \quad (1.1)$$

We can extract the *real* part x and *imaginary* part y of z using the operations

$$x = \operatorname{Re}\{z\} \quad (1.2)$$

$$y = \operatorname{Im}\{z\}. \quad (1.3)$$

1.3 Euler's relation and polar form

Recall Euler's relation:

$$e^{j\theta} = \cos \theta + j \sin \theta. \quad (1.4)$$

So:

$$\cos \theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad (1.5)$$

$$\sin \theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \quad (1.6)$$

Writing sines and cosines in this form will be very helpful for us later on, especially when we deal with Fourier series and Fourier transforms.

Euler's relation allows us to express complex numbers in *polar* form:

$$z = re^{j\theta}. \quad (1.7)$$

We can extract the *magnitude* (or absolute value or modulus) r and *angle* (or phase or argument) θ of z using the operations

$$r = |z| \quad (1.8)$$

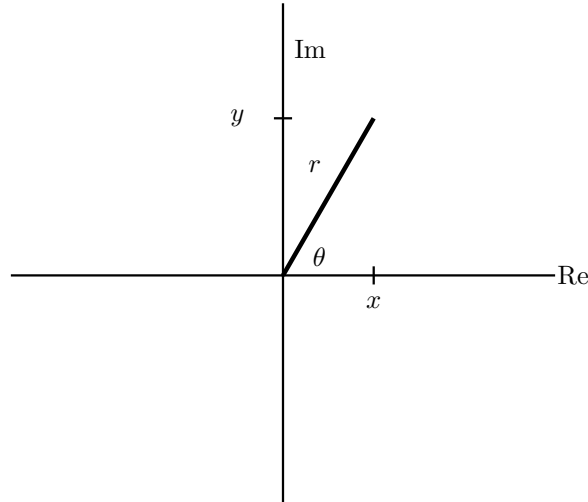
$$\theta = \angle z = \arg\{z\}. \quad (1.9)$$

The components of the two forms are related by

$$r = \sqrt{x^2 + y^2} \quad (1.10)$$

$$\tan \theta = \frac{y}{x}. \quad (1.11)$$

The following diagram shows the complex number $z = a + jb = re^{j\theta}$ in the complex plane:



The *complex conjugate* of a complex number z is the complex number z^* (also written \bar{z}) whose imaginary part is the opposite of the imaginary part of z . So, if

$$z = x + jy = e^{j\theta} \quad (1.12)$$

then

$$z^* = x - jy = e^{-j\theta} \quad (1.13)$$

An easy way to take the conjugate of a complicated arithmetic expression containing complex numbers is to replace all the j 's with $-j$'s. The conjugate is an alternative method for determining the magnitude of a complex number:

$$|z|^2 = z^* z \quad (1.14)$$

We can use the complex conjugate to extract the real and imaginary parts of a complex number z :

$$\operatorname{Re}\{z\} = \frac{1}{2}(z + z^*) \quad (1.15)$$

$$\operatorname{Im}\{z\} = \frac{1}{2j}(z - z^*) \quad (1.16)$$

1.4 Continuous phase vs. principal value of phase

Note that we can add any multiple of 2π to the phase of a complex number z without changing z 's value. So what happens when we compute $\arg\{e^{j\theta}\}$? Is it the same as $\arg\{e^{j(\theta+2\pi)}\}$? This question isn't a big deal right now, but later, when we take the phase of a function (such as the Fourier transform), we will need to distinguish between the *continuous phase* (denoted by $\arg\{\}$), which ensures that the phase is a continuous function, and the *principal value of the phase* (denoted by $\operatorname{ARG}\{\}$), where the returned phase θ satisfies $-\pi < \theta \leq \pi$.

