Solutions to Problems in Chapter 4

Problems with Solutions

Problem 4.1

Fourier Series of the Output Voltage of an Ideal Full-Wave Diode Bridge Rectifier

The nonlinear circuit in Figure 4.1 is a full-wave rectifier. It is often used as a first stage of a power supply to generate a constant voltage from the 60Hz sinusoidal line voltage for all kinds of electronic devices. Here the input voltage is not sinusoidal.

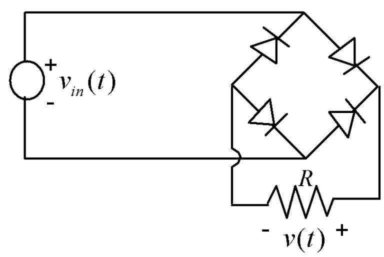

Figure 4.1: Full-wave rectifier circuit

The voltages are $v_{in}(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT) * \left(A\frac{2}{T}t \right)\left(u(t+\frac{T}{2}) - u(t-\frac{T}{2}) \right)$, and $v(t) = \left| v_{in}(t) \right|$.

Let T_1 be the fundamental period of the rectified voltage signal $v(t)$ and let $\omega_1 = 2\pi/T_1$ be its fundamental frequency. Find the fundamental period T_1. Sketch the input and output voltages $v_{in}(t), v(t)$.

Answer:

Let us first sketch the input voltage, which is the periodic sawtooth signal shown in Figure 4.2. Then the output voltage is simply the absolute value of the input signal which results in a triangular wave as shown in Figure 4.3.

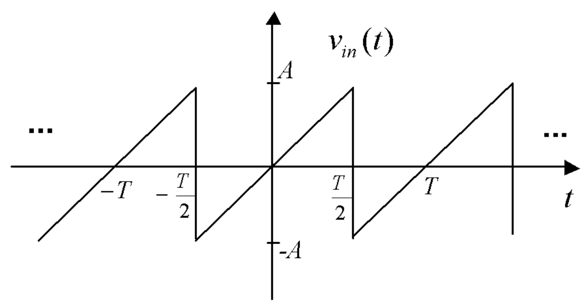

Figure 4.2: Input voltage of full-wave rectifier

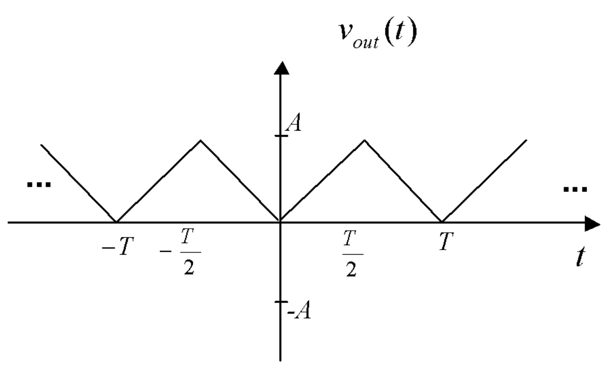

Figure 4.3: Output voltage of full-wave rectifier

We can see that the fundamental period of the output signal is the same as that of the input: $T_1 = T$.

(b) Compute the Fourier series coefficients of $v_{in}(t)$. Sketch the spectrum for $A = 1$.

Answer:

The DC component of the input is obviously 0:

$$a_0 = \frac{A}{T_1}\int_{-T_1/2}^{T_1/2} \frac{2}{T_1} t dt = \frac{A}{T_1^2}\left[t^2\right]_{-T_1/2}^{T_1/2} = \frac{A}{T_1^2}\left[\frac{T_1^2}{4} - \frac{T_1^2}{4}\right] = 0$$

for $k \neq 0$:

$$\begin{aligned} a_k &= \frac{A}{T_1}\int_{-T_1/2}^{T_1/2} \frac{2}{T_1} t e^{-jk\omega_1 t} dt \\ &= \frac{jA}{k\pi T_1}\left[\left(te^{-jk\omega_1 t}\right)_{-T_1/2}^{T_1/2} - \int_{-T_1/2}^{T_1/2} e^{-jk\omega_1 t} dt\right] \\ &= \frac{jA}{k\pi T_1}\left[\left(\frac{T_1}{2}e^{-jk\pi} + \frac{T_1}{2}e^{jk\pi}\right) - 0\right] \\ &= \frac{jA}{k\pi}\cos k\pi \\ &= \frac{jA(-1)^k}{k\pi} \end{aligned}$$

The spectrum is imaginary, so we can have a single plot to represent it as in Figure 4.4.

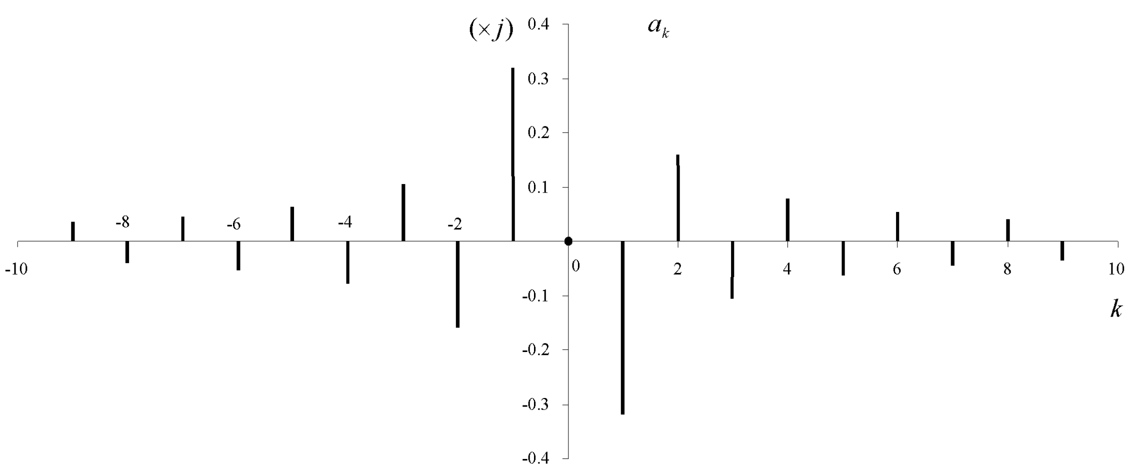

Figure 4.4: Line spectrum of input voltage, case $A = 1$

(c) Compute the Fourier series coefficients of $v(t)$. Write $v(t)$ as a Fourier series. Sketch the spectrum for $A = 1$.

