Solutions to Problems in Chapter 16

Problems with Solutions

Problem 16.1

Consider the DSB-AM/WC signal:

$$y(t) = [1 + 0.2x(t)]\cos(2\pi 10^5 t),$$

where the periodic modulating signal is $x(t) = \left|\sin(1000\pi t)\right|$.

(a) Sketch the modulating signal $x(t)$ and the spectrum of the modulated signal $Y(j\omega)$.

Answer:

The modulating signal $x(t)$ sketched in Figure 16.1 is a full-wave rectified sine wave.

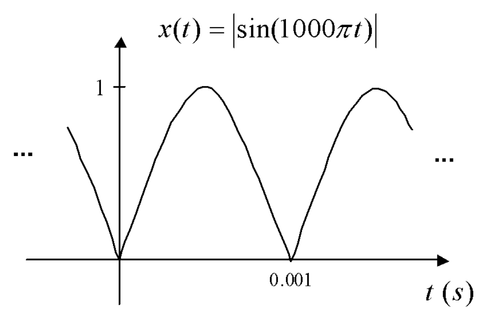


Figure 16.1: Modulating signal of Problem 16.1.

Let $T_1 = 0.001$ be the fundamental period of the signal $x(t)$, let $\omega_1 = 2\pi/T_1$ be its fundamental frequency, and $\omega_0 = 1000\pi = 0.5\omega_1$ be the fundamental frequency of $\sin(1000\pi t)$. The spectrum of the modulated signal is obtained by first computing the Fourier series coeffcients of the message signal:

$$
\begin{aligned}
a_k &= \frac{1}{T_1}\int_0^{T_1} v(t)e^{-jk\omega_1 t}dt \\
&= \frac{1}{T_1}\int_0^{T_1} \sin(\omega_0 t)e^{-jk\omega_1 t}dt \\
&= \frac{1}{2jT_1}\int_0^{T_1}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)e^{-jk\omega_1 t}dt \\
&= \frac{1}{2jT_1}\int_0^{\frac{\pi}{\omega_0}}\left(e^{j\omega_0(1-2k)t} - e^{-j\omega_0(1+2k)t}\right)dt \\
&= \frac{\omega_0}{2\pi j}\left[\frac{-2}{j\omega_0(1-2k)} - \frac{2}{j\omega_0(1+2k)}\right] \\
&= \frac{2}{\pi(1-4k^2)}
\end{aligned}
$$

Thus, the spectrum $X(j\omega)$ is given by:

$$
\begin{aligned}
X(j\omega) &= 2\pi\sum_{k=-\infty}^{+\infty} a_k\delta(\omega - k\omega_1) \\
&= \sum_{k=-\infty}^{+\infty}\frac{4}{(1-4k^2)}\delta(\omega - k2000\pi)
\end{aligned}
$$

and the spectrum of the modulated signal given below is sketched in Figure 16.2.

$$
Y(j\omega) = \left[\sum_{k=-\infty}^{+\infty}\frac{2}{(1-4k^2)}\delta(\omega - 200000\pi - k2000\pi) + \sum_{k=-\infty}^{+\infty}\frac{2}{(1-4k^2)}\delta(\omega + 200000\pi - k2000\pi)\right]
$$

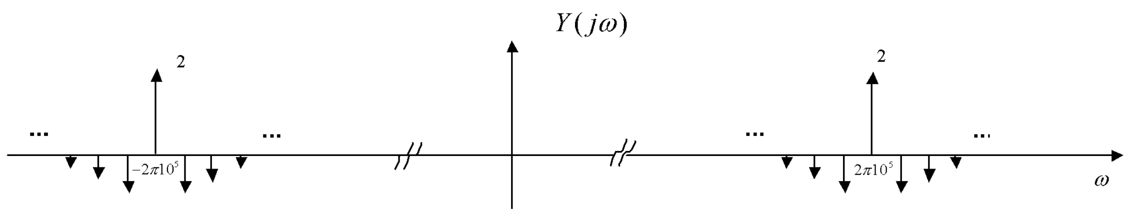


Figure 16.2: Spectrum of modulated signal of Problem 16.1.

(b) Design an envelope detector to demodulate the AM signal. That is, draw a circuit diagram of the envelope detector and compute the values of the circuit components. Justify all of your approximations and assumptions. Provide rough sketches of the carrier signal, the modulated signal and the signal at the output of the detector. What is the modulation index m of the AM signal?

Answer:

An envelope detector can be implemented with the following simple RC circuit with a diode.

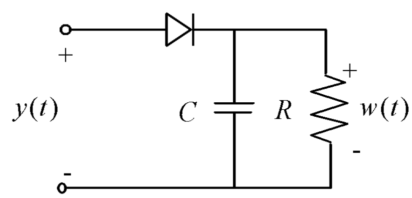


Figure 16.3: Envelope detector in Problem 16.1.

