Solutions to Problems in Chapter 6

Problems with Solutions

Problem 6.1

Compute the Laplace transforms of the following three signals (find the numerical values of ω_0 and θ in (c) first). Specify their regions of convergence. Find their Fourier transforms if they exist.

(a) $x_1(t) = 10e^{-(t-2)}u(t-2) + 10e^{0.5(t-2)}u(-t+2)$ as shown in Figure 6.1

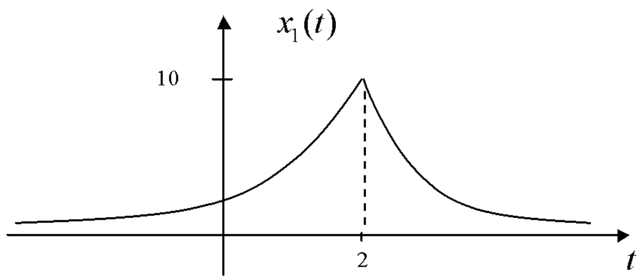


Figure 6.1: Double sided signal of Problem 6.1(a).

Answer:

Using the table and the time-shifting property, we get

$$\begin{aligned} x_1(t) &= 10e^{-(t-2)}u(t-2) + 10e^{0.5(t-2)}u(-t+2) \\ &= 10e^{-(t-2)}u(t-2) + 10e^{0.5(t-2)}u(-(t-2)) \end{aligned}$$

$$
\begin{aligned}
X_1(s) &= 10\underbrace{\frac{e^{-2s}}{s+1}}_{\text{Re}\{s\}>-1} - 10\underbrace{\frac{e^{-2s}}{s-0.5}}_{\text{Re}\{s\}<0.5} \\
&= 10e^{-2s}\frac{s-0.5-s-1}{(s+1)(s-0.5)},\ -1<\text{Re}\{s\}<0.5 \\
&= \frac{-15e^{-2s}}{(s+1)(s-0.5)},\ -1<\text{Re}\{s\}<0.5
\end{aligned}
$$

The Fourier transform exists since the ROC contains the imaginary axis, i.e., $s = j\omega$. It is given by:

$$
\begin{aligned}
X_1(j\omega) &= \frac{-15e^{-2j\omega}}{(j\omega+1)(j\omega-0.5)} \\
&= \frac{15e^{-2j\omega}}{(0.5+\omega^2)-0.5j\omega}
\end{aligned}
$$

(b) Signal $x_2(t)$ shown in Figure 6.2.

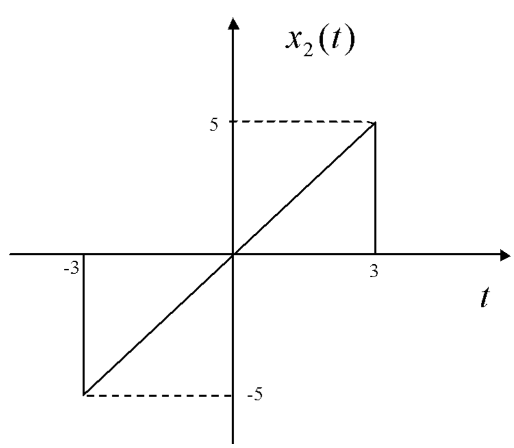


Figure 6.2: Sawtooth signal of Problem 6.1(b).

Answer:

We can compute this one by using the integral defining the Laplace transform. Here $x_2(t)$ has finite support, hence the Laplace transform integral converges for all s.

$$
\begin{aligned}
X_2(s) &= \frac{5}{3}\int_{-3}^{3} te^{-st}\,dt \\
&= -\frac{5}{3}\frac{te^{-st}}{s}\bigg|_{-3}^{3} + \frac{5}{3s}\int_{-3}^{3} e^{-st}\,dt \\
&= -\frac{5}{3}\frac{3e^{-3s}+3e^{3s}}{s} - \frac{5}{3}\frac{(e^{-3s}-e^{3s})}{s^2}, \ \forall s \\
&= -\frac{5e^{-3s}+5e^{3s}}{s} - \frac{5}{3}\frac{(e^{-3s}-e^{3s})}{s^2}, \ \forall s \\
&= \frac{(-15s-5)e^{-3s}-(15s-5)e^{3s}}{3s^2}, \ \forall s \\
&= \frac{(-5s-5/3)e^{-3s}-(5s-5/3)e^{3s}}{s^2}, \ \forall s
\end{aligned}
$$