1.5 Complex arithmetic

Suppose we have the complex numbers:

$$z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}, \quad (1.17)$$

$$z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}, \quad (1.18)$$

where x_i , y_i , r_i and θ_i are real numbers. Then, we define the sum of z_1 and z_2 to be the sum of the real and imaginary parts independently:

$$\operatorname{Re}\{z_1 + z_2\} = \operatorname{Re}\{z_1\} + \operatorname{Re}\{z_2\}, \quad (1.19)$$

$$\operatorname{Im}\{z_1 + z_2\} = \operatorname{Im}\{z_1\} + \operatorname{Im}\{z_2\}. \quad (1.20)$$

This can easily be verified in rectangular form:

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2) \quad (1.21)$$

$$= (x_1 + x_2) + j(y_1 + y_2). \quad (1.22)$$

This motivates us to picture complex numbers as vectors: individual components of vectors add independent, just like the real and imaginary components of complex numbers. However, multiplication is a bit harder to interpret in rectangular form:

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) \quad (1.23)$$

$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + y_1 x_2) \quad (1.24)$$

A recurrent problem-solving technique is to switch forms or representations when one becomes difficult to use. So, let's try this in polar form:

$$z_1 z_2 = (r_1 e^{j\theta_1}) (r_2 e^{j\theta_2}) \quad (1.25)$$

$$= (r_1 r_2) e^{j(\theta_1 + \theta_2)}. \quad (1.26)$$

Thus, the magnitude of the product of two complex numbers is the *product* of the magnitude of the factors, and the angle is the *sum* of the angles of the factors:

$$|z_1 z_2| = |z_1| |z_2|, \quad (1.27)$$

$$\angle\{z_1 z_2\} = \angle\{z_1\} + \angle\{z_2\}. \quad (1.28)$$

To summarize:

Addition of Multiplication of Complex Numbers: Let

$$z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}, \quad (1.29)$$

$$z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}, \quad (1.30)$$

be two complex numbers. Then, their *sum* is:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2), \quad (1.31)$$

and their *product* is:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + y_1 x_2) = r_1 r_2 e^{j(\theta_1 + \theta_2)}. \quad (1.32)$$

When we divide complex numbers in rectangular form, we multiply the top and bottom by the conjugate of the denominator. Likewise, exponentiation is usually done most easily when polar notation is used.

In 6.003, we will be dealing with complex numbers all the time. So, it will be important for you to be comfortable with switching between rectangular and polar forms as appropriate for a given concept or problem.

Problem 1.1

Try the following exercises to practice working with complex numbers.

- (a) Compute $\frac{1+j}{\sqrt{3}+j}$ using both rectangular arithmetic and by converting the problem first into polar form. Which method was less painful?
- (b) Compute the magnitude and angle of $e^j + e^{3j}$.
- (c) Simplify $(\sqrt{3} - j)^8$.
- (d) Compute $\int_0^\infty e^{-2t} \cos(\pi t) dt$.
- (e) Write the real part of:

$$\frac{1 - z^n}{1 - z} \tag{1.33}$$

in terms of the magnitude and phase of z , where n is a positive integer.

- (f) What did you learn from this problem?

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2 Signals

2.1 Linear transformations on the independent variable of signals in CT

There are many ways of transforming a CT signal into another. For instance, we can scale it, shift it in time, differentiate it, or perform a combination of these actions. Later in this chapter, we'll introduce the idea of transforming a signal as a *system*. To familiarize you with manipulating signals, we'll examine a particular type of transformation in this subsection: transformation on the independent variable of signals. More formally, let us for now restrict ourselves to transformations of the form:

$$x(t) \longrightarrow y(t) = x(f(t)), \quad (1.34)$$

where $x(t)$ is the starting signal given to us, $y(t)$ is the signal we end up with after the transformation, and $f(t)$ is a function of t . The arrow “ \longrightarrow ” denotes the action and direction of transformation. The function $f(t)$ can be any well-defined function, of course, but for the study of 6.003, we'll look at the class of *linear* functions $f(t) = at + b$, where a and b are arbitrary real constants. The resulting transformation of $x(t)$ into $y(t)$ is hence called “linear transformations on the independent variable.”

All such transformations can be decomposed into just three fundamental types of signal transformations on the independent variable: time shift, time scaling, and time reversal¹. They involve a change of the variable t into something else:

- Time shift: $t \longrightarrow t - t_0, \quad t_0 \in \mathbb{R}$
- Time scaling: $t \longrightarrow at, \quad a \in \mathbb{R}^+$
- Time reversal (or flip): $t \longrightarrow -t$

When applied to the signal $x(t)$, we obtain:

- Time shift: $x(t) \xrightarrow{t \rightarrow t - t_0} x_{\text{shift}}(t) = x(t - t_0)$

When the shift constant t_0 is positive, the effect is to move the signal $x(t)$ to the right by t_0 . In other words, each point on the signal $x(t)$ now falls t_0 later in time, so we call this transformation a *time delay*. Likewise, when it's negative, we call it a *time advance*.