Answer:

Let us first compute the DC component of the output signal:

$$a_0 = \frac{A}{T_1} \int_{-T_1/2}^{T_1/2} \frac{2}{T_1} |t| \, dt = \frac{4A}{T_1^2} \int_0^{T_1/2} t dt = \frac{4A}{T_1^2} \frac{T_1^2}{8} = \frac{1}{2} A .$$

For $k \neq 0$, the spectral coefficients are computed as follows:

$$\begin{aligned}
a_k &= \frac{A}{T_1} \int_{-T_1/2}^{T_1/2} \frac{2}{T_1} |t| e^{-jk\omega_1 t} dt \\
&= \frac{A}{T_1} \left[\int_0^{T_1/2} \frac{2}{T_1} t e^{-jk\omega_1 t} dt - \int_{-T_1/2}^{0} \frac{2}{T_1} t e^{-jk\omega_1 t} dt \right] \\
&= \frac{A}{T_1} \int_0^{T_1/2} \frac{2}{T_1} t (e^{-jk\omega_1 t} + e^{jk\omega_1 t}) dt \\
&= \frac{2A}{T_1} \int_0^{T_1/2} \frac{2}{T_1} t \cos(k\omega_1 t) dt \\
&= \frac{2A}{k\pi T_1} \int_0^{T_1/2} (k\omega_1) t \cos(k\omega_1 t) dt \\
&= \frac{2A}{k\pi T_1} \left[\left(t \sin(k\omega_1 t) \right)_0^{T_1/2} - \int_0^{T_1/2} \sin(k\omega_1 t) dt \right] \\
&= \frac{2A}{k\pi T_1} \left[\underbrace{\left(\frac{T_1}{2} \sin k\pi - 0 \right)}_{0} + \frac{1}{k\omega_1} \cos(k\omega_1 t)_0^{T_1/2} \right] \\
&= \frac{A}{(k\pi)^2} \left(\cos k\pi - 1 \right) \\
&= \frac{A}{(k\pi)^2} \left((-1)^k - 1 \right)
\end{aligned}$$

The spectrum of the triangular wave is real and even (because the signal is real and even), so we can have a single plot to represent it as in Figure 4.5.

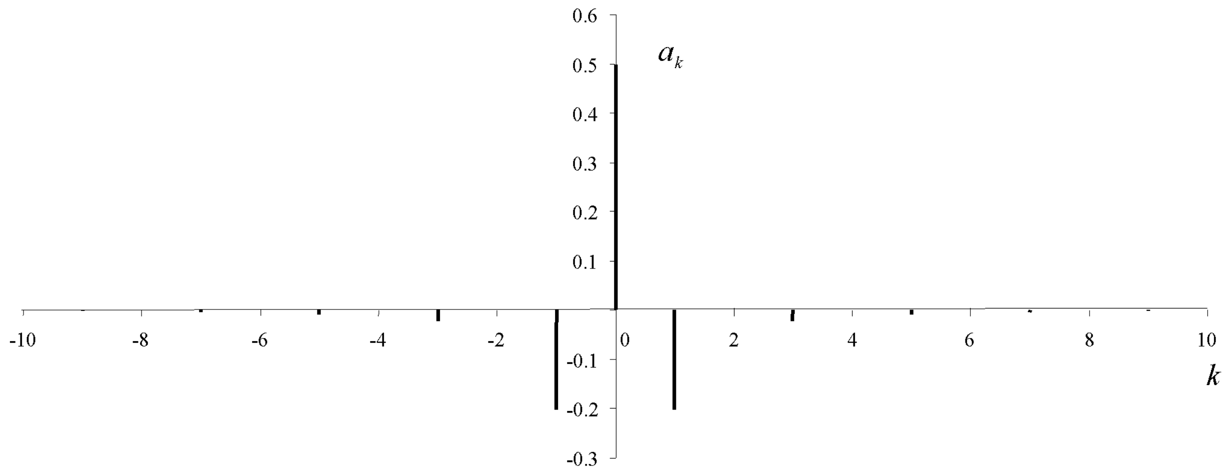

Figure 4.5: Line spectrum of output voltage, case $A = 1$

Thus,

$$v(t) = \sum_{k=-\infty}^{+\infty} \frac{A}{(k\pi)^2}\left((-1)^k - 1\right)e^{jk\omega_1 t}$$

is the Fourier series expansion of the full-wave rectified voltage.

(d) What is the average power of the input voltage $v_{in}(t)$ at frequencies higher than or equal to its fundamental frequency? Same question for the output voltage $v(t)$? Discuss the difference in power.