The Fourier transform of this signal is given by:

$$
\begin{aligned}
X_2(j\omega) &= \frac{(-5j\omega-5/3)e^{-3j\omega}-(5j\omega-5/3)e^{3j\omega}}{-\omega^2} \\
&= \frac{-5j\omega(e^{3j\omega}+e^{-3j\omega})+5/3(e^{3j\omega}-e^{-3j\omega})}{-\omega^2} \\
&= \frac{5j(e^{3j\omega}+e^{-3j\omega})}{\omega} - \frac{5/3(e^{3j\omega}-e^{-3j\omega})}{\omega^2} \\
&= \frac{10j\cos(3\omega)}{\omega} - \frac{10j\sin(3\omega)}{3\omega^2} \\
&= 10j\frac{3\omega\cos(3\omega)-\sin(3\omega)}{3\omega^2}
\end{aligned}
$$

It is odd and purely imaginary, as expected. Applying L'Hopital's rule twice, we can also check that $X_2(j0)$ is finite as it should.

(c) Damped sinusoid signal $x_3(t) = e^{-10t}\sin(\omega_0 t+\theta)u(t)$ of Figure 6.3

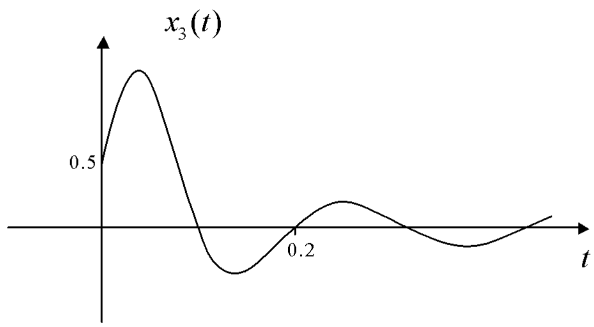


Figure 6.3: Damped sinousoid signal of Problem 1(c).

Answer:

Let us first find the values of the parameters θ and ω_0. We have

$$x_3(0) = 0.5 = \sin(\theta)$$
$$\theta = \arcsin(0.5)$$
$$\theta = \frac{\pi}{6}$$

and

$$x_3(0.2) = 0 = e^{-2}\sin(0.2\omega_0 + \frac{\pi}{6})$$
$$\Leftrightarrow 0.2\omega_0 + \frac{\pi}{6} = k\pi,\ \ k \in \mathbb{Z}$$
$$\text{here } k = 2$$
$$\omega_0 = 5\left(2\pi - \frac{\pi}{6}\right) = \frac{55\pi}{6}\,\text{rd/s}$$

The signal's Laplace transform can be obtained as follows:

$$
\begin{aligned}
x_3(t) &= e^{-10t}\sin\left(\frac{55\pi}{6}t+\frac{\pi}{6}\right)u(t) \\
&= e^{-10t}\left(\frac{e^{j\frac{55\pi}{6}t+j\frac{\pi}{6}}-e^{-j\frac{55\pi}{6}t-j\frac{\pi}{6}}}{2j}\right)u(t) \\
&= e^{-10t}\left(\frac{e^{j\frac{\pi}{6}}e^{j\frac{55\pi}{6}t}-e^{-j\frac{\pi}{6}}e^{-j\frac{55\pi}{6}t}}{2j}\right)u(t) \\
&= e^{-10t}\left(\frac{2j\,\mathrm{Im}\left\{e^{j\frac{\pi}{6}}e^{j\frac{55\pi}{6}t}\right\}}{2j}\right)u(t) \\
&= e^{-10t}\left(\cos\frac{\pi}{6}\sin\frac{55\pi}{6}t+\sin\frac{\pi}{6}\cos\frac{55\pi}{6}t\right)u(t) \\
&= e^{-10t}\left(\frac{\sqrt{3}}{2}\sin\frac{55\pi}{6}t+\frac{1}{2}\cos\frac{55\pi}{6}t\right)u(t) \\
&= \left(\frac{\sqrt{3}}{2}e^{-10t}\sin\frac{55\pi}{6}t+\frac{1}{2}e^{-10t}\cos\frac{55\pi}{6}t\right)u(t)
\end{aligned}
$$