- Time scaling: $x(t) \xrightarrow{t \rightarrow at} x_{\text{scale}}(t) = x(at)$

When the scaling factor a is greater than 1, the effect is to “squeeze” the signal toward $t = 0$: an arbitrary point on $x(\cdot)$ located at, say, $t = t_1$, namely $x(t_1)$, is now moved to the point $t'_1 = t_1/a$ on the resulting signal $x_{\text{scale}}(\cdot)$. Quick check: $x_{\text{scale}}(t'_1) = x_{\text{scale}}(t_1/a) = x(at_1/a) = x(t_1)$. Since t'_1 is closer to the vertical axis than t_1 is, this is called *time compression*. When the factor a is between 0 and 1, we call it *time expansion*. At first, the correspondence of a large factor to compression and a small factor to expansion seems counterintuitive, but the above example explains the nomenclature.

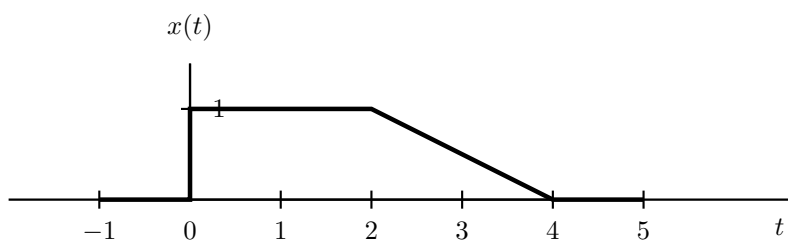
- Time reversal: $x(t) \xrightarrow{t \rightarrow -t} x_{\text{reverse}}(t) = x(-t)$

When more than one transformation is applied to a signal, one must be careful about the order in which it is done. The following example illustrates this.

¹Time reversal is really a special case of time scaling, but we'd like to think of it as conceptually different. It is sometimes easier to analyze a problem that involves a time scaling by a negative factor by breaking down that time scaling into a time scaling by the *absolute value* of the factor followed by a time reversal (or *vice versa*).

Example 1.2

We are given the signal $x(t)$:



Let us transform the given $x(t)$ to $x(-2t+6)$. We need to use all three types of transformations (a shift, a scale and a flip), but what in what order shall we do them? How do we do it? The following guide explains:

What to do when you need to do multiple transformations:

- We can do the transformations in *any* order. However, the amount and direction of the shift depends on whether it is performed before or after the reversal and the scale.
- We can think of cascaded transformations as repeated substitutions of the independent variable t .

To demonstrate these principles, let's do the transformation in three different orders.

• $x(t)$ $\xrightarrow{\text{Advance by 6}}$ $x(t+6)$ $\xrightarrow{\text{Reverse}}$ $x(-t+6)$ $\xrightarrow{\text{Compress by 2}}$ $x(-2t+6)$.

• $x(t)$ $\xrightarrow{\text{Compress by 2}}$ $x(2t)$ $\xrightarrow{\text{Advance by 3}}$ $x(2(t+3))$ $\xrightarrow{\text{Reverse}}$ $x(2(-t+3))$.

• $x(t)$ $\xrightarrow{\text{Reverse}}$ $x(-t)$ $\xrightarrow{\text{Delay by 6}}$ $x(-(t-6))$ $\xrightarrow{\text{Compress by 2}}$ $x(-(2t-6))$.

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We observe the following:

- We need to use the same operations: a time reversal, a time shift, and a time scaling.
- The operations are generally not commutative; switching the order may change the result. However, scaling and reversal are commutative operations.
- The time scale is always a compression by 2 (it's never an expansion).
- The time shift, on the other hand, depends on its order relative to the other operations:
 - If it is after compression, then it's a shift by 3, otherwise it's by 6.
 - If it is after time reversal, then it's a delay, otherwise it's an advance.

Because of the subtlety with the time shift, you may want to adopt a consistent order that you use whenever you encounter these problems. However, you should be able to do it in any order. Of course, you can always double-check your solution by plugging in values of t .