Answer:

Since $a_0 = 0$, and given the Parseval Theorem, the average power of the input voltage $v_{in}(t)$ at frequencies higher than or equal to its fundamental frequency is equal to its total average power computed in the time domain:

$$
\begin{aligned}
P_{in} &= \sum_{k=-\infty}^{+\infty} \left|a_k\right|^2 = \frac{1}{T}\int_{-T/2}^{T/2} \frac{4A^2}{T^2} t^2 dt \\
&= \frac{4A^2}{3T^3}\left[t^3\right]_{-T/2}^{T/2} \\
&= \frac{4A^2}{24T^3}\left[T^3 + T^3\right] \\
&= \frac{A^2}{3}
\end{aligned}
$$

The average power in all harmonic components of $v(t)$ excluding the DC component (call it P_{out}) is computed as follows:

$$
\begin{aligned}
P_{out} &= \sum_{k=-\infty}^{+\infty} \left|a_k\right|^2 - \left|a_0\right|^2 = \frac{1}{T}\int_{-T/2}^{T/2} \frac{4A^2}{T^2} t^2 dt - \left|a_0\right|^2 \\
&= \frac{A^2}{3} - \frac{A^2}{4} = \frac{A^2}{12}
\end{aligned}
$$

Note that the input and output signals have the same total average power, but some of the power in the input voltage (namely $A^2/4$) was transferred over to DC by the nonlinear circuit.

Problem 4.2

Given periodic signals with spectra $x(t) \overset{\mathcal{FS}}{\leftrightarrow} a_k$ and $y(t) \overset{\mathcal{FS}}{\leftrightarrow} b_k$, show the following properties:

(a) Multiplication: $x(t)y(t) \overset{\mathcal{FS}}{\leftrightarrow} \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$

Answer:

$$\begin{aligned}
x(t)y(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{-jk\omega_0 t} \sum_{n=-\infty}^{+\infty} b_n e^{-jn\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} a_k b_n e^{-j(k+n)\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} a_k b_{p-k} e^{-jp\omega_0 t} \\
&= \sum_{p=-\infty}^{+\infty} \underbrace{\left(\sum_{k=-\infty}^{+\infty} a_k b_{p-k} \right)}_{\text{FS coeff. } c_p} e^{-jp\omega_0 t}
\end{aligned}$$

Therefore, $x(t)y(t) \overset{\mathcal{FS}}{\leftrightarrow} \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$.

(b) Periodic convolution: $\int_T x(\tau)y(t-\tau)d\tau \overset{\mathcal{FS}}{\leftrightarrow} Ta_k b_k$

Answer:

$$\begin{aligned}
\int_T x(\tau)y(t-\tau)d\tau &= \int_T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 \tau} \sum_{p=-\infty}^{+\infty} b_p e^{jp\omega_0 (t-\tau)} d\tau \\
&= \int_T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 \tau} \sum_{p=-\infty}^{+\infty} b_p e^{-jp\omega_0 \tau} e^{jp\omega_0 t} d\tau \\
&= \int_T \sum_{k=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} a_k b_p e^{j(k-p)\omega_0 \tau} d\tau e^{jp\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} a_k b_p \int_T e^{j(k-p)\omega_0 \tau} d\tau\, e^{jp\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} a_k b_k T e^{jk\omega_0 t}
\end{aligned}$$

Therefore, $\int_T x(\tau)y(t-\tau)d\tau \overset{\mathcal{FS}}{\leftrightarrow} Ta_k b_k$

Problem 4.3

Fourier Series of the Output of an LTI System.

Consider the familiar rectangular waveform $x(t)$ of period T and *duty cycle* $\eta = \frac{2t_0}{T}$. This signal is the input to an LTI system with impulse response $h(t) = e^{-5t}\sin(10\pi t)u(t)$.

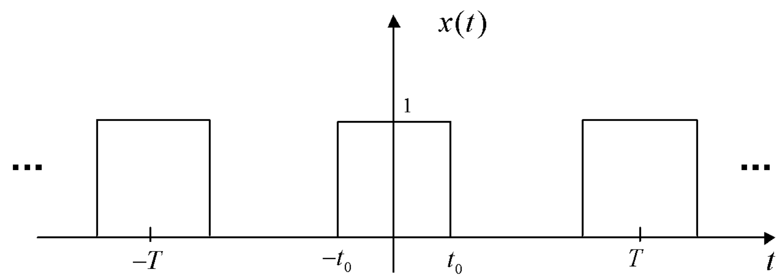

Figure 4.6: Rectangular wave input to LTI system

(a) Find the frequency response $H(j\omega)$ of the LTI system. Give expressions for its magnitude $|H(j\omega)|$ and phase $\angle H(j\omega)$ as functions of ω.

Answer:

The frequency response of the system is given by

$$
\begin{aligned}
H(j\omega) &= \int_0^{+\infty} e^{-5t}\sin(10\pi t)e^{-j\omega t}\,dt \\
&= \frac{1}{2j}\int_0^{+\infty} (e^{j10\pi t} - e^{-j10\pi t})e^{-(5+j\omega)t}\,dt \\
&= \frac{1}{2j}\int_0^{+\infty} (e^{-(5+j(\omega-10\pi))t} - e^{-(5+j(\omega+10\pi))t})\,dt \\
&= \frac{1}{2j(5+j(\omega-10\pi))} - \frac{1}{2j(5+j(\omega+10\pi))} \\
&= \frac{(5+j(\omega+10\pi))-(5+j(\omega-10\pi))}{2j(5+j(\omega-10\pi))(5+j(\omega+10\pi))} \\
&= \frac{10\pi}{25-\omega^2+100\pi^2+j10\omega}
\end{aligned}
$$

Magnitude: $|H(j\omega)| = \dfrac{10\pi}{\left[(25-\omega^2+100\pi^2)^2+100\omega^2\right]^{0.5}}$

Phase: $\angle H(j\omega) = \arctan\left(\dfrac{-10\omega}{25-\omega^2+100\pi^2}\right)$

(b) Find the Fourier series coefficients a_k of the input voltage $x(t)$ for $T=1s$ and a 60% duty cycle.

Answer:

The period given corresponds to a signal frequency of 1Hz, and the 60% duty cycle means that $\eta=\frac{3}{5}$ so that the spectral coefficients of the rectangular wave are given by $a_k=\frac{3}{5}\mathrm{sinc}(\frac{3k}{5})$.