The output voltage of the detector, when it goes from one peak at voltage v_1 to the next when it intersects the modulated carrier at voltage v_2 after approximately one period $T = 10\mu s$ of the carrier, is given by: $v_2 \cong v_1 e^{-T/RC}$. Since the time constant $\tau = RC$ of the detector should be large with respect to $T = 10\mu s$, we can use a first-order approximation of the exponential such that $v_2 \cong v_1(1 - T/RC)$. This is a line of negative slope $-\frac{v_1}{RC}$ between the initial voltage v_1 and the final voltage v_2 so that:

$$\frac{v_2 - v_1}{T} \cong -v_1/RC.$$

This slope must be "more negative" than the maximum negative slope of the envelope of $y(t)$, thus we have to solve the following minimization problem:

$$\min_t \frac{d(0.2\sin(1000\pi t))}{dt} = -200\pi$$

Taking the worst-case $v_1 = 1$, we must have:

$$-\frac{1}{RC} < -200\pi$$
$$\Leftrightarrow$$
$$RC < \frac{1}{200\pi} = 0.00159$$

We could take $R = 1k\Omega,\ C = 1\mu F$ to get $RC = 0.001$.

Let K be the maximum amplitude of $0.2x(t)$, i.e., $|0.2x(t)| < 0.2 = K$ and let $A = 1$. The modulation index m is computed as $m = K/A = 0.2$.

Problem 16.2

The system shown in Figure 16.4 demodulates the noisy modulated continuous-time signal $y_n(t) = y(t) + n(t)$ composed of the sum of:

- signal $y(t)$ which is a lower SSB-AM/SC of $x(t)$ (assume a magnitude of one half that of $X(j\omega)$),
- a noise signal $n(t)$.

The carrier signal is $\cos(\omega_c t)$ where $\omega_c = 100000\pi$ rd/s. The antialiasing filter $H_a(j\omega)$ is a perfect unity-gain lowpass filter with cutoff frequency ω_a.

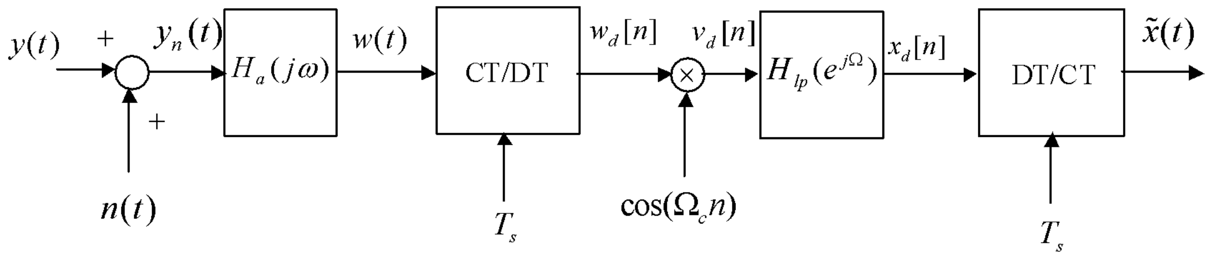


Figure 16.4: Discrete-time SSB-AM demodulator in Problem 16.2.

The modulating signal $x(t)$ has a triangular spectrum $X(j\omega)$ as shown inFigure 16.5. The spectrum $N(j\omega)$ of the noise signal is also shown in Figure 16.5.

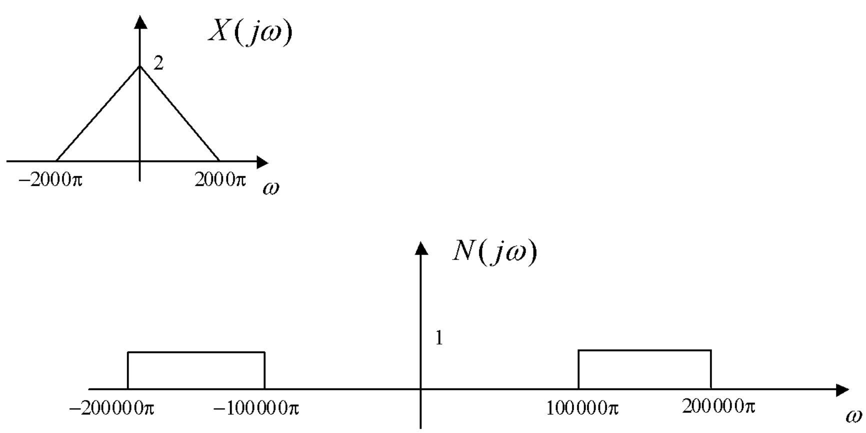


Figure 16.5: Fourier transforms of signal and noise in Problem 16.2.