Using the table:

$$
\begin{aligned}
X_3(s) &= \frac{\sqrt{3}}{2}\underbrace{\frac{\frac{55\pi}{6}}{(s+10)^2+\left(\frac{55\pi}{6}\right)^2}}_{\mathrm{Re}\{s\}>-10}+\frac{1}{2}\underbrace{\frac{s+10}{(s+10)^2+\left(\frac{55\pi}{6}\right)^2}}_{\mathrm{Re}\{s\}>-10} \\
&= \frac{1}{2}\frac{s+\frac{55\pi\sqrt{3}}{6}+10}{(s+10)^2+\left(\frac{55\pi}{6}\right)^2},\quad \mathrm{Re}\{s\}>-10 \\
&= \frac{1}{2}\frac{s+59.88}{s^2+20s+929.3},\quad \mathrm{Re}\{s\}>-10
\end{aligned}
$$

The Fourier transform of $x_3(t)$ exists since the imaginary axis lies in the ROC.

$$X_3(j\omega) = \frac{1}{2}\frac{j\omega + 59.88}{929.3 - \omega^2 + 20j\omega}$$

Problem 6.2

For the causal LTI system $H(s) = \frac{1}{s+1}, \mathrm{Re}\{s\} > -1$ shown in Figure 6.4, find the output responses $y_1(t)$ and $y_3(t)$ to the input signals $x_1(t)$ and $x_3(t)$ of Problem 1.

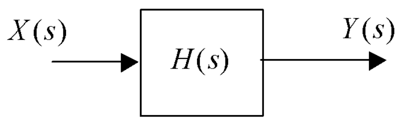


Figure 6.4: Causal LTI system of Problem 2.

The Laplace transform of the output response $y_1(t)$ is given by

$$\begin{aligned} Y_1(s) &= H(s)X_1(s) \\ &= \frac{-15e^{-2s}}{(s+1)^2(s-0.5)}, \quad -1 < \mathrm{Re}\{s\} < 0.5 \\ &= e^{-2s}\left[\underbrace{\frac{A}{(s+1)^2}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{B}{s+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{C}{s-0.5}}_{\mathrm{Re}\{s\}<0.5}\right] \end{aligned}$$

The coefficients are computed as follows:

$$\begin{aligned} A &= \left.\frac{-15}{(s-0.5)}\right|_{s=-1} = 10 \\ C &= \left.\frac{-15}{(s+1)^2}\right|_{s=0.5} = -6.667 \\ B &= \left.\frac{-15}{(s+1)(s-0.5)}\right|_{s=+\infty} + 6.667 = 6.667 \end{aligned}$$

Thus

$$Y_1(s) = e^{-2s}\left[\underbrace{\frac{10}{(s+1)^2}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{6.667}{s+1}}_{\mathrm{Re}\{s\}>-1} - \underbrace{\frac{6.667}{s-0.5}}_{\mathrm{Re}\{s\}<0.5}\right]$$

and taking the inverse Laplace transform, we get

$$y_1(t) = 10(t-2)e^{-(t-2)}u(t-2) + 6.667e^{-(t-2)}u(t-2) + 6.667e^{0.5(t-2)}u(-t+2)\,.$$

The Laplace transform of the output response $y_3(t)$ is given by:

$$\begin{aligned} Y_3(s) &= H(s)X_3(s) \\ &= \frac{1}{2}\frac{s+59.88}{(s+1)(s^2+20s+929.3)}, \quad \mathrm{Re}\{s\} > -1 \\ &= \underbrace{\frac{A}{s+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{B(s+10)+28.80C}{s^2+20s+929.3}}_{\mathrm{Re}\{s\}>-10} \end{aligned}$$

We find the coefficients by first multiplying on both sides by the common denominator, and then by identifying the coefficients of the polynomials

$$\begin{aligned} s+59.88 &= 2A\left(s^2+20s+929.3\right) + 2\left[B(s+10)+28.80C\right](s+1) \\ &= 2(A+B)s^2 + \left(40A+22B+57.6C\right)s + 1858.6A + 20B + 57.6C \end{aligned}$$

We obtain the linear matrix equation:

$$\begin{bmatrix} 1 & 1 & 0 \\ 40 & 22 & 57.6 \\ 1858.6 & 20 & 57.6 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 59.88 \end{bmatrix}$$

from which we compute

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0.0323 \\ -0.0323 \\ 0.0073 \end{bmatrix}$$

Thus,

$$Y_3(s) = \underbrace{\frac{0.0323}{s+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{-0.0323(s+10)+28.8(0.0073)}{s^2+20s+929.3}}_{\mathrm{Re}\{s\}>-10},$$

and taking the inverse Laplace transform, we get

$$y_3(t) = 0.0323e^{-t}u(t) - 0.0323e^{-10t}\cos(28.8t)u(t) + 0.0073e^{-10t}\sin(28.8t)u(t).$$

Problem 6.3

Find all possible ROCs for the following transfer function and give the corresponding impulse responses. Specify for each ROC whether the corresponding system is causal and stable.

$$H(s) = \frac{1}{(s^2+3s+3)(s-3)}$$

Answer:

The complex poles are found by identifying the damping ratio and undamped natural frequency of the second-order denominator factor with the standard second-order polynomial $s^2 + 2\varsigma\omega_n + \omega_n^2$. Here $\omega_n = \sqrt{3},\ \varsigma = \frac{\sqrt{3}}{2}$. Thus the poles are

$$
\begin{aligned}
p_1 &= 3 \\
p_2 &= -\varsigma\omega_n + j\omega_n\sqrt{1-\varsigma^2} = -\frac{3}{2} + j\frac{\sqrt{3}}{2} \\
p_3 &= p_2^* = -\frac{3}{2} - j\frac{\sqrt{3}}{2}
\end{aligned}
$$

There are three possible ROCs: $\mathrm{Re}\{s\} < -1.5, \quad -1.5 < \mathrm{Re}\{s\} < 3, \quad \mathrm{Re}\{s\} > 3$.

The partial fraction expansion of $H(s)$ yields:

$$
H(s) = \left(-\frac{1}{42} + j\frac{\sqrt{3}}{14}\right)\frac{1}{s + \frac{3}{2} - j\frac{\sqrt{3}}{2}} + \left(-\frac{1}{42} - j\frac{\sqrt{3}}{14}\right)\frac{1}{s + \frac{3}{2} + j\frac{\sqrt{3}}{2}} + \frac{1}{21}\frac{1}{s-3}.
$$

Using the table and simplifying, we find the following impulse responses:

$ROC_1 = \{s \in \mathbb{C} : \mathrm{Re}\{s\} > 3\}$:

$$
h(t) = \left(-\frac{1}{42} + j\frac{\sqrt{3}}{14}\right)e^{(-3/2 + j\sqrt{3}/2)t}u(t) + \left(-\frac{1}{42} - j\frac{\sqrt{3}}{14}\right)e^{(-3/2 - j\sqrt{3}/2)t}u(t) + \frac{1}{21}e^{3t}u(t)
$$

$ROC_2 = \{s \in \mathbb{C} : -1.5 < \mathrm{Re}\{s\} < 3\}$:

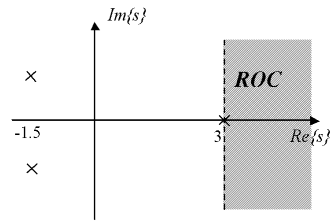
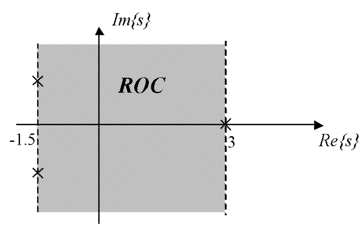
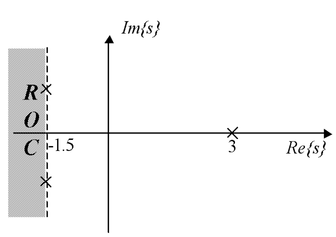
$$
h(t) = \left(-\frac{1}{42} + j\frac{\sqrt{3}}{14}\right)e^{(-3/2 + j\sqrt{3}/2)t}u(t) + \left(-\frac{1}{42} - j\frac{\sqrt{3}}{14}\right)e^{(-3/2 - j\sqrt{3}/2)t}u(t) - \frac{1}{21}e^{3t}u(-t)
$$

$ROC_3 = \{s \in \mathbb{C} : \mathrm{Re}\{s\} < -1.5\}$:

$$
h(t) = \left(\frac{1}{42} - j\frac{\sqrt{3}}{14}\right)e^{(-3/2 + j\sqrt{3}/2)t}u(-t) + \left(\frac{1}{42} + j\frac{\sqrt{3}}{14}\right)e^{(-3/2 - j\sqrt{3}/2)t}u(-t) - \frac{1}{21}e^{3t}u(-t)
$$

These are further simplified to their real form in Table 6.1.

Table 6.1: Impulse responses in Problem 3 for the three possible ROCs

ROC	$h(t)$	Causal	Stable
	$h(t)=\frac{-1}{21}e^{-1.5t}\left[3\sqrt{3}\sin\frac{\sqrt{3}}{2}t+\cos\frac{\sqrt{3}}{2}t\right]u(t)$ $+\frac{1}{21}e^{3t}u(t)$	YES. ROC is a right half-plane	NO. The $j\omega$-axis does not lie in the ROC
	$h(t)=\frac{-1}{21}e^{-1.5t}\left[3\sqrt{3}\sin\frac{\sqrt{3}}{2}t+\cos\frac{\sqrt{3}}{2}t\right]u(t)$ $-\frac{1}{21}e^{3t}u(-t)$	NO.	YES.
	$h(t)=\frac{1}{21}e^{-1.5t}\left[3\sqrt{3}\sin\frac{\sqrt{3}}{2}t+\cos\frac{\sqrt{3}}{2}t\right]u(-t)$ $-\frac{1}{21}e^{3t}u(-t)$	NO.	NO.

Exercises

Problem 6.4

Compute the step response of the LTI system $H(s)=\dfrac{6(s+1)}{s(s+3)},\ \operatorname{Re}\{s\}>0$.

Problem 6.5

Compute the output $y(t)$ of the LTI system $H(s)=\dfrac{100}{s^2+10s+100},\ \operatorname{Re}\{s\}>-5$ for the input signal $x(t)=e^{4t}u(-t)$.

Answer:

The Laplace transform of the input is looked up in Table D.4: $X(s)=-\dfrac{1}{s-4},\ \operatorname{Re}\{s\}<4$. The output signal is the inverse transform of:

$$
\begin{aligned}
Y(s)=H(s)X(s)&=\frac{-100}{(s^2+10s+100)(s-4)},\quad -5<\operatorname{Re}\{s\}<4\\
&=\underbrace{\frac{A(s+5)+B\sqrt{75}}{(s+5)^2+75}}_{\operatorname{Re}\{s\}>-5}+\underbrace{\frac{C}{s-4}}_{\operatorname{Re}\{s\}<4}\\
&=\underbrace{\frac{0.641(s+5)+0.667\sqrt{75}}{(s+5)^2+75}}_{\operatorname{Re}\{s\}>-5}-\underbrace{\frac{0.641}{s-4}}_{\operatorname{Re}\{s\}<4}
\end{aligned}
$$

which yields: $y(t)=\left[0.641e^{-5t}\cos(\sqrt{75}t)+0.667e^{-5t}\sin(\sqrt{75}t)\right]u(t)+0.641e^{4t}u(-t)$.