(c) Compute the Fourier series coefficients b_k of the output signal $y(t)$ (for the input described in (b) above), and sketch its power spectrum.

Answer:

$$b_k = H(jk\omega_0)a_k = \tfrac{3}{5}\,\mathrm{sinc}(\tfrac{3k}{5})\frac{10\pi}{25-(k\omega_0)^2+100\pi^2+j10k\omega_0}$$

$$= \frac{6\pi\,\mathrm{sinc}(\tfrac{3k}{5})}{25-(k2\pi)^2+100\pi^2+j20\pi k}$$

The power spectrum of the output signal is given by the expression below and shown in Figure 4.7.

$$\left|b_k\right|^2 = \frac{36\pi^2\,\mathrm{sinc}^2(\tfrac{3k}{5})}{\left[25-(k2\pi)^2+100\pi^2\right]^2+400\pi^2k^2}$$

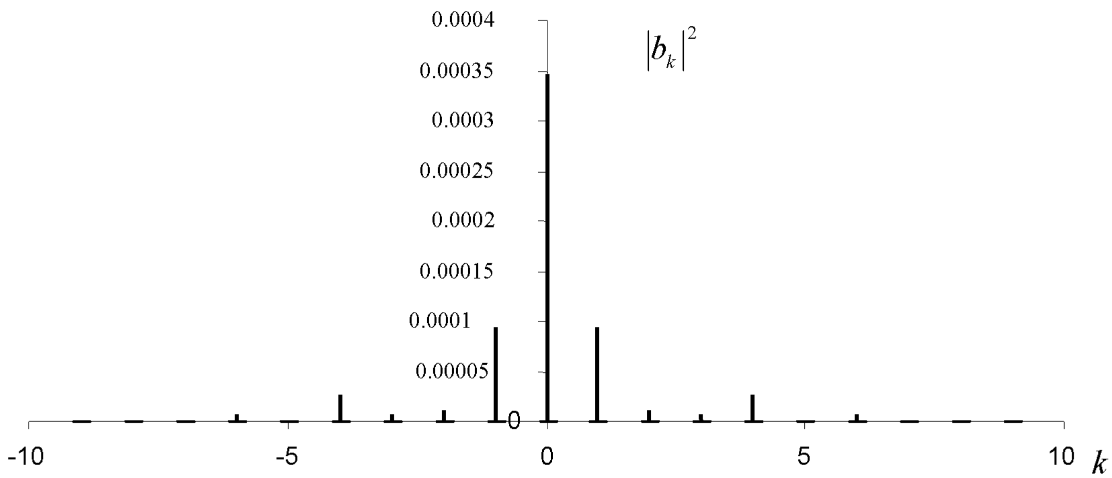

Figure 4.7: Power spectrum of output signal

(d) Using Matlab, plot an approximation to the output signal over three periods by summing the first 100 harmonics of $y(t)$, i.e., by plotting $\tilde{y}(t) = \sum_{-100}^{+100} b_k e^{jk\frac{2\pi}{T}t}$.

Answer:

$$\begin{aligned}
\tilde{y}(t) &= \sum_{k=-100}^{+100} b_k e^{jk2\pi t} \\
&= \sum_{k=-100}^{+100} \tfrac{3}{5}\,\mathrm{sinc}(\tfrac{3k}{5}) \frac{10\pi}{25-(k2\pi)^2+100\pi^2+j20k\pi} e^{jk2\pi t} \\
&= b_0 + \sum_{k=1}^{100} \tfrac{6}{5}\,\mathrm{sinc}(\tfrac{3k}{5}) \frac{10\pi}{\sqrt{\left[25-(k2\pi)^2+100\pi^2\right]^2+400\pi^2k^2}} \cos\left[k2\pi t + \arctan\left(\frac{-20k\pi}{25-(k2\pi)^2+100\pi^2}\right)\right] \\
&= \frac{6\pi}{25+100\pi^2} + \sum_{k=1}^{100} \tfrac{6}{5}\,\mathrm{sinc}(\tfrac{3k}{5}) \frac{10\pi}{\sqrt{\left[25-(k2\pi)^2+100\pi^2\right]^2+400\pi^2k^2}} \cos\left[k2\pi t + \arctan\left(\frac{-20k\pi}{25-(k2\pi)^2+100\pi^2}\right)\right]
\end{aligned}$$

Figure 4.8 shows a plot of $\tilde{y}(t)$ from 0s to 3s.

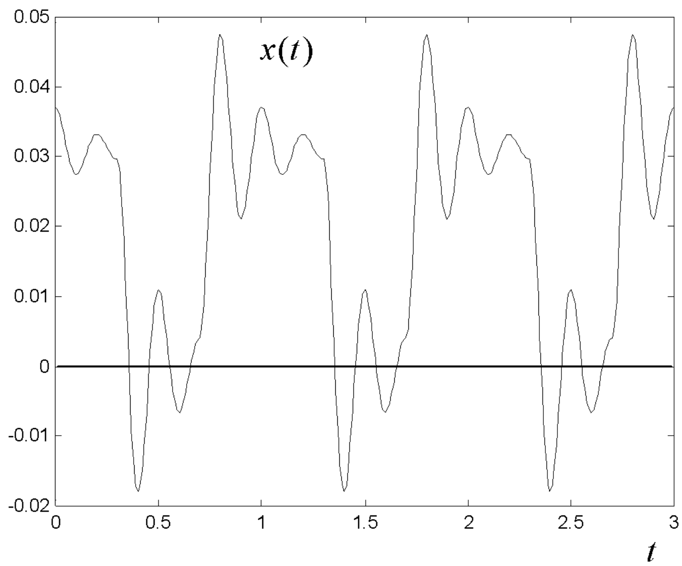

Figure 4.8: Approximation of output signal using truncated 100-harmonic Fourier series

Problem 4.4

Digital Sine Wave Generator

A programmable digital signal generator generates a sinusoidal waveform by filtering the staircase approximation to a sine wave shown in Figure 4.9.