(a) Find the minimum antialiasing filter's cutoff frequency ω_a that will avoid any unrepairable distortion of the modulated signals due to the additive noise $n(t)$. Sketch the spectra $Y_n(j\omega)$ and

$W(j\omega)$ of signals $y_n(t)$ and $w(t)$ for the frequency ω_a that you found. Indicate the important frequencies and magnitudes on your sketch.

Answer:

Minimum cutoff frequency of antialiasing filter: $\omega_a = 100000\pi$. The spectra $Y_n(j\omega)$ and $W(j\omega)$ are sketched in Figure 16.6.

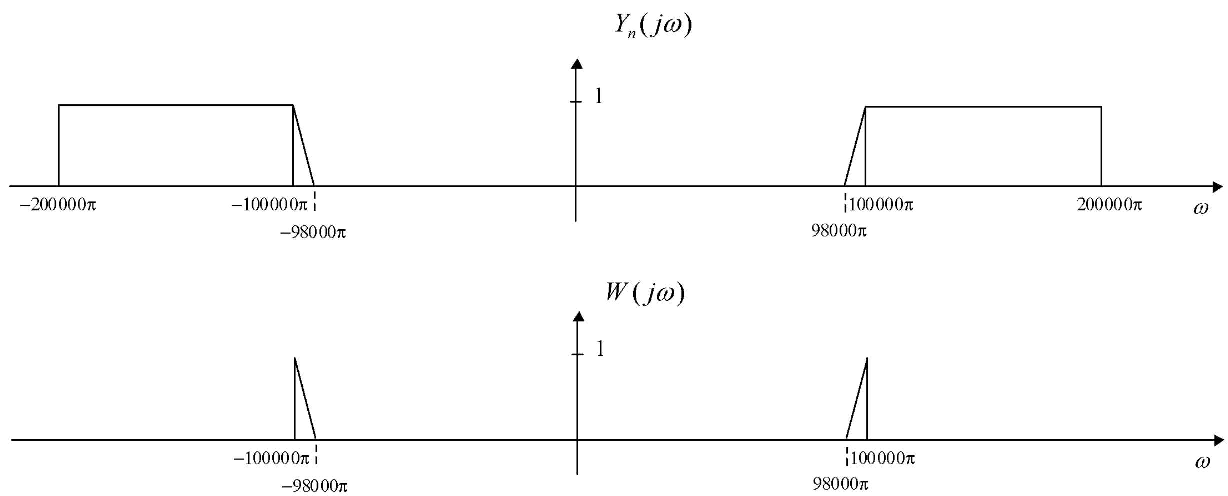


Figure 16.6: Fourier transforms of signal and noise in Problem 16.2.

(b) Find the minimum sampling frequency $\omega_s = \dfrac{2\pi}{T_s}$ and its corresponding sampling period T_s that would allow perfect reconstruction of the modulated signal. Give the corresponding cutoff frequency Ω_1 of the perfect unity-gain lowpass filter and the demodulation frequency Ω_c (what does the discrete-time synchronous demodulation amounts to here?) Using these frequencies, sketch the spectra $W_d(e^{j\Omega}), V_d(e^{j\Omega}), X_d(e^{j\Omega})$.

Answer:

Sampling frequency: $\omega_s = 2\omega_a = 200000\pi,\ T_s = \dfrac{2\pi}{\omega_s} = \dfrac{1}{100000} = 10^{-5}\,\text{s}$.

Demodulation frequency: $\Omega_c = \pi$. The synchronous demodulation amounts to multiplying the signal by $(-1)^n$.

Cutoff frequencies: $\Omega_1 = 2000\pi T_s = 0.02\pi$. The DTFT's $W_d(e^{j\Omega}), V_d(e^{j\Omega}), X_d(e^{j\Omega})$ are shown in Figure 16.7.

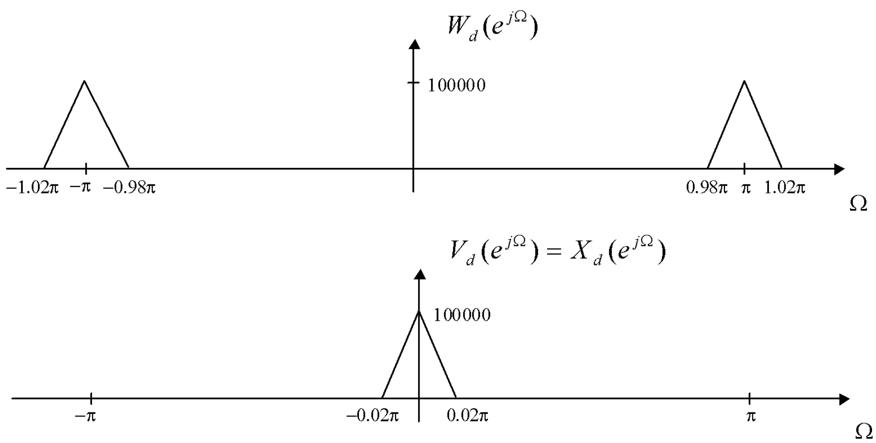
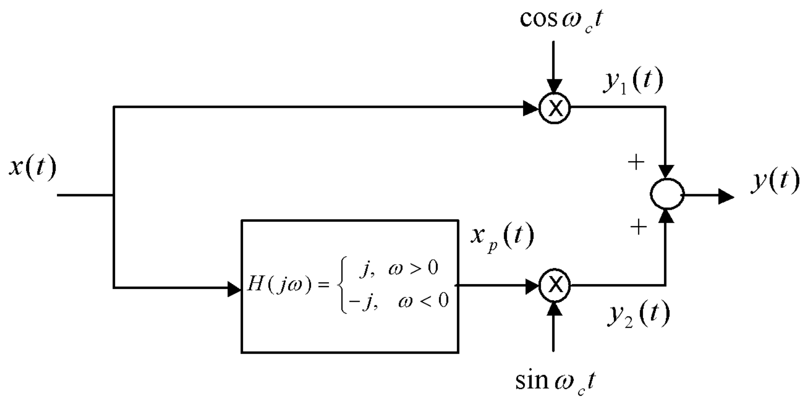