Let us show that the operations are generally not commutative; switching the order may change the result. If we do the first series of transformations (advance, reverse, compress) in the opposite order, we get:

$$x(t) \xrightarrow{\text{Compress by 2}} x(2t) \xrightarrow{\text{Reverse}} x(2(-t)) \xrightarrow{\text{Advance by 6}} x(2(-(t+6))) = x(-2t-12). \quad (1.35)$$

The final result from this series of transformations is not the same as before.

A common mistake when considering independent variable transformations is to misinterpret the English description, leading to incorrect final results. For instance, the first series of transformations in this example has an advance followed by a time reversal and a compression. When told to perform this sequence in English words on $x(t)$ and to say what is the dot for $x(\cdot)$ at each stage, some students mistakenly implement those words as (quotes indicate what they think the transformation is):

$$x(t) \xrightarrow{\text{"Advance by 6"}} x(t+6) \xrightarrow{\text{"Reverse"}} x(-(t+6)) \xrightarrow{\text{"Compress by 2"}} x(2(-(t+6))), \quad (1.36)$$

when it is actually:

$$x(t) \xrightarrow{\text{Advance by 6}} x(t+6) \xrightarrow{\text{Reverse}} x(-t+6) \xrightarrow{\text{Compress by 2}} x(-(2t)+6). \quad (1.37)$$

The incorrect version applies transformations to *the entire argument* of the function $x(\cdot)$, which results in the signal that would come from reversing the order of the transformations as shown earlier, whereas the correct version *replaces the independent variable* t by some function of t . In fact, the incorrect solution has an interesting interpretation. When a minus sign is

To clarify the “replace t ” idea, let’s define the following auxiliary functions:

$$g(u) = u + 6, \tag{1.38}$$

$$h(u) = -u, \tag{1.39}$$

$$k(u) = 2u. \tag{1.40}$$

Note that $g(\cdot)$ represents a time advance, $h(\cdot)$ represents a time reversal, and $k(\cdot)$ represents a time compression. We can emphasize the “replace t ” concept by rewriting Eq. 1.37 as:

$$x(t) \xrightarrow{\text{Advance by 6}} x(\underbrace{g(t)}_{\text{was } t}) = x(t + 6) \tag{1.41}$$

$$x(t + 6) \xrightarrow{\text{Reverse}} x(\underbrace{h(t)}_{\text{was } t} + 6) = x(-t + 6) \tag{1.42}$$

$$x(-t + 6) \xrightarrow{\text{Compress by 2}} x(-\underbrace{k(t)}_{\text{was } t} + 6) = x(-2t + 6). \tag{1.43}$$

Note that at each step, we replace t with the appropriate auxiliary function of t . We can also express the final signal $x(-2t + 6)$ as:

$$x(-2t + 6) = x(-k(t) + 6) = x(h(k(t)) + 6) = x(g(h(k(t)))). \tag{1.44}$$

Of course, we can unravel this nested expression in a different order to arrive at the same result:

$$x(g(h(k(t)))) = x(g(h(2t))) = x(g(-2t)) = x(-2t + 6). \tag{1.45}$$

Here, we begin *inside* and act on t and move *outwards*, whereas before, the nest was simplified in the opposite direction. The operations are applied *in reverse order* to t : we apply $k(\cdot)$ to t to get $k(t)$, then apply $h(\cdot)$ to get $h(k(t))$, then apply $g(\cdot)$ to get $g(h(k(t)))$, then finally feed this as the argument to $x(\cdot)$ to get $x(g(h(k(t))))$.

This exercise suggest the following theory as to why some students get $x(2(-(t + 6)))$ when asked to do an advance, then a reversal, then a compression. They assemble the functions $g(\cdot)$, $h(\cdot)$ and $k(\cdot)$ to t in the order in which they see and by putting the first function on the inside and the last function on the outside. They get:

$$x(k(h(g(t)))), \tag{1.46}$$

which simplifies to:

$$x(k(h(g(t)))) = x(k(h(t + 6))) = x(k(-(t + 6))) = x(2(-(t + 6))). \tag{1.47}$$

Thus, some students think of the expression $x(g(h(k(t))))$ as a nest, where $k(\cdot)$ (which represents time compression) is applied *first* on the variable t , even though we are applying the time compression operation *last* to the signal. We read the order of transformations as the *order in which replacements of t are made*, as opposed to *the order in which function are successively applied to t* , which yields the classic incorrect result. The result from applying successive transformations on the original signal (which is interpreted as replacements of t) is the same as the result of applying the corresponding transformations in reverse order to the argument of $x(\cdot)$ instead.

