

Solutions to Problems in Chapter 15

Problems with Solutions

Problem 15.1

The system shown in Figure 15.1 filters the continuous-time noisy signal $x_n(t) = x(t) + n(t)$ composed of the sum of a signal $x(t)$ and a noise $n(t)$.

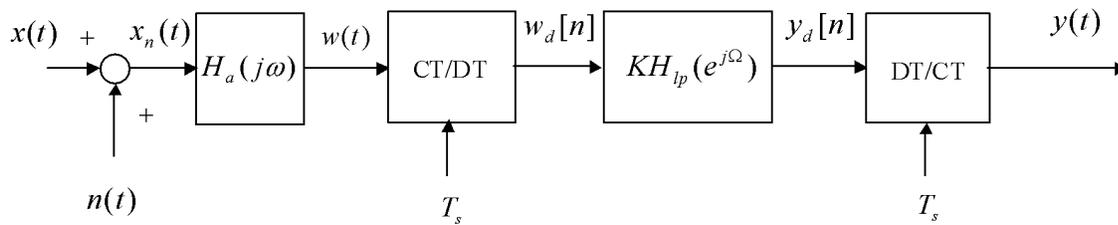


Figure 15.1: Sampled-data system with antialiasing filter of Problem 15.1.

The signal and noise have their respective spectrum $X(j\omega)$, $N(j\omega)$ shown in Figure 15.2. The frequency response of the antialiasing low-pass filter is also shown in the figure.

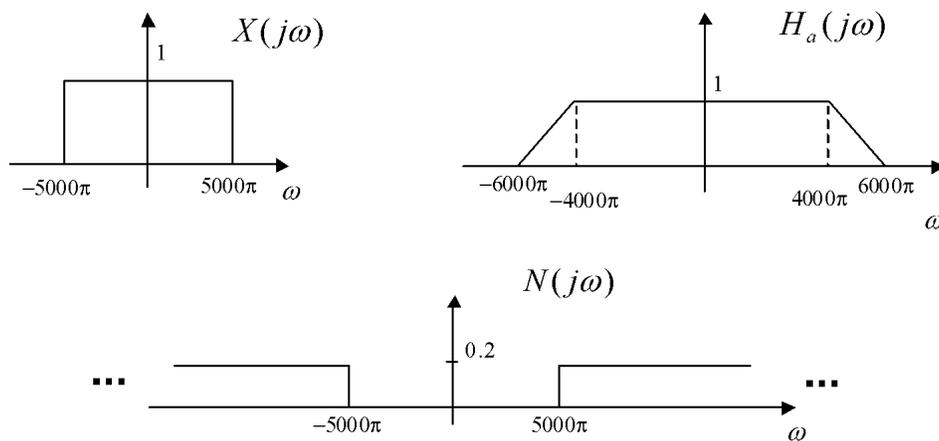


Figure 15.2: Antialiasing filter and signal and noise spectra in Problem 15.1.

(a) Let the sampling frequency be $\omega_s = 11000\pi$ rd/s. Sketch the spectrum $W(j\omega)$ of the signal $w(t)$. Also sketch the spectrum $W_d(e^{j\Omega})$. Indicate the important frequencies and magnitudes on your sketch. Discuss the results.

Answer:

The Nyquist frequency is $\frac{\omega_s}{2} = 5500\pi$. The spectra $W(j\omega)$ and $W_d(e^{j\Omega})$ are shown in Figure 15.3.

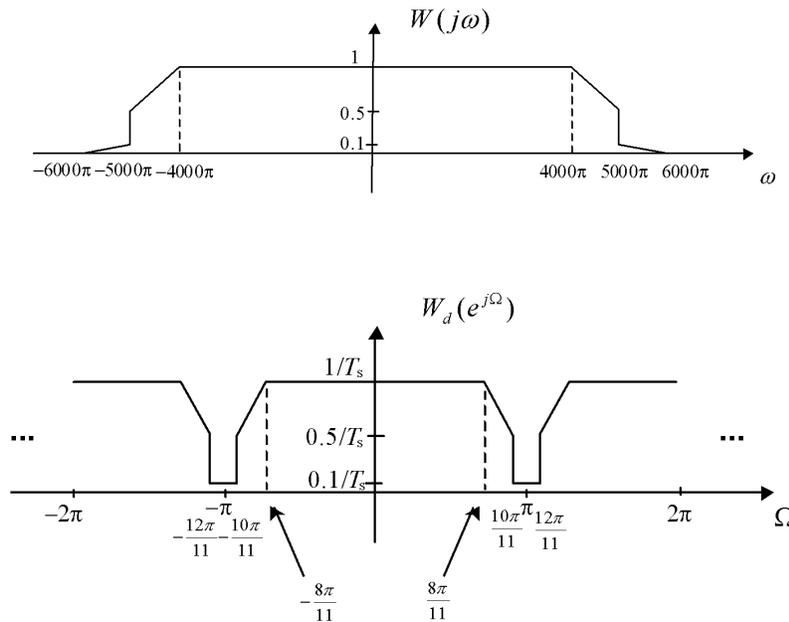


Figure 15.3: FT of output of antialiasing filter and DTFT of signal after CTDT, Problem 15.1.

(b) Design an ideal discrete-time lowpass filter $KH_{lp}(e^{j\Omega})$ (give its cutoff frequency Ω_c and gain K) so that the signal $x(t)$ (possibly distorted by linear filtering) can be approximately recovered at the output. Sketch the spectrum $Y_d(e^{j\Omega})$ with this filter.

Answer:

From Figure 15.3, the ideal lowpass would have a cutoff frequency $\Omega_c = \frac{10\pi}{11}$ in order to remove the remaining noise, and unity gain since the DT/CT operator already contains a gain of T_s . The frequency response of the low-pass filter $H_{lp}(e^{j\Omega})$ is shown in Figure 15.4, and the DTFT of the output of the filter $Y_d(e^{j\Omega})$ is in Figure 15.5.

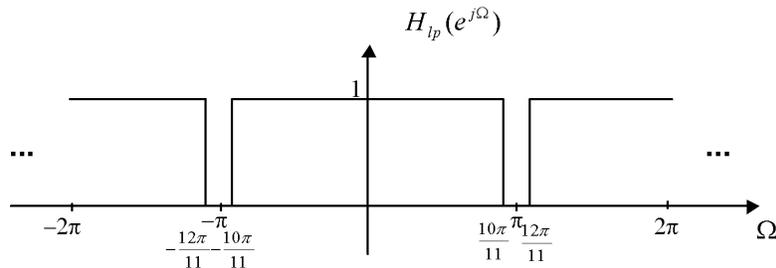


Figure 15.4: Frequency response of ideal low-pass filter, Problem 15.1.