Figure 16.7: Fourier transforms of signal and noise in Problem 16.2.

Problem 16.3

Discrete-time SSB-AM modulator

You have to design an upper SSB modulator in discrete-time as an implementation of the continuous-time modulator shown in Figure 16.8.



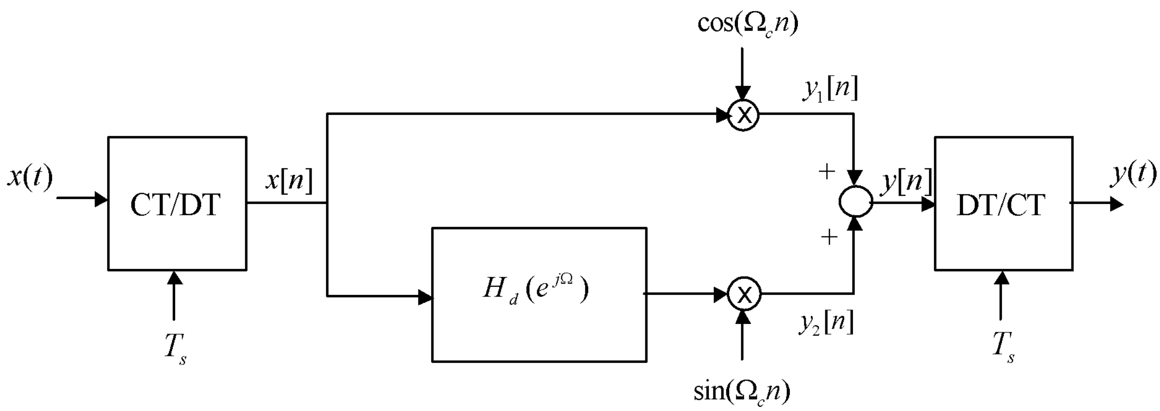


Figure 16.8: Continuous-time and discrete-time upper SSB-AM modulators of Problem 16.3.

The modulating signal has the Fourier transform shown in Figure 16.9, and the carrier frequency is $\omega_c = 2\,000\,000\pi$ rd/s (1MHz).

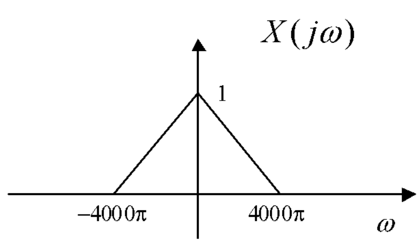


Figure 16.9: Spectrum of the message signal in Problem 16.3.

Design the modulation system (with ideal components) for the slowest possible sampling rate. Find an expression for the ideal discrete-time phase shift filter $H_d(e^{j\Omega})$ and compute its impulse response $h_d[n]$. Explain how you could implement an approximation to this ideal filter, and describe modifications to the overall system so that it would work in practice.

Answer:

First, the ideal phase-shift filter should have the following frequency response:

$$H_d(e^{j\Omega}) = \begin{cases} j, & 0 < \Omega < \pi \\ -j, & -\pi < \Omega < 0 \end{cases}$$

The inverse Fourier transform yields:

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Omega})e^{j\Omega n}\,d\Omega = \frac{1}{\pi n}[1-(-1)^n].$$

To compute the slowest sampling frequency, we start from the desired upper SSB-AM signal at the output, whose bandwidth is $\omega_M = 2\,004\,000\pi$ which should correspond to the highest discrete-time frequency $\Omega = \pi$. Thus, we can compute the sampling period from the relationship $\frac{\Omega}{T} = \omega \;\Rightarrow\; \frac{\pi}{\omega_M} = T$ which yields $T = 4.99\times 10^{-7}\,s$.