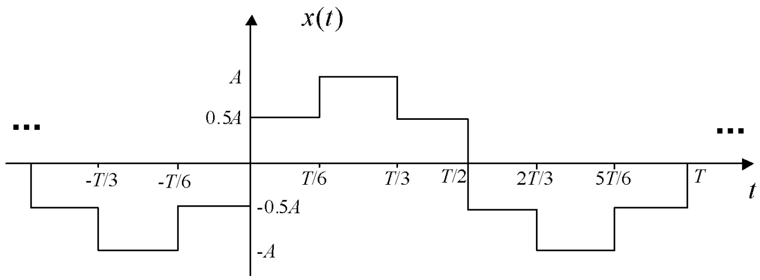

Figure 4.9: Staircase approximation to a sinusoidal wave in Problem 4.4.

(a) Find the Fourier series coefficients a_k of the periodic signal $x(t)$. Show that the even harmonics vanish. Express $x(t)$ as a Fourier series.

Answer:

First of all, the average over one period is 0, so $a_0 = 0$. For $k \neq 0$,

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\frac{2\pi}{T}t}dt$$

$$= -\frac{A}{2T}\int_{-\frac{T}{2}}^{-\frac{T}{3}} e^{-jk\frac{2\pi}{T}t}dt - \frac{A}{T}\int_{-\frac{T}{3}}^{-\frac{T}{6}} e^{-jk\frac{2\pi}{T}t}dt - \frac{A}{2T}\int_{-\frac{T}{6}}^{0} e^{-jk\frac{2\pi}{T}t}dt$$

$$+\frac{A}{2T}\int_{\frac{T}{3}}^{\frac{T}{2}} e^{-jk\frac{2\pi}{T}t}dt + \frac{A}{T}\int_{\frac{T}{6}}^{\frac{T}{3}} e^{-jk\frac{2\pi}{T}t}dt + \frac{A}{2T}\int_{0}^{\frac{T}{6}} e^{-jk\frac{2\pi}{T}t}dt$$

$$= \frac{A}{2T}\int_{0}^{\frac{T}{6}}\left(e^{-jk\frac{2\pi}{T}t} - e^{jk\frac{2\pi}{T}t}\right)dt + \frac{A}{2T}\int_{\frac{T}{3}}^{\frac{T}{2}}\left(e^{-jk\frac{2\pi}{T}t} - e^{jk\frac{2\pi}{T}t}\right)dt + \frac{A}{T}\int_{\frac{T}{6}}^{\frac{T}{3}}\left(e^{-jk\frac{2\pi}{T}t} - e^{jk\frac{2\pi}{T}t}\right)dt$$

$$= \frac{-jA}{T}\int_{0}^{\frac{T}{6}}\sin\left(k\frac{2\pi}{T}t\right)dt - \frac{j2A}{T}\int_{\frac{T}{6}}^{\frac{T}{3}}\sin\left(k\frac{2\pi}{T}t\right)dt - \frac{jA}{T}\int_{\frac{T}{3}}^{\frac{T}{2}}\sin\left(k\frac{2\pi}{T}t\right)dt$$

$$= \frac{jA}{T}\left(\frac{T}{2\pi k}\right)\cos\left(k\frac{2\pi}{T}t\right)\Big|_{0}^{\frac{T}{6}} + \frac{j2A}{T}\left(\frac{T}{2\pi k}\right)\cos\left(k\frac{2\pi}{T}t\right)\Big|_{\frac{T}{6}}^{\frac{T}{3}} + \frac{jA}{T}\left(\frac{T}{2\pi k}\right)\cos\left(k\frac{2\pi}{T}t\right)\Big|_{\frac{T}{3}}^{\frac{T}{2}}$$

$$= \frac{jA}{2\pi k}\left[\cos\left(k\frac{\pi}{3}\right) - 1 + 2\cos\left(k\frac{2\pi}{3}\right) - 2\cos\left(k\frac{\pi}{3}\right) + \cos(k\pi) - \cos\left(k\frac{2\pi}{3}\right)\right]$$

$$= \frac{jA}{2\pi k}\left[-\cos\left(k\frac{\pi}{3}\right) + \cos\left(k\frac{2\pi}{3}\right) - 1 + \cos(k\pi)\right]$$

Note that the coefficients are purely imaginary, which is consistent with our real, odd signal.

The even spectral coefficients are for $k = 2m,\ m = 1, 2, \ldots$:

$$a_k = a_{2m} = \frac{jA}{2\pi 2m}\left[-\cos\left(m\frac{2\pi}{3}\right) + \cos\left(m\frac{4\pi}{3}\right) - 1 + \cos(m2\pi)\right]$$

$$= \frac{jA}{2\pi 2m}\left[-\cos\left(-m\frac{\pi}{3} + m\pi\right) + \cos\left(m\frac{\pi}{3} + m\pi\right)\right]$$

$$= \frac{jA}{2\pi 2m}\left[\cos\left(m\frac{\pi}{3}\right) - \cos\left(m\frac{\pi}{3}\right)\right] = 0$$

Figure 4.10 shows a plot of $x(t)$ computed using 250 harmonics in the Matlab script Fourierseries.m which can be found in the Chapter 4 folder on the CD-ROM.

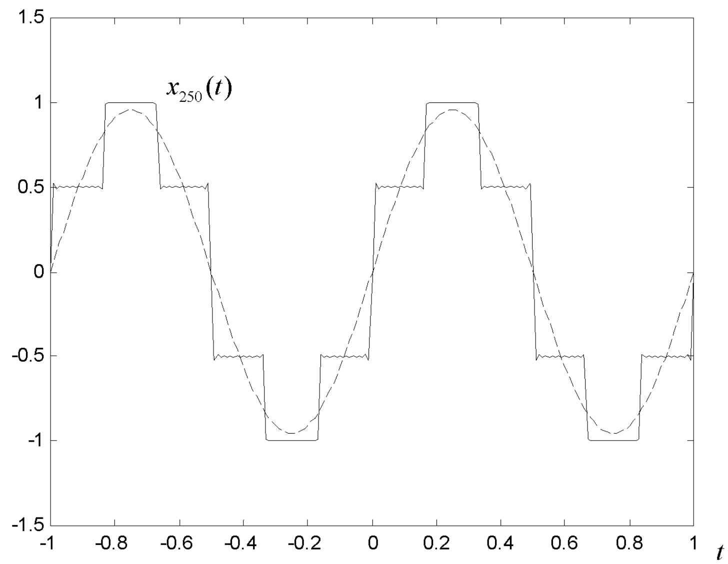

Figure 4.10: Truncated Fourier series approximation to staircase signal and first harmonic in Problem 4.4(a).

The Fourier series representation of $x(t)$ is

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T}t}=\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\frac{jA}{2\pi k}\left[-\cos\left(k\frac{\pi}{3}\right)+\cos\left(k\frac{2\pi}{3}\right)-1+\cos\left(k\pi\right)\right]e^{jk\frac{2\pi}{T}t}.$$

(b) Write $x(t)$ using the real form of the Fourier series.

$$x(t)=a_0+2\sum_{k=1}^{+\infty}[B_k\cos(k\omega_0 t)-C_k\sin(k\omega_0 t)]$$

Recall that the C_k coefficients are the imaginary parts of the a_k's. Hence

$$x(t)=\sum_{k=1}^{+\infty}\frac{-A}{\pi k}\left[-\cos\left(k\frac{\pi}{3}\right)+\cos\left(k\frac{2\pi}{3}\right)-1+\cos\left(k\pi\right)\right]\sin(k\omega_0 t)$$

(c) Design an ideal lowpass filter that will produce the perfect sinusoidal waveform $y(t) = \sin\frac{2\pi}{T}t$ at its output with $x(t)$ as its input. Sketch its frequency response and specify its gain K and cutoff frequency ω_c.

Answer:

The frequency response of the lowpass filter is shown in Figure 4.11.

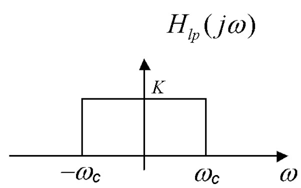

Figure 4.11: Frequency response of lowpass filter

The cutoff should be between the fundamental and the second harmonic, say $\omega_c = \frac{3\pi}{T}$. The gain should be:

$$
\begin{aligned}
K &= \frac{-\pi}{A}\left[-\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{2\pi}{3}\right)-2\right]^{-1} \\
&= \frac{-\pi}{A}\left[-\frac{1}{2}-\frac{1}{2}-2\right]^{-1} = \frac{\pi}{3A}
\end{aligned}.
$$

(d) Now suppose that the first-order lowpass filter whose differential equation is given below is used to filter $x(t)$.

$$
\tau\frac{dy(t)}{dt} + y(t) = Bx(t)
$$

where the time constant is chosen to be $\tau = \dfrac{T}{2\pi}$. Give the Fourier series representation of the output $y(t)$. Compute the total average power in the fundamental components P_{1tot} and in the third harmonic components P_{3tot}. Find the value of the DC gain B such that the output $w(t)$ produced by the fundamental harmonic of the real Fourier series of $x(t)$ has unit amplitude.

Answer:

$$H(s) = \frac{B}{\tau s + 1}$$

$$H(j\omega) = \frac{B}{\tau j\omega + 1}$$

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\frac{2\pi}{T}) e^{jk2\pi t} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{B}{jk+1} \frac{jA}{2\pi k}\left[-\cos\left(k\frac{\pi}{3}\right) + \cos\left(k\frac{2\pi}{3}\right) - 1 + \cos(k\pi)\right] e^{jk\frac{2\pi}{T}t}$$

Power:

$$\begin{aligned} P_{1tot} &= 2\left|\frac{B}{j+1}\frac{jA}{2\pi}\left[-\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right) - 1 + \cos(\pi)\right]\right|^2 \\ &= 2\left|\frac{B}{j+1}\frac{j3A}{2\pi}\right|^2 = A^2B^2\frac{9}{4\pi^2} \end{aligned}$$

$$\begin{aligned} P_{3tot} &= 2\left|\frac{B}{j3+1}\frac{jA}{2\pi}\left[-\cos(\pi) + \cos(2\pi) - 1 + \cos(3\pi)\right]\right|^2 \\ &= 2\left|\frac{B}{j3+1}\frac{jA}{2\pi}[-2]\right|^2 = 2\frac{4A^2B^2}{40\pi^2} = \frac{A^2B^2}{5\pi^2} \end{aligned}$$

For the filter's DC gain B, we found that the gain at ω_0 should be:

$$\frac{\pi}{3A} = \left|H(j\omega_0)\right| = \frac{B}{\left|\tau j\omega_0 + 1\right|} = \frac{B}{\sqrt{\left(\tau\omega_0\right)^2 + 1}}$$

$$\Leftrightarrow B = \frac{\pi\sqrt{\left(\tau\omega_0\right)^2 + 1}}{3A}$$

Exercises

Problem 4.5

The output voltage of a half-wave rectifier is given by: $v(t) = \begin{cases} v_{in}(t), & v_{in}(t) > 0 \\ 0, & v_{in}(t) \le 0 \end{cases}$.

Suppose that the periodic input voltage signal is $v_{in}(t) = A\sin(\omega_1 t),\ \omega_1 = 2\pi/T_1$. Find the fundamental period T and the fundamental frequency ω_0 of the half-wave rectified voltage signal $v(t)$. Compute the Fourier series coefficients of $v(t)$ and write the voltage as a Fourier series.

Answer:

The fundamental period of $v(t)$ is $T = T_1$ and the fundamental frequency is $\omega_0 = \omega_1$.

$$\begin{aligned}
a_k &= \frac{1}{T}\int_0^T v(t)e^{-jk\omega_0 t}\,dt = \frac{A}{T}\int_0^{\frac{T}{2}} \sin(\omega_0 t)e^{-jk\omega_0 t}\,dt = \frac{A}{2jT}\int_0^{\frac{T}{2}} \left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)e^{-jk\omega_0 t}\,dt \\
&= \frac{A}{2jT}\int_0^{\frac{\pi}{\omega_0}} \left(e^{j\omega_0(1-k)t} - e^{-j\omega_0(1+k)t}\right)dt = \frac{A\omega_0}{4\pi j}\left[\frac{e^{j(1-k)\pi} - 1}{j\omega_0(1-k)} + \frac{e^{-j(1+k)\pi} - 1}{j\omega_0(1+k)}\right] \\
&= -\frac{A}{4\pi}\left[\frac{-e^{-jk\pi} - 1}{(1-k)} + \frac{-e^{-jk\pi} - 1}{(1+k)}\right] = -\frac{A}{4\pi}\left[\frac{-2(e^{-jk\pi} + 1)}{1-k^2}\right] \\
&= \frac{A}{2\pi}\left[\frac{(e^{-jk\pi} + 1)}{1-k^2}\right] = \frac{A}{2\pi}\left[\frac{(-1)^k + 1}{1-k^2}\right]
\end{aligned}$$

Note that we have to make sure that this expression is finite for $k = \pm 1$ (L'Hopital's rule)

$$a_1 = \frac{A}{2\pi}\left[\frac{\frac{d}{dk}(e^{-jk\pi}+1)}{\frac{d}{dk}(1-k^2)}\right]_{k=1} = \frac{A}{2\pi}\left[\frac{-j\pi e^{-jk\pi}}{-2k}\right]_{k=1} = -\frac{jA}{4}$$

$$a_{-1} = \frac{A}{2\pi}\left[\frac{-j\pi e^{-jk\pi}}{-2k}\right]_{k=-1} = \frac{jA}{4}$$

Notice that these are imaginary whereas all the other coefficients are real!

Thus, $v(t) = \sum_{k=-\infty}^{+\infty} \frac{A}{2\pi}\left[\frac{(-1)^k+1}{1-k^2}\right]e^{jk\omega_0 t}$ is the Fourier series expansion of the half-wave rectified sinusoid.

Problem 4.6

Suppose that the voltages in the full-wave bridge rectifier circuit of Figure 4.1 are $v_{in}(t) = A\sin(\omega_0 t),\ \omega_0 = 2\pi/T$, and $v(t) = |v_{in}(t)|$. Let $T_1 = T/2$ be the fundamental period of the rectified voltage signal $v(t)$ and let $\omega_1 = 2\pi/T_1$ be its fundamental frequency.

(a) Compute the Fourier series coefficients of $v(t)$ and write $v(t)$ as a Fourier series.

(b) Express $v(t)$ as a real Fourier series of the form $v(t) = a_0 + 2\sum_{k=1}^{+\infty}[B_k \cos(k\omega_1 t) - C_k \sin(k\omega_1 t)]$

Problem 4.7

Fourier Series of a Train of RF Pulses

Consider the following signal $x(t)$ of fundamental frequency $\omega_0 = \frac{2\pi}{T}$, a periodic train of radio frequency (RF) pulses. Over one period from $-T/2$ to $T/2$, the signal is given by:

$$x(t) = \begin{cases} A\cos(\omega_c t), & -T_1 < t < T_1 \\ 0, & -T/2 < t < -T_1 \\ 0, & T_1 < t < T/2 \end{cases}$$

This signal could be used to test a transmitter-receiver radio communication system. Assume that the pulse frequency is an integer multiple of the signal frequency, i.e., $\omega_c = N\omega_0$. Compute the Fourier series coefficients of $x(t)$.

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}\,dt = \frac{A}{T}\int_{-T_1}^{T_1}\cos(N\omega_0 t)e^{-jk\omega_0 t}\,dt \\
&= \frac{A}{2T}\int_{-T_1}^{T_1}\left(e^{jN\omega_0 t} + e^{-jN\omega_0 t}\right)e^{-jk\omega_0 t}\,dt = \frac{A}{2T}\int_{-T_1}^{T_1}\left(e^{j\omega_0(N-k)t} - e^{-j\omega_0(N+k)t}\right)dt \\
&= \frac{A}{2T}\left[\frac{e^{j\omega_0(N-k)T_1} - e^{-j\omega_0(N-k)T_1}}{j\omega_0(N-k)} + \frac{e^{-j\omega_0(N+k)T_1} - e^{j\omega_0(N+k)T_1}}{j\omega_0(N+k)}\right] \\
&= \frac{A}{2T}\left[\frac{2T_1\sin[\omega_0(N-k)T_1]}{T_1\omega_0(N-k)} + \frac{2T_1\sin[\omega_0(N+k)T_1]}{T_1\omega_0(N+k)}\right] \\
&= \frac{AT_1}{T}\left\{\mathrm{sinc}[2T_1 f_0(N-k)] + \mathrm{sinc}[2T_1 f_0(N+k)]\right\} \\
&= \frac{AT_1}{T}\left\{\mathrm{sinc}[2T_1(f_c - kf_0)] + \mathrm{sinc}[2T_1(f_c + kf_0)]\right\} \\
&= \frac{AT_1}{T}\left\{\mathrm{sinc}[2T_1(kf_0 - f_c)] + \mathrm{sinc}[2T_1(kf_0 + f_c)]\right\}
\end{aligned}$$

Problem 4.8

(a) Compute and sketch (magnitude and phase) the Fourier series coefficients of the sawtooth signal of Figure 4.12.

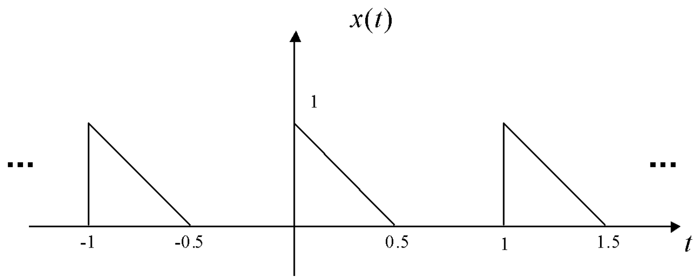

Figure 4.12: Periodic sawtooth signal in Problem 4.8(a).

(b) Express $x(t)$ as its real Fourier series of the form:

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty}[B_k \cos(k\omega_1 t) - C_k \sin(k\omega_1 t)]$$

(c) Use MATLAB to plot, superimposed on the same figure, approximations to the signal over two periods by summing the first 5, and the first 50 harmonic components of $x(t)$, i.e., by plotting $\tilde{x}(t) = \sum_{-N}^{N} a_k e^{jk\frac{2\pi}{T}t}$. Discuss your results.

(d) The sawtooth signal $x(t)$ is the input to an LTI system with impulse response $h(t) = e^{-t}\sin(2\pi t)u(t)$. Let $y(t)$ denote the resulting periodic output. Find the frequency response $H(j\omega)$ of the LTI system. Give expressions for its magnitude $|H(j\omega)|$ and phase $\angle H(j\omega)$ as functions of ω. Find the Fourier series coefficients b_k of the output $y(t)$. Use your computer program of (c) to plot an approximation to the output signal over two periods by summing the first 50 harmonic components of $y(t)$. Discuss your results.

Problem 4.9

Consider the sawtooth signal $y(t)$ in Figure 4.13.

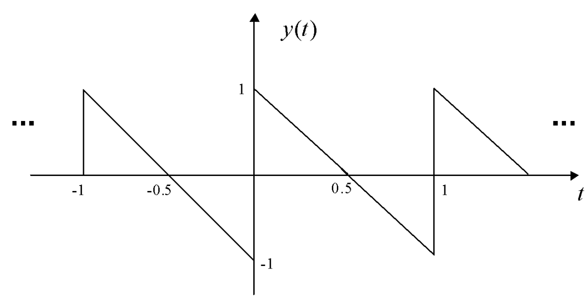

Figure 4.13: Periodic sawtooth signal in Problem 4.9.

(a) Compute the Fourier series coefficients of $y(t)$ using a direct calculation.

Answer:

The Fourier series coefficients are:

$$a_0 = \frac{1}{T}\int_0^1 x(t)dt = 0$$

For $k \neq 0$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_0^1 x(t)e^{-jk2\pi t}dt = \int_0^1 (1-2t)e^{-jk2\pi t}dt \\
&= \frac{-1}{jk2\pi}\left[(1-2t)e^{-jk2\pi t}\right]_0^1 - \frac{1}{jk\pi}\int_0^1 e^{-jk2\pi t}dt \\
&= \frac{-1}{jk2\pi}\left[-e^{-jk2\pi} - 1\right] + \frac{1}{2(jk\pi)^2}\left[e^{-jk2\pi t}\right]_0^1 \\
&= -j\frac{1}{k\pi} - \frac{1}{2(k\pi)^2}\left[e^{-jk2\pi} - 1\right] \\
&= \frac{-j}{k\pi}
\end{aligned}
$$

(b) Compute the Fourier series coefficients of $y(t)$ using properties of Fourier series, your result of Problem 4.8(a) for $x(t)$, and the fact that $y(t) = x(t) - x(-t)$.

Answer:

We can use the following property: $x(-t) \leftrightarrow a_{-k}$. Thus, the Fourier series coefficients b_k of $y(t)$ can be obtained as follows.

$$
\begin{aligned}
b_k = a_k - a_{-k} &= \frac{-j}{k2\pi} + \frac{\left[1-(-1)^k\right]}{2(k\pi)^2} - \frac{j}{k2\pi} - \frac{\left[1-(-1)^{-k}\right]}{2(-k\pi)^2} \\
&= \frac{-2j}{k2\pi} = \frac{-j}{k\pi}
\end{aligned}
$$

Problem 4.10

Compute and sketch (magnitude and phase) the Fourier series coefficients of the following signals:

(a) Signal $x(t)$ shown in Figure 4.14.

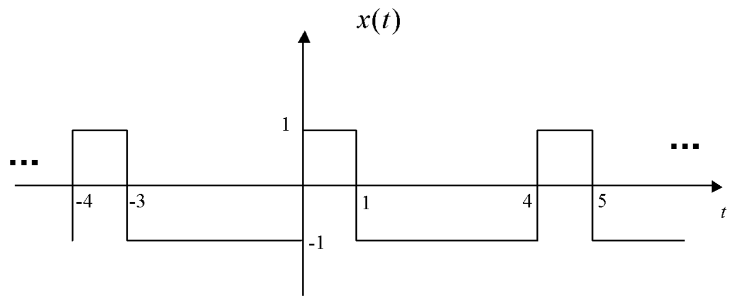

Figure 4.14: Periodic rectangular signal in Problem 4.10(a).

(b) $x(t) = \sin(10\pi t) + \cos(20\pi t)$