Problem 6.6

Suppose that the LTI system described by $H(s) = \frac{2}{(s+3)(s-1)}$ is known to be stable. Is this system causal? Compute its impulse response $h(t)$.

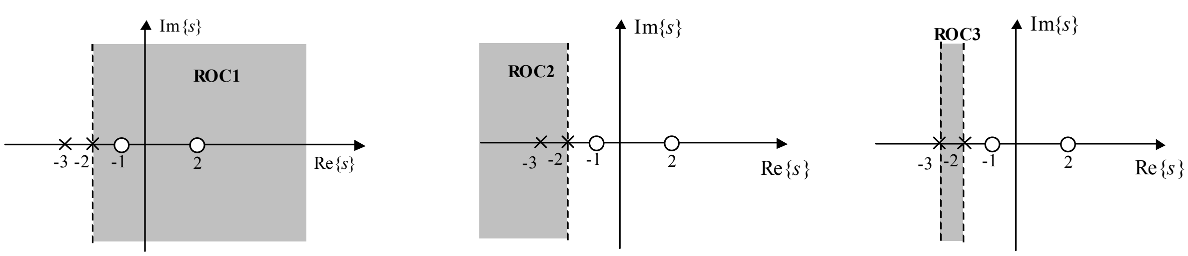
Problem 6.7

Consider an LTI system with transfer function $H(s) = \frac{s^2 - s - 2}{s^2 + 5s + 6}$. Sketch all possible regions of convergence of $H(s)$ on pole-zero plots and compute the associated impulse responses $h(t)$. Indicate for each impulse response whether it corresponds to a system that is causal/stable.

Answer:

$$H(s) = \frac{(s-2)(s+1)}{(s+2)(s+3)}$$

There are three possible ROC's shown below that can be associated with this transfer function. Only one ROC leads to a stable system: ROC1.



Impulse response with ROC1: stable and causal.

$$H(s) = \frac{(s-2)(s+1)}{(s+2)(s+3)}$$
$$= \underbrace{1}_{\forall s} + \underbrace{\frac{4}{s+2}}_{\mathrm{Re}\{s\}>-2} - \underbrace{\frac{10}{s+3}}_{\mathrm{Re}\{s\}>-3}$$

$$h(t) = \delta(t) + \left[4e^{-2t} - 10e^{-3t}\right]u(t)$$

Impulse response with ROC2: unstable, anticausal

$$H(s) = \frac{(s-2)(s+1)}{(s+2)(s+3)}$$
$$= \underbrace{1}_{\forall s} + \underbrace{\frac{4}{s+2}}_{\mathrm{Re}\{s\}<-2} - \underbrace{\frac{10}{s+3}}_{\mathrm{Re}\{s\}<-3}$$

$$h(t) = \delta(t) + \left[-4e^{-2t} + 10e^{-3t}\right]u(-t)$$

Impulse response with ROC3: unstable, noncausal (impulse response is two-sided.)

$$H(s) = \frac{(s-2)(s+1)}{(s+2)(s+3)}$$
$$= \underbrace{1}_{\forall s} + \underbrace{\frac{4}{s+2}}_{\mathrm{Re}\{s\}<-2} - \underbrace{\frac{10}{s+3}}_{\mathrm{Re}\{s\}>-3}$$

$$h(t) = \delta(t) - 4e^{-2t}u(-t) - 10e^{-3t}u(t)$$

Problem 6.8

Consider an LTI system with transfer function $H(s) = \frac{s(s-1)}{s^2 + \sqrt{2}s + 1}$. Sketch all possible regions of convergence (ROC's) of $H(s)$ on a pole-zero plot and compute the associated impulse responses $h(t)$. Indicate for each impulse response whether it corresponds to a system that is causal/stable.