This system could be implemented in practice using an FIR approximation to the ideal phase-shift filter. Starting from $h_d[n]$, a windowed impulse response of length $M+1$ time-delayed by $M/2$ to make it causal would work. However, the resulting delay of $M/2$ samples introduced in the lower path of the modulator should be balanced out by the introduction of an equivalent delay block $z^{-M/2}$ in the upper path.

Exercises

Problem 16.4

The superheterodyne receiver consists of taking the product of the AM radio signal with a carrier whose frequency is tuned by a variable-frequency oscillator. Then, the resulting signal is filtered by a fixed bandpass filter centered at the intermediate frequency (IF) ω_{IF}. The goal is to have a fixed high-quality filter, which is cheaper than a high-quality tunable filter. The output of the IF bandpass filter is then demodulated with an oscillator at constant frequency to tune in to your favorite radio station. Suppose that the input signal is an AM wave of bandwidth 10kHz and carrier frequency ω_c that may lie anywhere in the range 0.535 MHz -1.605MHz (typical of AM radio broadcasting). Find the range of tuning that must be provided in the local oscillator ω_{VCO} in order to achieve this requirement.

Problem 16.5

In the operation of the superheterodyne receiver of Figure 16.22, it should be clear from Figure 16.23 that even though the receiver is tuned in to station 1, the "ghost" spectrum of a radio station at a carrier frequency higher than ω_{c1} could appear in the passband of the IF bandpass filter if an additional filter is not implemented. Determine that ghost carrier frequency.

Answer:

Ghost carrier frequency: $\omega_{cx} = \omega_{c1} + 2\omega_{IF}$.

Problem 16.6

Design an envelope detector to demodulate the AM signal:

$$y(t) = [1 + 0.4x(t)]\cos(2\pi 10^3 t),$$

where $x(t)$ is the periodic modulating signal shown in Figure 16.10. That is, draw a circuit diagram of the envelope detector and compute the values of the circuit components. Justify all of your approximations and assumptions. Provide rough sketches of the carrier signal, the modulated signal and the signal at the output of the detector. What is the modulation index m of the AM signal?

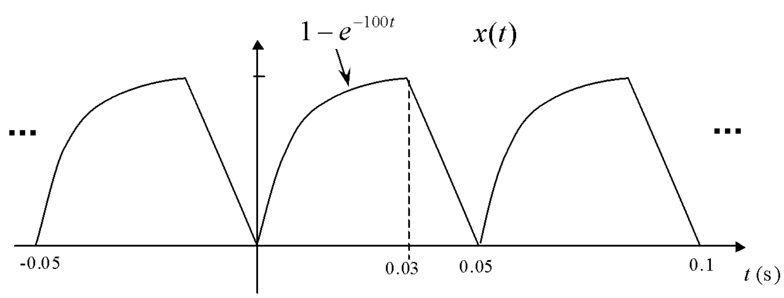


Figure 16.10: Modulating signal in Problem 16.6.

Problem 16.7

The sampled-data system shown in Figure 16.11 provides amplitude modulation of the continuous-time signal $x(t)$. The impulse train signal is $p[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$. The antialiasing filter $H_a(j\omega)$ is an ideal unity-gain lowpass filter with cutoff frequency ω_a, and the perfect discrete-time highpass filter $H_{hp}(e^{j\Omega})$ has gain N (which is also the period of $p[n]$) and cutoff frequency $\Omega_1 = 0.6\pi$.

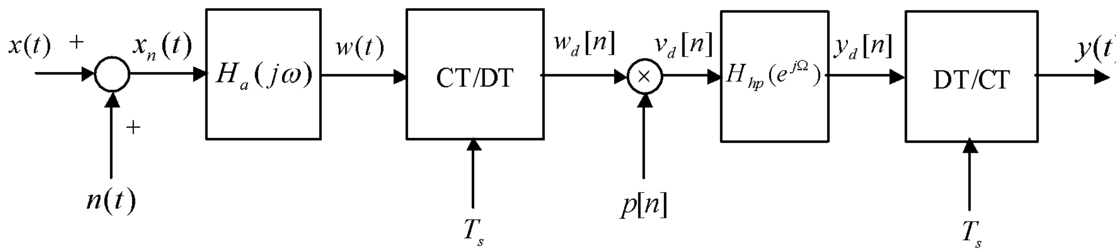


Figure 16.11: Modulation system in Problem 16.7.

The modulating (or message) signal $x(t)$ has spectrum $X(j\omega)$ as shown in Figure 16.12. The spectrum of the noise signal $N(j\omega)$ is also shown in the figure.