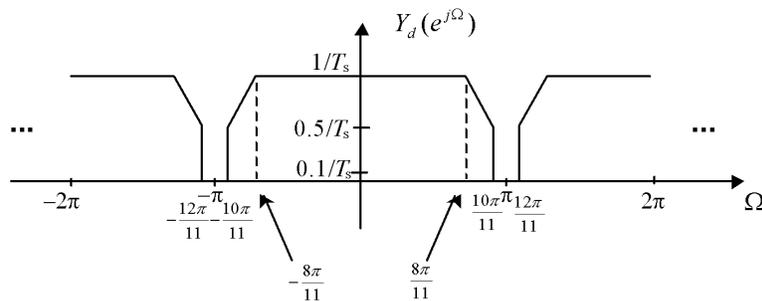


Figure 15.5: DTFT of output of ideal low-pass filter, Problem 15.1.

(c) Sketch the spectrum $Y(j\omega)$ and compute the ratio of linear distortion in the output $y(t)$ with respect to the input signal, i.e., compute $100 \times \frac{E_{error}}{E_x}$ in percent, where

E_{error} : energy of $x(t) - y(t)$, E_x : energy of $x(t)$.

Answer:

The Fourier transforms of the output $Y(j\omega)$ and the error signal $X(j\omega) - Y(j\omega)$ are sketched in Figure 15.6.

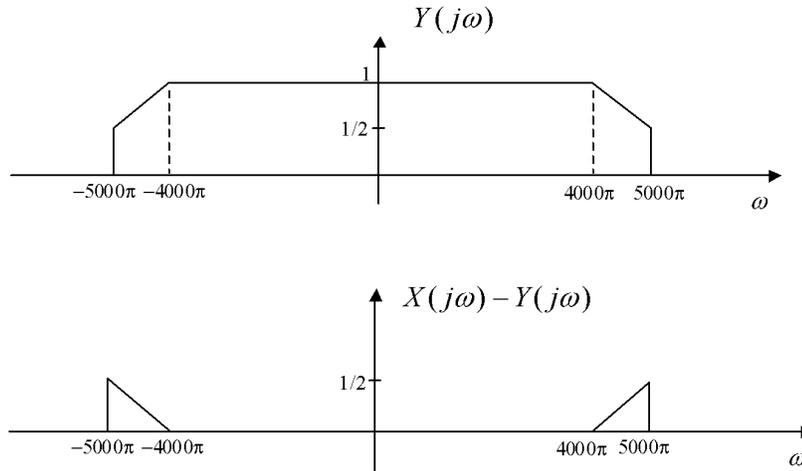


Figure 15.6: Fourier transforms of continuous-time output signal and error signal, Problem 15.1.

The energy of the signal and the energy of the error signal are computed using the Parseval Equality as follows:

$$\begin{aligned}
 E_x &= \frac{1}{2\pi} \int_{-5000\pi}^{5000\pi} |X(j\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-5000\pi}^{5000\pi} d\omega = 5000 \\
 E_{error} &= 2 \frac{1}{2\pi} \int_{4000\pi}^{5000\pi} |X(j\omega) - Y(j\omega)|^2 d\omega = \frac{1}{\pi} \int_{4000\pi}^{5000\pi} \left| \frac{1}{2000\pi} (\omega - 4000\pi) \right|^2 d\omega \\
 &= \frac{1}{2000^2 \pi^3} \int_{4000\pi}^{5000\pi} (\omega - 4000\pi)^2 d\omega = \frac{1}{(3)(2000^2) \pi^3} \left[(\omega - 4000\pi)^3 \right]_{4000\pi}^{5000\pi} \\
 &= \frac{1}{(3)(2000^2) \pi^3} \left[(5000\pi - 4000\pi)^3 - 0 \right] = \frac{1000^3 \pi^3}{(3)(2000^2) \pi^3} = \frac{1000}{12} = 83.3
 \end{aligned}$$

Ratio of linear distortion:

$$\frac{E_{error}}{E_x} = \frac{1000/12}{5000} \times 100\% = 1.67\%$$

Problem 15.2

Consider the discrete-time system shown in Figure 15.7, where $\downarrow N$ represents decimation by N . The operator $\{\uparrow N\}_{lp}$ denotes upsampling by N followed by an ideal unity-gain lowpass filter with cutoff frequency $\frac{\pi}{N}$. This system transmits a signal $x[n]$ which comes in at a rate of 100 kilosamples/s over a channel with limited bit-rate capacity. At each time n , the 16-bit quantizer basically rounds off the real value $x_2[n]$ to a 16-bit value $y[n]$.

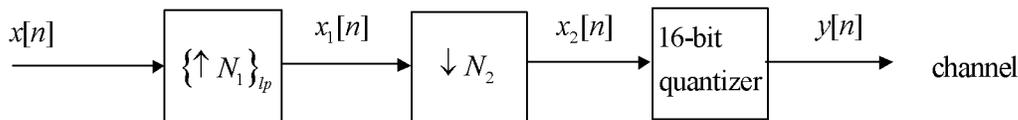


Figure 15.7: Block diagram of discrete-time system, Problem 15.2.

(a) Given that the input signal $x[n]$ is bandlimited to 0.13π radians, find the integers N_1, N_2 that will minimize the sample rate for transmission over the channel. Give the bit rate that you obtain.

Answer:

Bandwidth of signal $x[n]$: $\omega_m = \frac{13\pi}{100} = \frac{13(2\pi)}{200}$. We want the DTFT of $x_2[n]$ to cover as much

as the frequency interval $[-\pi, \pi]$ as possible using upsampling and decimation. We can

upsample first by a factor $N_1 = 13$, which compresses the bandwidth to $\frac{\pi}{100}$, and then we can

decimate by a factor $N_2 = 100$ to expand the spectrum up to π . The resulting sample rate is

reduced by a factor $\frac{N_2}{N_1} = \frac{100}{13}$ to 13 kilosamples/s. With a 16-bit quantizer, the bit rate needed to transmit signal $y[n]$ is $16\text{bits/sample} \times 13000\text{samples/s} = 208000\text{ bits/s}$.

(b) With the values obtained in (a), assume a triangular spectrum for $x[n]$ of unit amplitude at dc, and sketch the spectra $X(e^{j\omega})$, $X_1(e^{j\omega})$, $X_2(e^{j\omega})$, indicating the important frequencies and magnitudes.