It is interesting to observe that by applying the steps *in reverse order* yields the answer from the incorrect interpretation:

$$x(t) \longrightarrow x(k(t)) = x(2t) \longrightarrow x(k(h(t)) = x(k(2t)) = x(-2t) \quad (1.48)$$

$$\longrightarrow x(k(h(g(t)))) = x(k(h(t + 6))) = x(k(-(t + 6))) = x(2(-(t + 6))). \quad (1.49)$$

It is now easy to see the source of the incorrect interpretation: the auxiliary functions are being applied to t *in the opposite order* in which we are actually using them to transform the signal. We want $x(g(h(k(t)))) = x(-2t + 6)$, which we confirmed is the result of a time advance (corresponds to $g(\cdot)$), then a time reversal ($h(\cdot)$), then a time compression ($k(\cdot)$). Thus, some students think of the expression $x(g(h(k(t))))$ as a nest, where $k(\cdot)$ (time compression) is applied *first*, even though we are applying the time compression operation *last*. We read the order of transformations as the *order in which replacements of t are made*, as opposed to *the order in which function are successively applied to t* , which yields the classic incorrect result. The result from applying successive transformations on the original signal (which is interpreted as replacements of t) is the same as the result of applying the corresponding transformations in reverse order to the argument of $x(\cdot)$ instead.

We can also clarify transformations by defining a new signal at each step by its relationship to the previous signal. For the same series just described, let's define:

$$y(t) = x(t + 6), \quad (1.50)$$

$$z(t) = y(-t), \quad (1.51)$$

$$w(t) = z(2t). \quad (1.52)$$

Note that $x(\cdot) \longrightarrow y(\cdot)$ represents a time advance, $y(\cdot) \longrightarrow z(\cdot)$ represents a time reversal, and $z(\cdot) \longrightarrow w(\cdot)$ represents a time compression. We can write the transformation process as:

$$x(t) \xrightarrow{\text{Advance by 6}} y(t) = x(t + 6) \quad (1.53)$$

$$y(t) \xrightarrow{\text{Reverse}} z(t) = y(-t) = x(-t + 6) \quad (1.54)$$

$$z(t) \xrightarrow{\text{Compress by 2}} w(t) = z(2t) = y(-2t) = x(-2t + 6). \quad (1.55)$$

So, the final result is $w(t) = z(2t) = y(-2t) = x(-2t + 6)$, as expected. With this picture, each step is then completely modularized from the other steps and there is no confusion.

2.2 Transformation of signals in DT

Transformations in discrete time are analogous to those in continuous time. However, there are a few subtle points to consider. For instance, can we time shift $x[n]$ by a non-integer delay, say to $x[n - \frac{1}{2}]$? If we compress the signal $x[n]$ to $x[2n]$, do we lose half the information stored? Finally, if we expand $x[n]$ to $x[\frac{1}{2}n]$, how do we “fill in the blanks?” These are some interesting questions to think about, and we will examine them further when we study sampling. It turns out that one useful method of executing a non-integer delay is by *interpolating* the DT samples (“connecting the dots”) into a CT signal, shifting the CT signal, then resampling to get the final shifted DT signal. A similar trick can be used for DT expansion. We will discuss the actual methods of interpolation later.

2.3 Periodic signals

When signals are invariant under certain transformations, they exhibit *symmetry properties*. If a signal is left unchanged by a time shift T_0 , then it is *periodic*. In other words, periodic signals repeat themselves in time and have the property that there exists a T in CT (or N in DT) such that

$$x(t) = x(t + T) \tag{1.56}$$

$$x[n] = x[n + N] \tag{1.57}$$

for all t (or n). The constant T (or N) is called the *period*. Because all multiples of T (or N) also satisfy the periodicity condition above, we will usually refer to the *smallest* positive period T_0 (or N_0) as the *fundamental period*. What if we have a constant CT signal? Is it periodic? It would be considered periodic, but the fundamental period is undefined. In DT, we have no problem; constant signals have a fundamental period of 1.

One must be careful when determining whether a DT signal is periodic. Complex exponentials certainly *look* periodic, but are they? For instance, it is tempting to say that the signal:

$$x[n] = \cos n \tag{1.58}$$

is periodic. In fact, it is not; let’s see why it isn’t. Suppose $x[n]$ were a periodic signal. Then, there exists some integer N such that:

$$\cos n = \cos(n + N) \tag{1.59}$$

The sine function repeats itself every 2π , and only for interval that are multiples of 2π , so N must be a multiple of 2π . Since π is irrational, this cannot be the case, so $\cos n$ is aperiodic.

In general, a DT exponential of the form $x[n] = Ae^{\omega_0 n + \phi}$ is periodic if and only if there exists some integer N_0 such that the exponent shifts by an integer multiple of 2π after N_0 . Why? A periodic signal cycles back to the same value after one period, so the angle of the exponential must be offset by an integer multiple of 2π . Thus:

Periodicity of Complex Exponentials:

- A **continuous-time** exponential of the form:

$$x(t) = Ae^{\omega_0 t + \phi} \quad (1.60)$$

is *always* periodic with fundamental period:

$$N_0 = \frac{2\pi}{\omega_0}. \quad (1.61)$$

- In DT, it's more complicated. A **discrete-time** exponential of the form:

$$x[n] = Ae^{\omega_0 n + \phi} \quad (1.62)$$

is periodic if and only if the frequency ω_0 is a rational multiple of π of the form:

$$\omega_0 = 2\pi \frac{m}{N_0}, \quad (1.63)$$

where m and N_0 are integers that share no common factors. The fundamental period is:

$$N_0 = 2\pi \frac{m}{\omega_0}. \quad (1.64)$$

We should also consider whether the sum, product, or other combination of two periodic signals is periodic:

Combining Periodic Signals:

- Adding, multiplying, or combining periodic **DT** signals **always** creates a periodic signal. The period is the lowest common multiple (LCM) of the periods of the individual signals.
- However, the result of combining periodic **CT** signals **may not be** periodic. If any of the ratios of the fundamental periods of the individual signals is irrational, then the signal may not be periodic. If those ratios are all rational, then the period is the lowest common multiple (LCM) of the periods of the individual signals.
- In both CT and DT, it is possible that the *fundamental* period of the overall signal is smaller than the LCM of the fundamental periods of the individual signals.

Problem 1.3

Determine whether the following signals are periodic, and determine the fundamental period if they are.

(a) $x_a(t) = \cos\left(\frac{3\pi}{2}t\right)$.

(b) $x_b[n] = \cos\left(\frac{3\pi}{2}n\right)$.

(c) $x_c(t) = \cos\left(\frac{3\pi}{2}t\right) + 3\sin\left(\frac{\pi}{3}t\right)$.

(d) $x_d[n] = \cos\left(\frac{3\pi}{2}n\right) + 3\sin\left(\frac{\pi}{3}n\right)$.

(e) $x_e(t) = \cos\left(\frac{3\pi}{2}t\right) + 3\sin t$.

(f) $x_f[n] = \cos(3n)$.

(g) $x_g(t) = r(t) \cdot \sin(\pi t)$, where

$$r(t) = \begin{cases} 1, & \text{if the largest integer smaller than } t \text{ is even,} \\ -1, & \text{otherwise.} \end{cases}$$

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2.4 The DT unit impulse and unit step functions

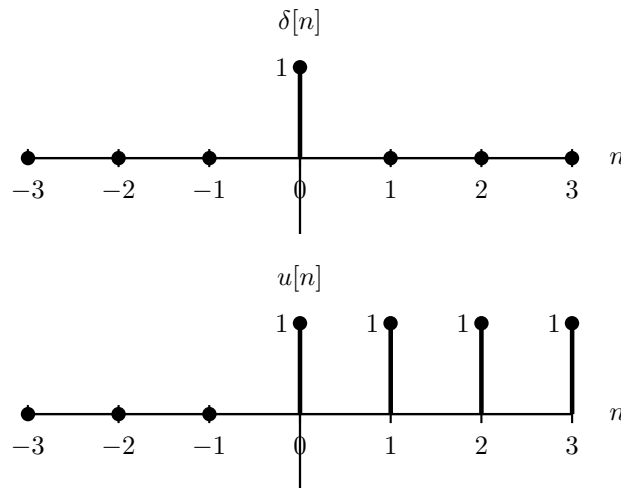
The unit impulse and unit step functions will form the basis (literally) of how we describe signals in the time domain. In DT, the *unit impulse* (also called the *unit sample*) is defined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \quad (1.65)$$

The *unit step* is defined as

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \quad (1.66)$$

They look like this:



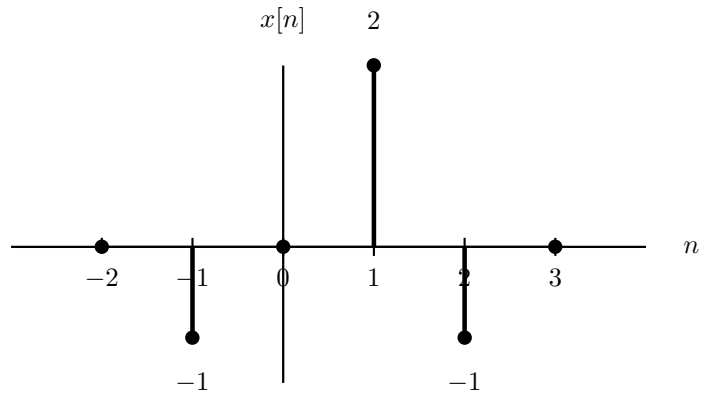
The two signals are related: the unit impulse is the *first difference* of the unit step, and the unit step is the *running sum* of the unit impulse:

$$\delta[n] = u[n] - u[n-1] \quad (1.67)$$

$$u[n] = \sum_{m=-\infty}^n \delta[m] \quad (1.68)$$

Problem 1.4

Express the following signal $x[n]$ solely in terms of shifted scaled superpositions of (a) the unit impulse, and (b) the unit step.



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2.5 The CT unit impulse and unit step functions

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3 Systems

In 6.003, we are concerned with four system properties: stability, causality, linearity and time-invariance.

In general, the output of a cascaded series of systems where the output of one is the input to the next one is different if the order of the systems is changed. However, there are cases in which switching the order does not change the final output; such systems are *commutative*. It turns out that if systems possess linearity or time-invariance, they are commutative with certain signal operations. In particular, a system H is time-invariant if the output resulting from a shift the input in time by t_0 followed by a run through H is the same as the output resulting from switching the order of the two operations (first put it through H then shift by time). Similarly, a system H is linear if the output resulting from the putting the superimposition of individual signals through H is the same as the output resulting from a run of each individual signal through the system and superimposing the individual outputs together. Here is a diagram to illustrate this concept for time-invariance:

$$\begin{array}{ccc}
 x(t) & \xrightarrow{H} & y(t) = H\{x(t)\} \\
 \text{time delay by } t_0 \downarrow & & \text{time delay by } t_0 \downarrow \\
 x(t - t_0) & \xrightarrow{H} & \underbrace{y(t - t_0) = H\{x(t - t_0)\}} \\
 & & \text{Equality holds iff time-invariant}
 \end{array}$$

Problem 1.5

Which of the properties of stability, causality, linearity, and time-invariance do the following systems have?

(a) $y(t) = \int_{-\infty}^t x(\tau) \, d\tau.$

(b) $y(t) = \int_0^t x(\tau) \, d\tau.$

(c) $y(t) = x(t^2).$

(d) $y[n] = (x[n])^2.$

(e) $y[n] = \text{median}(x[n-2], x[n-1], x[n]).$

(Workspace)

Problem 1.6

Indicate which of the following statements are true and which ones are false.

- (a) **TRUE** **FALSE** The overall system resulting from the series cascade of two linear systems is also linear.
- (b) **TRUE** **FALSE** The overall system resulting from the series cascade of two stable systems is also stable.
- (c) **TRUE** **FALSE** The overall system resulting from the series cascade of two time-invariant systems is also time-invariant.
- (d) **TRUE** **FALSE** The overall system resulting from the series cascade of two time-varying (non-TI) systems must not be time-invariant.

(Workspace)