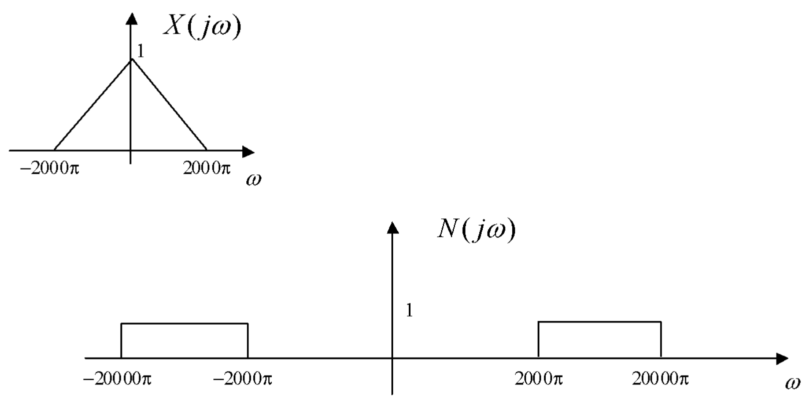
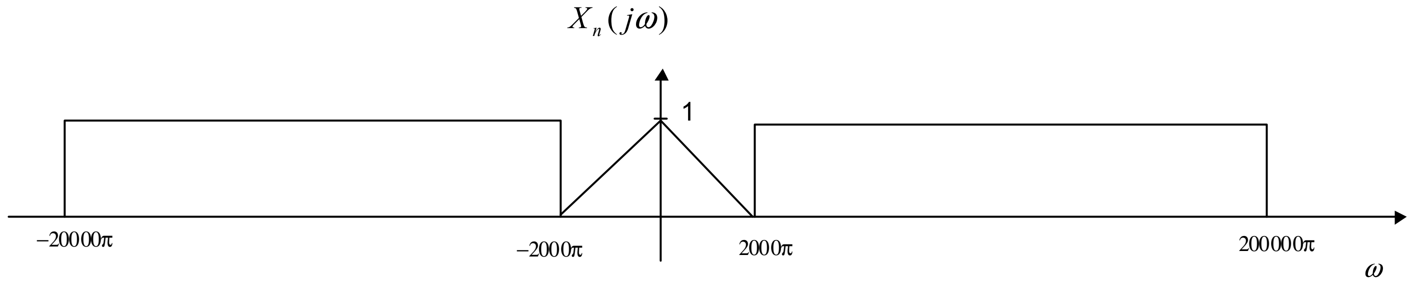


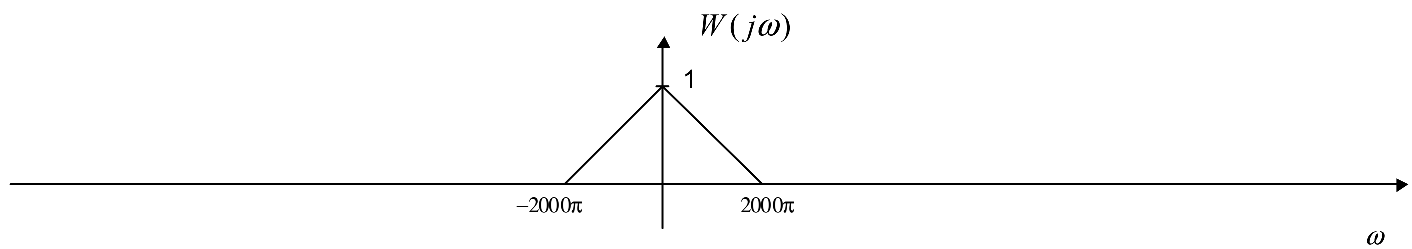
Figure 16.12: Fourier transforms of signal and noise in Problem 16.7.

(a) Find the minimum antialiasing filter's cutoff frequency ω_a that will avoid any unrepairable distortion of the signal $x(t)$ due to the additive noise $n(t)$. Sketch the spectra $X_n(j\omega)$ and $W(j\omega)$ of signals $x_n(t)$ and $w(t)$ for the frequency ω_a that you found. Indicate the important frequencies and magnitudes on your sketches.

Answer:

Minimum $\omega_a = 2000\pi$.