Answer:

The spectra $X(e^{j\omega})$, $X_1(e^{j\omega})$, $X_2(e^{j\omega})$ are sketched in Figure 15.8.

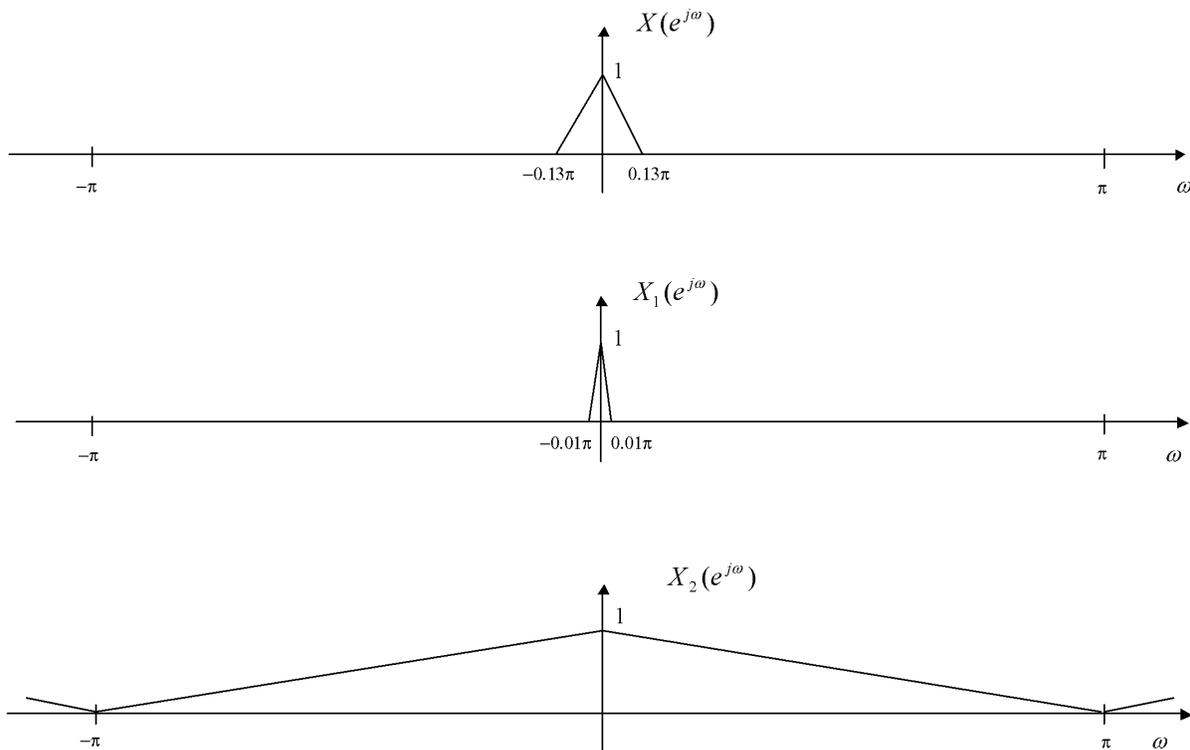


Figure 15.8: DTFTs of upsampled-filtered and decimated signals, Problem 15.2.

Exercises

Problem 15.3

The signals below are sampled with sampling period T_s . Determine the bounds on T_s that guarantee there will be no aliasing.

$$(a) x(t) = \cos(10\pi t) \frac{\sin(\pi t)}{2t}.$$

Answer:

This signal is an AM modulation of the impulse response of a lowpass filter with cutoff frequency π . The spectrum is shifted to $\pm 10\pi$ rad/s. Therefore, no aliasing will occur if $\omega_s > 2(11\pi)$, i.e., $T_s < \frac{1}{11}$. Also, No aliasing would result for lower sampling periods violating the sampling theorem, but keeping the original double-rectangle spectrum intact.

$$(b) x(t) = e^{-4t} u(t) * \frac{\sin(Wt)}{\pi t}$$

Answer:

The convolution results in the product of the two spectra. The spectrum of $e^{-4t} u(t)$ extends to $\pm\infty$, but the spectrum of $\frac{\sin(Wt)}{\pi t}$ is a perfect lowpass with bandwidth W . Hence, no aliasing will occur if $\omega_s > 2W$, i.e., $T_s < \frac{\pi}{W}$.

Problem 15.4

A continuous-time voltage signal lies in the frequency band $|\omega| < 5\pi$. This signal is contaminated by a large sinusoidal signal of frequency 120π . The contaminated signal is sampled at a sampling rate of $\omega_s = 13\pi$.

(a) After sampling, at what frequency does the interfering sinusoidal signal appear?

(b) Now suppose the contaminated signal is passed through an antialiasing filter consisting of an

RC circuit with frequency response $H(j\omega) = \frac{1}{RCj\omega + 1}$. Find the value of the time constant RC

required so that the contaminated sinusoid is attenuated by a factor of 1000 (60dB) prior to sampling.

Problem 15.5

Consider the sampling system in Figure 15.9 with an ideal unit-gain, lowpass antialiasing filter

$H_{lp}(j\omega)$ with cutoff frequency ω_c , and where the sampling frequency is $\omega_s = \frac{2\pi}{T_s}$.

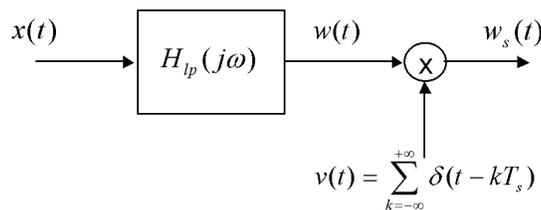


Figure 15.9: Sampling system in Problem 15.5.

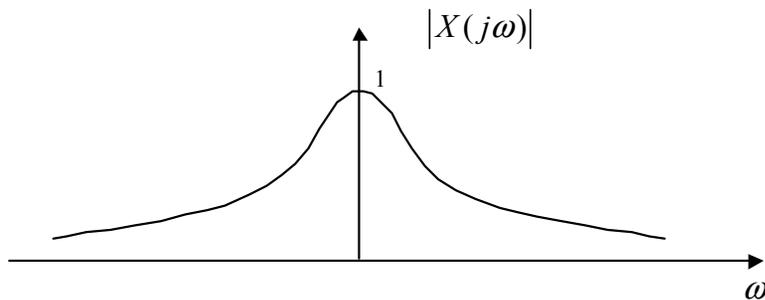
The input signal is $x(t) = e^{-t}u(t)$, and the sampling period is set at $T_s = \frac{\pi}{\sqrt{3}}$.

(a) Give a mathematical expression for $X(j\omega)$, the Fourier transform of the input signal, and sketch it (magnitude and phase.) Design the antialiasing filter (i.e., find its cutoff frequency) so that its bandwidth is maximized while avoiding aliasing of its output $w(t)$ in the sampling operation. With this value of ω_c , sketch the magnitudes of the Fourier transforms $W(j\omega)$ and $W_s(j\omega)$.

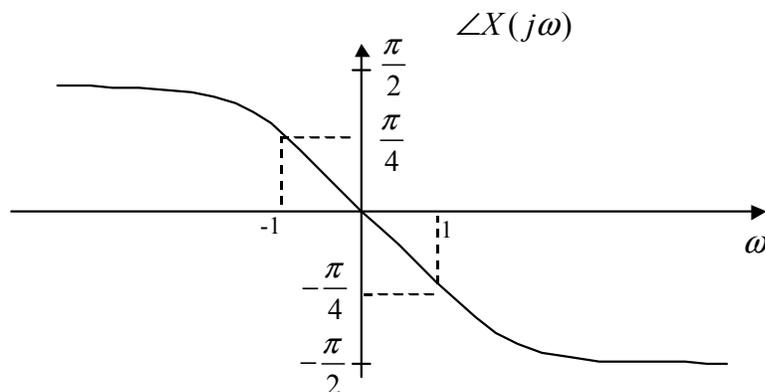
Answer:

The Fourier transform of the input signal $x(t) = e^{-t}u(t)$ is $X(j\omega) = \frac{1}{j\omega + 1}$.

Magnitude:

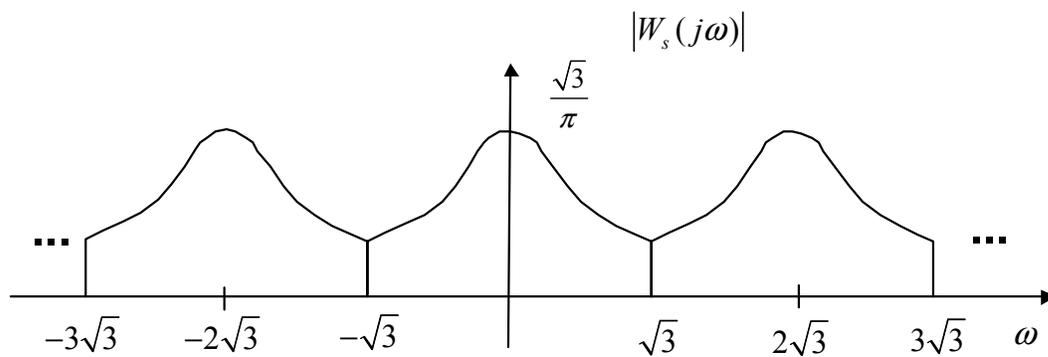
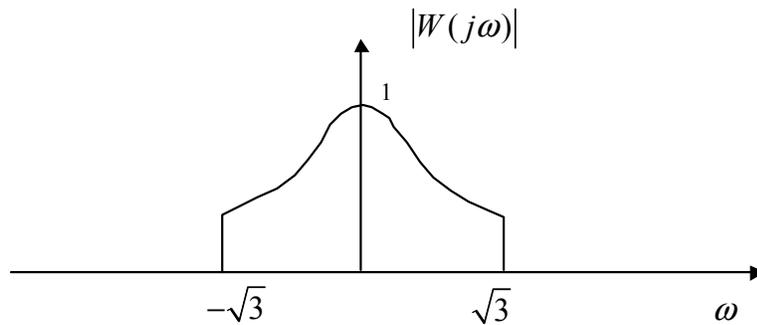


Phase:



The cutoff frequency of the antialiasing filter must be set equal to half of the sampling frequency to satisfy the sampling theorem. Therefore, $\omega_c = \frac{1}{2} \omega_s = \frac{1}{2} \frac{2\pi}{T_s} = \sqrt{3}$. With this cutoff frequency,

the magnitudes of $W(j\omega)$ and $W_s(j\omega)$ are shown below.



(b) Compute the ratio of total energies $r = 100 \frac{E_{\infty w}}{E_{\infty x}}$ in percent, where $E_{\infty w}$ is the total energy in

signal $w(t)$ and $E_{\infty x}$ is the total energy in signal $x(t)$. This ratio gives us an idea of how similar

$w(t)$ is to $x(t)$ before sampling. How similar is it? (Hint: $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$)

Answer:

The total energy in the input signal is easy to compute in the time domain:

$$E_{\infty x} = \int_0^{\infty} |e^{-t}|^2 dt = \int_0^{\infty} e^{-2t} dt = -\frac{1}{2} [e^{-2t}]_0^{\infty} = \frac{1}{2}$$

On the other hand, the total energy in the filtered signal is easier to compute in the frequency domain:

$$\begin{aligned} E_{\infty w} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |W(j\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \left| \frac{1}{1+j\omega} \right|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \left| \frac{1}{1+j\omega} \right|^2 d\omega = \frac{1}{2\pi} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{1+\omega^2} d\omega \\ &= \frac{1}{2\pi} [\arctan(\omega)]_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{2\pi} [\arctan(\sqrt{3}) - \arctan(-\sqrt{3})] \\ &= \frac{1}{2\pi} \left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right) \right] = \frac{1}{3} \end{aligned}$$

Ratio of total energies: $r = 100 \frac{E_{\infty w}}{E_{\infty x}} \% = 100 \frac{1/3}{1/2} \% = 67\%$. This ratio indicates that quite a bit of

energy is lost in the antialiasing filter and therefore $w(t)$ is not very similar to $x(t)$. To do a

better job, one would have to increase both the cutoff frequency of the filter and the sampling frequency.

Problem 15.6

Consider the sampling system in Figure 15.10 where the sampling frequency is $\omega_s = \frac{2\pi}{T_s}$.

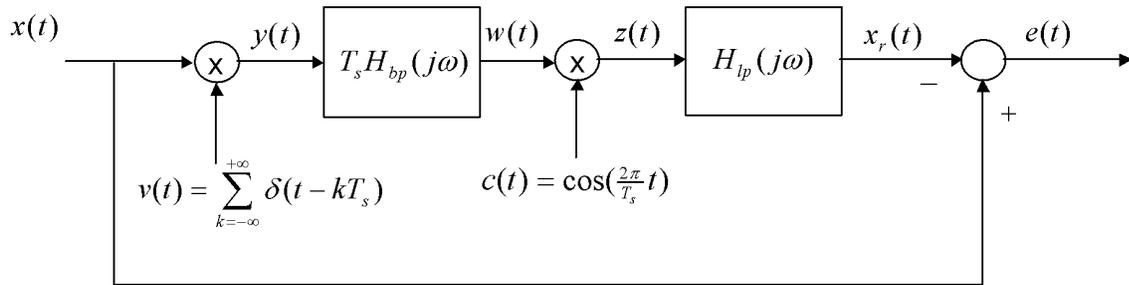


Figure 15.10: Sampling system in Problem 15.6.

The input signal is $x(t) = \frac{W^2}{2\pi} \text{sinc}^2\left(\frac{W}{2\pi}t\right)$ and the spectra of the ideal bandpass filter and the

ideal lowpass filter are shown in Figure 15.11.

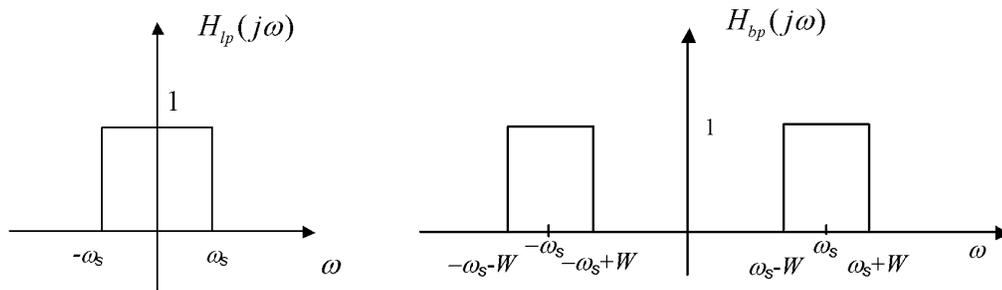


Figure 15.11: Frequency responses of ideal low-pass and band-pass filters in Problem 15.6.

(a) Compute and sketch $X(j\omega)$, the Fourier transform of the input signal. For what range of sampling frequencies $\omega_s = \frac{2\pi}{T_s}$ is the sampling theorem satisfied for the first sampler?

(b) Assume that the sampling theorem is satisfied with the slowest sampling frequency ω_s in the range found in (a). Sketch the spectra $Y(j\omega)$, $W(j\omega)$ and $Z(j\omega)$ for this case.

(c) Sketch the Fourier transforms $X_r(j\omega)$, $E(j\omega)$. Compute the total energy in the error signal $e(t) := x(t) - x_r(t)$.

Problem 15.7

Consider the sampling system of Figure 15.12, where the input signal is $x(t) = \frac{W}{\pi} \text{sinc}\left(\frac{W}{\pi}t\right)$

and the spectra of the ideal bandpass filter and the ideal lowpass filter are shown in Figure 15.13.

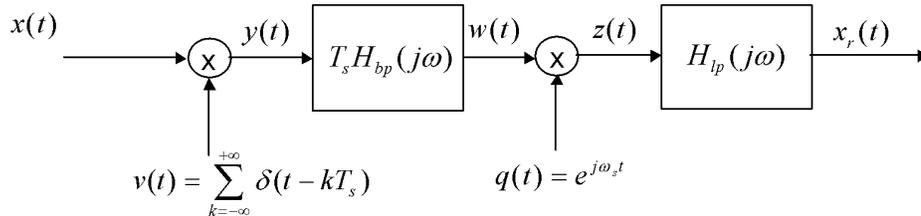


Figure 15.12: Sampling system in Problem 15.7.

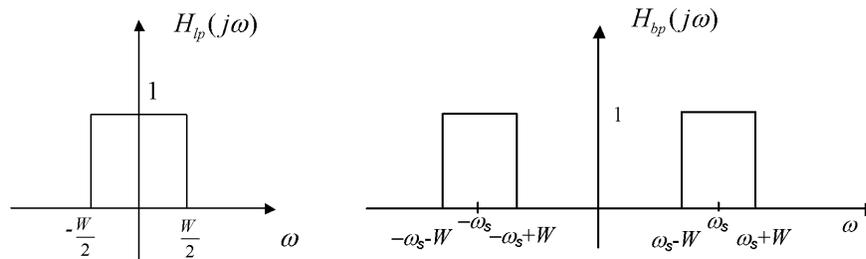


Figure 15.13: Frequency responses of the ideal low-pass and band-pass filters in Problem 15.7.

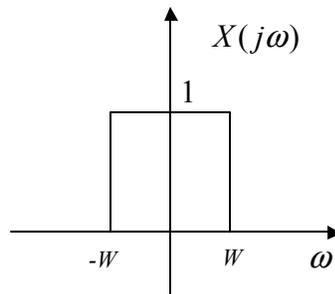
(a) Find and sketch $X(j\omega)$, the Fourier transform of the input signal $x(t) = \frac{W}{\pi} \text{sinc}\left(\frac{W}{\pi}t\right)$. For

what range of sampling frequencies $\omega_s = \frac{2\pi}{T_s}$ is the sampling theorem satisfied?

Answer:

From the table, we find the DTFT of $x(t) = \frac{W}{\pi} \text{sinc}\left(\frac{W}{\pi}t\right)$ to be a rectangular spectrum of

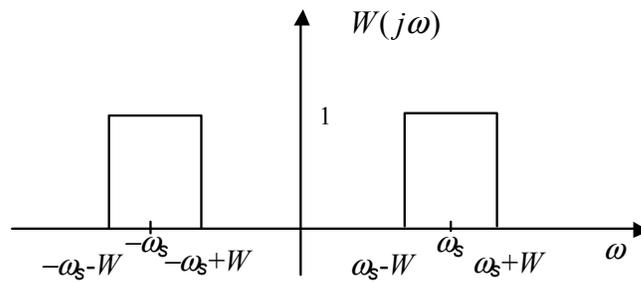
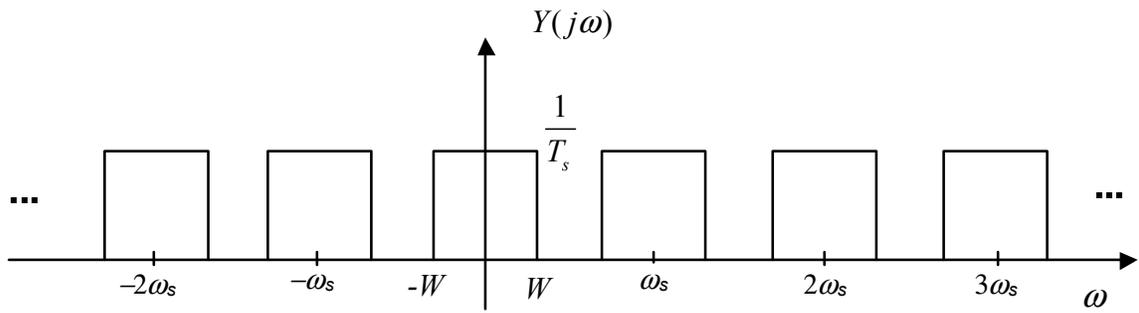
bandwidth W :



Hence, the sampling theorem is satisfied for $\omega_s > 2W$.

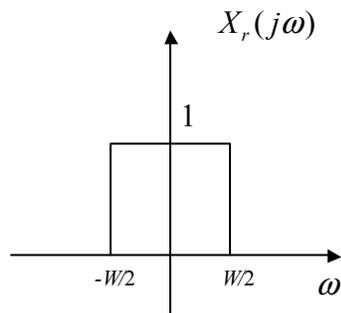
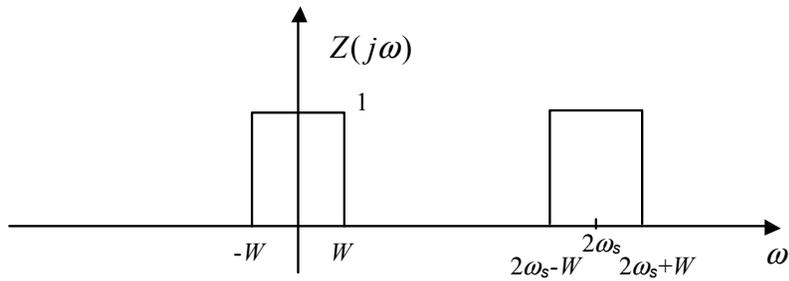
(b) Assume that the sampling theorem is satisfied with for the remaining questions. Sketch the Fourier transforms $Y(j\omega)$ and $W(j\omega)$.

Answer:



(c) Sketch the Fourier transforms $Z(j\omega)$ and $X_r(j\omega)$.

Answer:



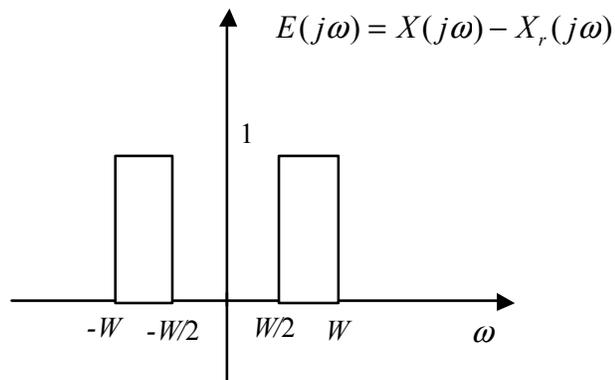
(d) Using the Parseval Equality, find the total energy of the error signal $e(t) := x(t) - x_r(t)$ defined as the difference between the input signal $x(t)$ and the "reconstructed" output signal $x_r(t)$.

Answer:

Total energy of the error signal is:

$$E_{\infty e} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |E(j\omega)|^2 d\omega = \frac{W}{2\pi}$$

where the Fourier transform of the error signal is shown below.



Problem 15.8

The system depicted in Figure 15.14 is used to implement a continuous-time bandpass filter. The discrete-time filter $H_d(e^{j\Omega})$ has frequency response on $[-\pi, \pi]$ given as:

$$H_d(e^{j\Omega}) = \begin{cases} 1, & \Omega_a \leq |\Omega| \leq \Omega_b \\ 0, & \text{otherwise} \end{cases}$$

Find the sampling period T_s , and frequencies Ω_a , Ω_b , W_1 so that the equivalent continuous-time frequency response $G(j\omega)$ satisfies $G(j\omega) = 1$ for $100\pi < |\omega| < 200\pi$, and $G(j\omega) = 0$ elsewhere. In solving this problem, choose W_1 as small as possible, and choose T_s as large as possible.

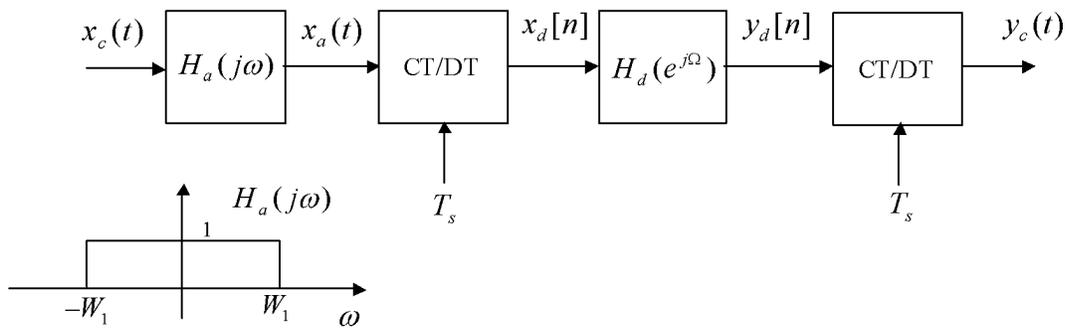


Figure 15.14: System implementing a continuous-time bandpass filter in Problem 15.8.

Problem 15.9

The signal $x[n]$ with DTFT depicted in Figure 15.15 is decimated to obtain $x_{\downarrow 4}[n] = x[4n]$.

Sketch $X_{\downarrow 4}(e^{j\omega})$.

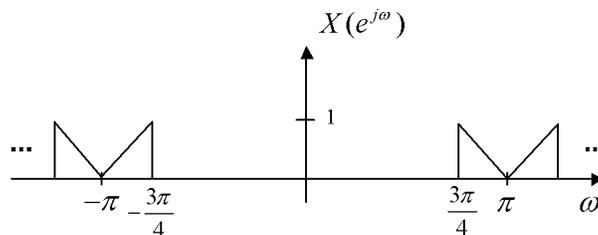
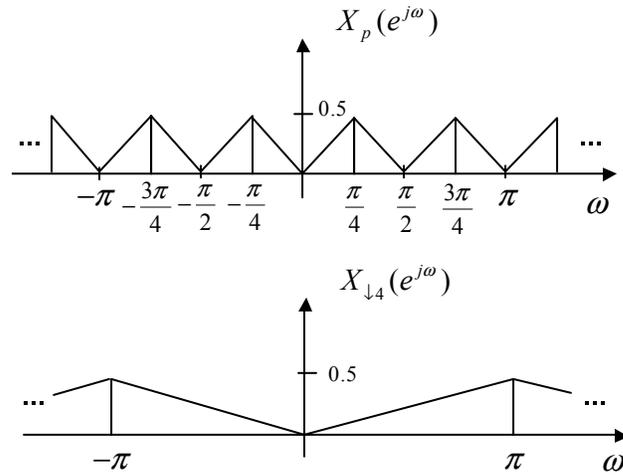


Figure 15.15: DTFT of signal in Problem 15.9.

Answer:



Problem 15.10

Suppose that you need to transmit a discrete-time signal whose DTFT is shown in Figure 15.16 with sample rate 1MHz. The specification is that the signal should be transmitted at the lowest possible rate, but in real time, i.e., the signal at the receiver must be exactly the same with a sample rate of 1MHz. Design a system (draw a block diagram) that meets these specs. Both before transmission and after reception, you can use upsampling-filtering denoted as $\{\uparrow N\}_{lp}$, decimation denoted as $\downarrow N$, ideal filtering, modulation, demodulation and summing junctions. (You might want to read the section on amplitude modulation and synchronous demodulation in Chapter 16 first.)

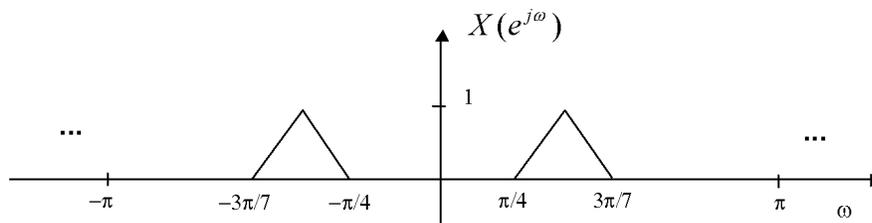


Figure 15.16: Spectrum of discrete-time signal to be transmitted in Problem 15.10.

Problem 15.11

Decimated multirate system

Consider the decimated multirate system shown in Figure 15.17, used for voice data compression. This system transmits a signal $x[n]$ over two low-bit-rate channels. The sampled

voice signal is bandlimited to $\omega_M = \frac{2\pi}{5}$.

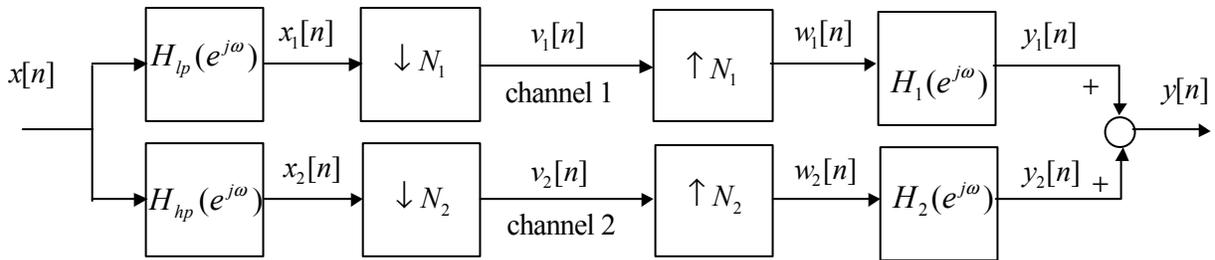


Figure 15.17: Decimated multirate system in Problem 15.11.

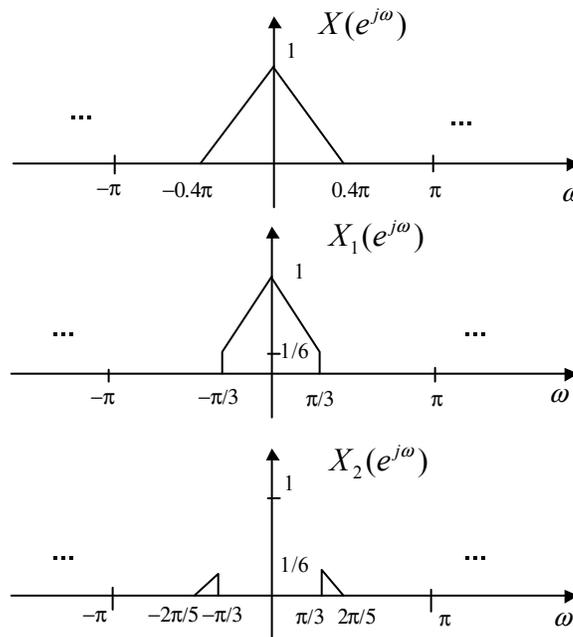
Numerical values:

- Input lowpass filters' cutoff frequency $\omega_{clp} = \frac{\pi}{3}$,
- Input highpass filter's cutoff frequency $\omega_{chp} = \frac{\pi}{3}$,

- signal's spectrum over $[-\pi, \pi]$: $X(e^{j\omega}) = \begin{cases} 1 - \frac{5}{2\pi}|\omega|, & |\omega| \leq \frac{2\pi}{5} \\ 0, & \frac{2\pi}{5} < |\omega| < \pi \end{cases}$.

(a) Sketch the spectra $X(e^{j\omega})$, $X_1(e^{j\omega})$, $X_2(e^{j\omega})$, indicating the important frequencies and magnitudes.

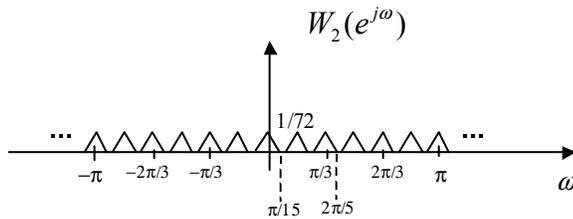
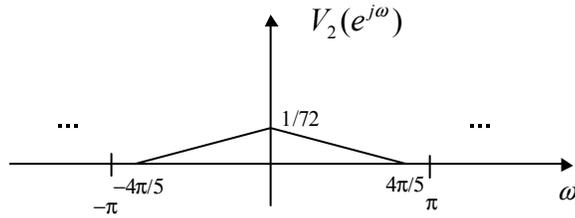
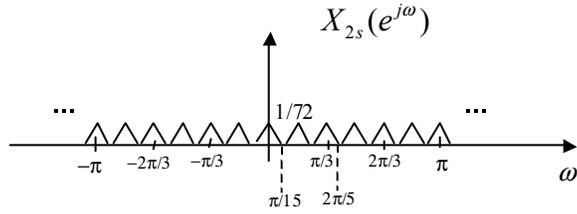
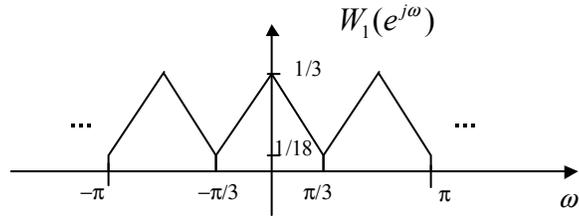
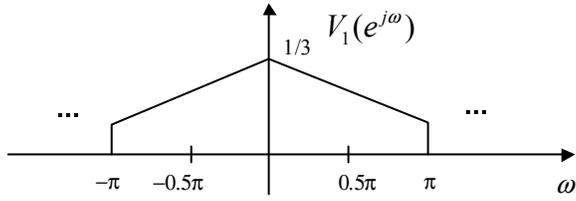
Answer:



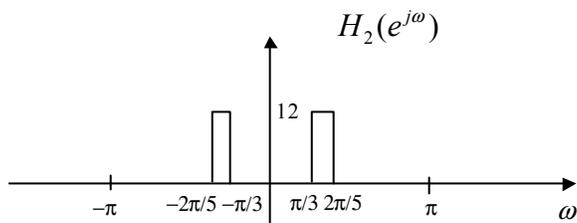
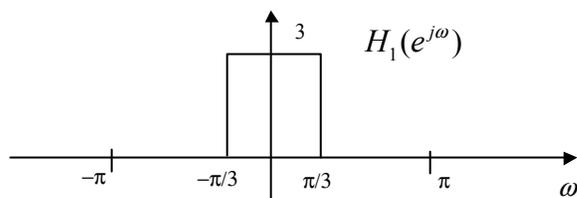
(b) Find the maximum decimation factors N_1 and N_2 avoiding aliasing and sketch the corresponding spectra $V_1(e^{j\omega})$, $V_2(e^{j\omega})$, $W_1(e^{j\omega})$, $W_2(e^{j\omega})$, indicating the important frequencies and magnitudes. Specify what the ideal output filters $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ should be for perfect signal reconstruction. Sketch their frequency responses, indicating the important frequencies and magnitudes.

Answer:

Maximum decimation factors: $N_1 = 3$, and $N_2 = 12$ since the copies of the spectrum $X_2(e^{j\omega})$ can be interleaved without aliasing.



Filter $H_1(e^{j\omega})$ should be low-pass and filter $H_2(e^{j\omega})$ should be bandpass:



(c) Compute the ratio of total energies at the output between the high-frequency subband $y_2[n]$

and the output signal $y[n]$, i.e., compute $r := 100 \frac{E_{\infty y_2}}{E_{\infty y}} \%$, where the energy is given by

$$E_{\infty y} := \sum_{n=-\infty}^{+\infty} |y[n]|^2.$$

Answer:

Using Parseval's relationship:

$$\begin{aligned} E_{\infty y} &= \sum_{n=-\infty}^{+\infty} |y[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(e^{j\omega})|^2 d\omega \\ &= \frac{1}{\pi} \int_0^{2\pi/5} \left| 1 - \frac{5}{2\pi} \omega \right|^2 d\omega = \frac{1}{\pi} \int_0^{2\pi/5} \left(1 - \frac{5}{\pi} \omega + \frac{25}{4\pi^2} \omega^2 \right) d\omega \\ &= \frac{1}{\pi} \left(\omega - \frac{5}{2\pi} \omega^2 + \frac{25}{12\pi^2} \omega^3 \right)_0^{2\pi/5} = \frac{1}{\pi} \left(\frac{2\pi}{5} - \frac{2\pi}{5} + \frac{25(8\pi^3)}{12\pi^2(125)} \right)_0^{2\pi/5} \\ &= \frac{1}{\pi} \left(\frac{2\pi}{15} \right) = \frac{2}{15} = 0.1333 \end{aligned}$$

$$\begin{aligned}
E_{\infty y_2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y_2(e^{j\omega})|^2 d\omega \\
&= \frac{1}{\pi} \int_{\pi/3}^{2\pi/5} \left| 1 - \frac{5}{2\pi} \omega \right|^2 d\omega \\
&= \frac{1}{\pi} \left(\omega - \frac{5}{2\pi} \omega^2 + \frac{25}{12\pi^2} \omega^3 \right) \Big|_{\pi/3}^{2\pi/5} \\
&= \frac{1}{\pi} \left(\frac{2\pi}{5} - \frac{2\pi}{5} + \frac{25(8\pi^3)}{12\pi^2(125)} - \frac{\pi}{3} + \frac{5\pi}{18} - \frac{25(\pi^3)}{12\pi^2(27)} \right) \\
&= \frac{1}{\pi} \left(\frac{2\pi}{15} - \frac{\pi}{18} - \frac{25\pi}{12(27)} \right) = \frac{2}{15} - \frac{1}{18} - \frac{25}{324} = 0.000617
\end{aligned}$$

$$r = 100 \frac{E\{y_2\}}{E\{y\}} = 100 \frac{0.000617}{0.1333} = 0.46\%$$

(d) Now suppose each of the two signal subbands $v_1[n]$ and $v_2[n]$ are quantized with a different number of bits to achieve good data compression, without losing intelligibility of the voice message. According to your result in (c), it would seem to make sense to use less bits to quantize the high-frequency subband. Suppose that the filter bank operates at an input/output sample rate of 10kHz, and you decide to use 12 bits to quantize $v_1[n]$ and 4 bits to quantize $v_2[n]$. Compute the bit rates of channel 1 and channel 2, and the overall bit rate of the system.

Answer:

After decimation by 3, channel 1 operates at 3.333 kilosamples/s which results in a bit rate of $3333.33333 * 12 = 40\,000$ bits/s. And after decimation by 2, channel 2 operates at 5 kilosamples/s which results in a bit rate of $5000 * 12 = 60,000$ bits/s.

Therefore, the overall bit rate of the filter bank is 100,000 bits/s.