Problem 6.9

System identification

Suppose we know that the input of an LTI system is $x(t) = e^t u(-t)$. The output was measured to be $y(t) = e^{-t}\sin(t)u(t) + e^{-t}u(t) + 2e^t u(-t)$. Find the transfer function $H(s)$ of the system, its region of convergence, and sketch its pole-zero plot. Is the system causal? Is it stable? Justify your answers.

Answer:

First take the Laplace transforms of the input and output signals using Table D.4:

$$X(s) = -\frac{1}{s-1}, \quad \operatorname{Re}\{s\} < 1$$

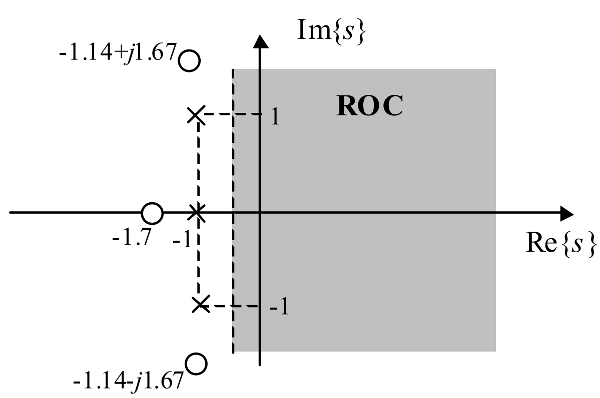
$$\begin{aligned}
Y(s) &= \underbrace{\frac{1}{(s+1)^2+1}}_{\operatorname{Re}\{s\}>-1} + \underbrace{\frac{1}{s+1}}_{\operatorname{Re}\{s\}>-1} - \underbrace{\frac{2}{s-1}}_{\operatorname{Re}\{s\}<1} \\
&= \frac{(s^2-1)-(s^2+2s+2)(s+3)}{(s^2+2s+2)(s+1)(s-1)}, \quad -1 < \operatorname{Re}\{s\} < 1 \\
&= \frac{-(s^3+4s^2+8s+7)}{(s^2+2s+2)(s+1)(s-1)}, \quad -1 < \operatorname{Re}\{s\} < 1
\end{aligned}$$

Then, the transfer function is simply:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{-(s^3+4s^2+8s+7)}{(s^2+2s+2)(s+1)(s-1)}}{-\dfrac{1}{s-1}} = \frac{s^3+4s^2+8s+7}{(s^2+2s+2)(s+1)}.$$

To determine the ROC of $H(s)$, first note that the ROC of $Y(s)$ should contain the intersection of the ROC's of $H(s)$ and $X(s)$. There are two possible ROC's for $H(s)$: (a) an open left half-plane to the left of $\mathrm{Re}\{s\} = -1$, (b) an open right half-plane to the right of $\mathrm{Re}\{s\} = -1$. But since the ROC of $X(s)$ is an open left half-plane to the left of $\mathrm{Re}\{s\} = 1$, the only possible choice is (b). Hence, the ROC of $H(s)$ is $\mathrm{Re}\{s\} > -1$.

The system is causal as the transfer function is rational and the ROC is a right half-plane. It is also stable as all three poles $p_{1,2} = -1 \pm j,\ p_3 = -1$ are in the open left half-plane. The zeros are computed as $z_{1,2} = -1.14 \pm j1.67,\ z_3 = -1.7$. The pole-zero plot of the system is shown below.



Problem 6.10

(a) Find the impulse response of the system $H(s) = \frac{3s^2 - 3s - 6}{s^3 + 12s^2 + 120s + 200}$, $\text{Re}\{s\} > -2$. Hint: this system has a pole at –2.

(b) Find the settling value of the step response of $H(s)$ given in (a).