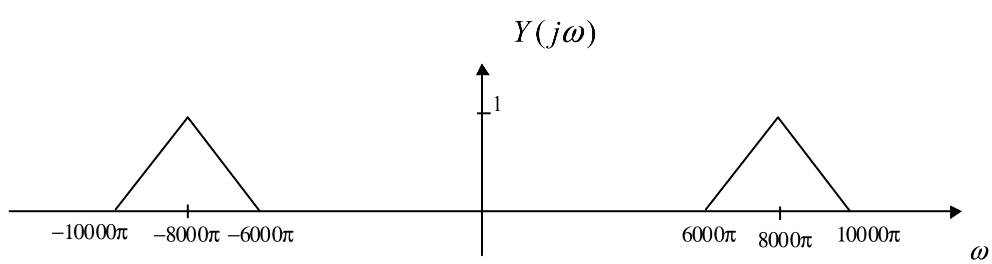


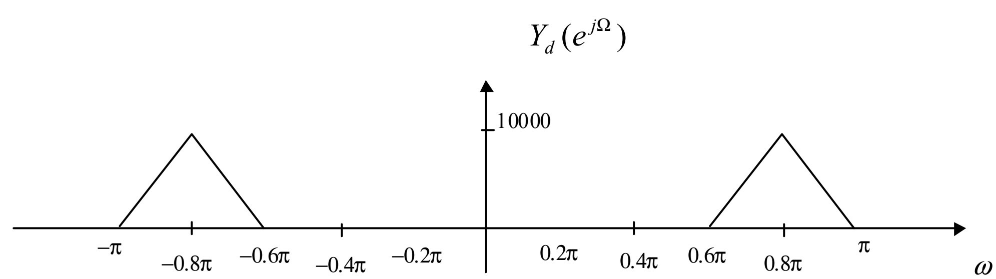
(b) Find the minimum sampling frequency $\omega_s = \dfrac{2\pi}{T_s}$ and its corresponding sampling period T_s that would allow amplitude modulation of the signal $x(t)$ at a carrier frequency of $\omega_c = 8000\pi$ rd/s. Find the corresponding discrete-time sampling period N of $p[n]$ for the system to work. Using these frequencies, sketch the spectra $W_d(e^{j\Omega}), V_d(e^{j\Omega}), Y_d(e^{j\Omega})$ and $Y(j\omega)$.

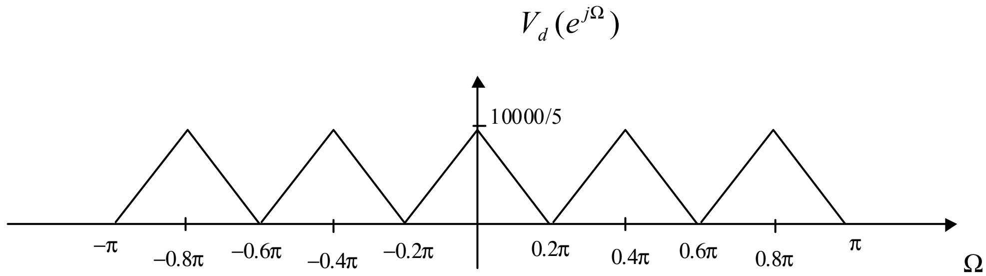
Answer:

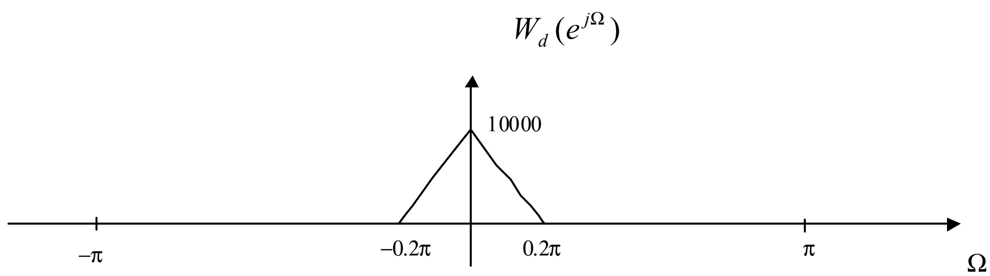
The minimum sampling frequency has to be consistent with the bandwidth of the AM signal at the output. The highest possible frequency components obtained at the output are at $0.5\omega_s$, and this frequency can be made equal to the bandwidth of the AM signal which is $\omega_c + \omega_m = 8000\pi + 2000\pi = 10000\pi$ rd/s. Therefore the minimum sampling frequency is $\omega_s = 20000\pi$ rd/s, for which $T_s = \frac{2\pi}{\omega_s} = \frac{1}{10000} = 10^{-5} s$.

With $\omega_s = 20000\pi$ rd/s we have the modulation frequency $\omega_c = 8000\pi$ mapping to the discrete-time frequency $\Omega_c = 8000\pi T = 0.8\pi$, while the frequency band occupied by the spectrum of the modulated signal should be $[\frac{3}{5}\pi, \pi]$ and $[-\pi, -\frac{3}{5}\pi]$. Thus, we can fit five copies of the spectrum of the modulating signal $x(t)$ in an interval of 2π, and hence we can use a period of $N = 5$. Working backwards is easier to get the spectra:









Problem 16.8

The system shown in Figure 16.13 is a lower single-sideband, suppressed-carrier AM modulator implemented in discrete time. The message signal $x(t)$ is corrupted by an additive noise $n(t)$: $x_n(t) = x(t) + n(t)$ before sampling. We want the modulator to operate at a carrier frequency $\omega_c = 2\pi \times 10^6$ rd/s (1MHz).

The antialiasing filter $H_a(j\omega)$ is a perfect unity-gain lowpass filter with cutoff frequency ω_a. The antialiased signal $w(t)$ is first converted to a discrete-time signal $w_d[n]$ via the CT/DT operator. Signal $w_d[n]$ is modulated with frequency Ω_c and bandpass filtered by $H_{bp}(e^{j\Omega})$ to create the lower sidebands. Finally, the D/C operator produces the continuous-time lower SSB/SC AM signal $y(t)$.

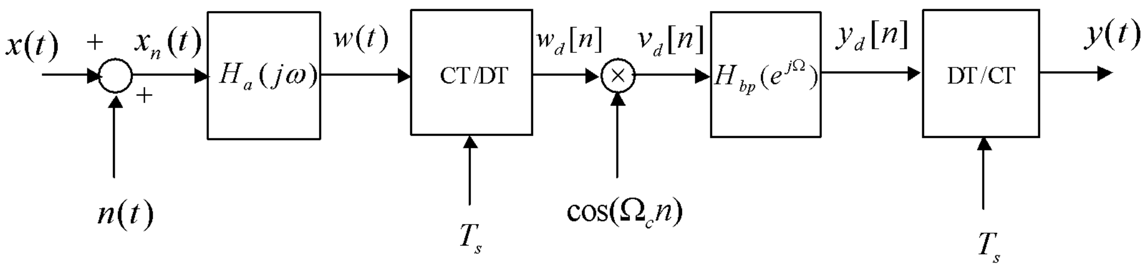
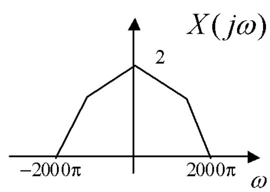


Figure 16.13: Discrete-time SSB modulator of Problem 16.8.

The Fourier transforms of the modulating signal $x(t)$ and the noise $X(j\omega)$ and $N(j\omega)$ are shown in Figure 16.14.



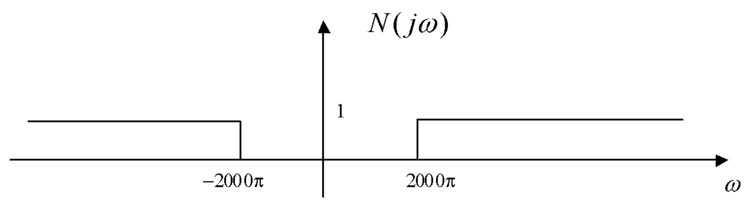


Figure 16.14: Fourier transforms of signal and noise in Problem 16.8.

(a) The antialiasing filter's cutoff frequency is given as $\omega_a = 3000\pi$. Sketch the spectra $X_n(j\omega)$ and $W(j\omega)$ of signals $x_n(t)$ and $w(t)$. Indicate the important frequencies and magnitudes on your sketch.

(b) Find the minimum sampling frequency ω_s and corresponding sampling period T_s that will produce the required modulated signal. Find the discrete-time modulation frequency Ω_c that will result in a continuous-time carrier frequency $\omega_c = 2\pi \times 10^6$ rd/s. Give the cutoff frequencies $\Omega_1 < \Omega_2$ of the bandpass filter to obtain a lower SSB-AM/SC signal. Using these frequencies, sketch the spectra $W_d(e^{j\Omega}), V_d(e^{j\Omega}), Y_d(e^{j\Omega})$ and $Y(j\omega)$.

Problem 16.9

A DSB/SC AM signal $y(t)$ is generated with a carrier signal $c_{transm}(t) = \cos(\omega_c t)$. Suppose that the oscillator of the synchronous AM demodulator at the receiver in Figure 16.15 is slightly off:

$$c_{rec}(t) = \cos((\omega_c + \Delta)t).$$

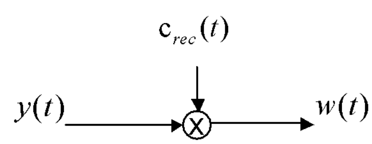


Figure 16.15: Synchronous demodulator at the receiver in Problem 16.9.

(a) Denoting the modulating signal as $x(t)$, write the expression for $w(t)$, and for $z(t)$ after perfect unity-magnitude lowpass filtering at ω_c:

Answer:

$$\begin{aligned} w(t) &= x(t)\cos(\omega_c t)\cos((\omega_c + \Delta)t) \\ &= 0.5x(t)[\cos(2\omega_c + \Delta)t + 0.5\cos(-\Delta t)] \\ &= 0.5x(t)\cos(2\omega_c + \Delta)t + 0.5x(t)\cos(\Delta t) \end{aligned}$$

$$z(t) = 0.5x(t)\cos(\Delta t)$$

(b) Let $\omega_c = 2\pi 10^6$, $\Delta = 2\pi 10^3$ rd/s. Plot $z(t)$ for the modulating signal $x(t) = \sin(20000\pi t)$.

Answer: