Solutions to Problems in Chapter 13

Problems with Solutions

Problem 13.1

Sketch the pole-zero plot and compute the impulse response $h[n]$ of the system with transfer function:

$$H(z) = \frac{z(1-0.8z^{-1})}{(z^2 - 0.8z + 0.64)(1+2z^{-1})}$$

and with ROC: $0.8 < |z| < 2$. Specify whether or not the system is causal and stable.

Answer:

$$H(z) = \frac{z(z-0.8)}{(z^2 - 0.8z + 0.64)(z+2)} = \frac{z^{-1}(1-0.8z^{-1})}{(1-0.8e^{j\frac{\pi}{3}}z^{-1})(1-0.8e^{-j\frac{\pi}{3}}z^{-1})(1+2z^{-1})}, \quad 0.8 < |z| < 2$$

$$= \frac{A}{\underbrace{(1-0.8e^{j\frac{\pi}{3}}z^{-1})}_{|z|>0.8}} + \frac{A^*}{\underbrace{(1-0.8e^{-j\frac{\pi}{3}}z^{-1})}_{|z|>0.8}} + \frac{C}{\underbrace{(1+2z^{-1})}_{|z|<2}}$$

The coefficients are given by:

$$A=\left.\frac{z^{-1}(1-0.8z^{-1})}{(1-0.8e^{-j\frac{\pi}{3}}z^{-1})(1+2z^{-1})}\right|_{z=0.8e^{j\frac{\pi}{3}}}=\frac{1.25e^{-j\frac{\pi}{3}}(1-e^{-j\frac{\pi}{3}})}{(1-e^{-j\frac{2\pi}{3}})(1+2.5e^{-j\frac{\pi}{3}})}=\frac{1.25e^{-j\frac{\pi}{3}}(1-e^{-j\frac{\pi}{3}})}{(3.5-e^{-j\frac{2\pi}{3}}+2.5e^{-j\frac{\pi}{3}})}=$$

$$=\frac{1.25\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}-(-\frac{1}{2}-j\frac{\sqrt{3}}{2})\right)}{(5.25-1.5j\frac{\sqrt{3}}{2})}=\frac{\frac{5}{4}}{\frac{1}{4}(21-j3\sqrt{3})}=\frac{5(21+j3\sqrt{3})}{468}=0.224-0.0555j$$

$$C=\left.\frac{z^{-1}(1-0.8z^{-1})}{(1-0.8e^{j\frac{\pi}{3}}z^{-1})(1-0.8e^{-j\frac{\pi}{3}}z^{-1})}\right|_{z=-2}=\frac{-0.7}{(1-0.4+0.16)}=-0.92$$

Thus,

$$H(z)=\underbrace{\frac{0.224-0.0555j}{(1-0.8e^{j\frac{\pi}{3}}z^{-1})}}_{|z|>0.8}+\underbrace{\frac{0.224+0.0555j}{(1-0.8e^{-j\frac{\pi}{3}}z^{-1})}}_{|z|>0.8}-\underbrace{\frac{0.92}{(1+2z^{-1})}}_{|z|<2}$$

The inverse z-transform is obtained using the table:

$$h[n]=(0.05+j0.1)\left(0.8e^{j\frac{\pi}{3}}\right)^{n}u[n]+(0.05-j0.1)\left(0.8e^{-j\frac{\pi}{3}}\right)^{n}u[n]-1.84(-2)^{n}u[-n-1]$$

$$=2\,\mathrm{Re}\left\{e^{j\frac{\pi}{3}n}(0.05+j0.1)\right\}(0.8)^{n}u[n]-1.84(-2)^{n}u[-n-1].$$

$$=2(0.8)^{n}\left(0.05\cos\frac{\pi}{3}n-0.1\sin\frac{\pi}{3}n\right)u[n]-1.84(-2)^{n}u[-n-1]$$

The pole-zero plot is in Figure 13.1.

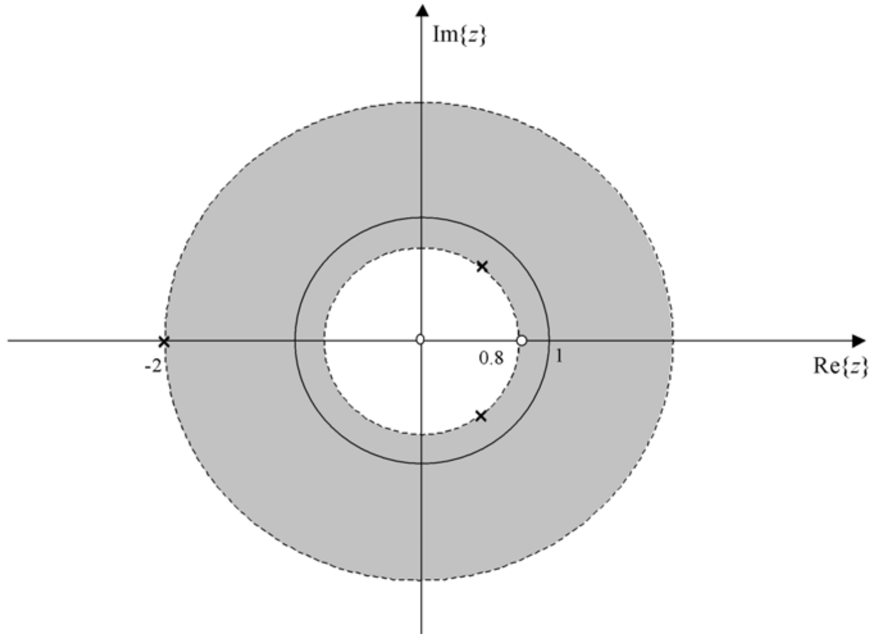


Figure 13.1: Pole-zero plot of transfer function in Problem 13.1.

The system is not causal since the ROC is a ring , but it is stable as it includes the unit circle.

Problem 13.2

Compute the inverse z-transform of $X(z)=\dfrac{z}{(1-0.5z^{-1})}$, $|z|<0.5$ using the power series expansion method.

Answer:

$$X(z)=\frac{z}{(1-0.5z^{-1})}=\frac{2z^2}{2z-1},$$

long division yields:

$$
\begin{array}{r}
-2z^2-4z^3-(2)^3z^4-\ldots \\
-1+2z \overline{\big) \quad 2z^2 \qquad\qquad\qquad} \\
\underline{2z^2-4z^3} \\
4z^3 \\
\underline{4z^3-8z^4} \\
8z^4
\end{array}
$$

Note that the resulting power series converges because the ROC implies $|2z|<1$. The signal is

$$
\begin{aligned}
x[n] &= -2\delta[n+2]-4\delta[n+3]-8\delta[n+4]\ldots \\
&= -(\frac{1}{2})^{n+1}u[-n-2].
\end{aligned}
$$

Problem 13.3

Consider the stable LTI system defined by its transfer function

$$H(z)=\frac{z^2+z-2}{z^2+z+0.5}$$

(a) Sketch the pole-zero plot for this transfer function, and give its ROC. Is the system causal?

Answer:

The poles are $p_1=-0.5+j0.5, p_2=-0.5-j0.5$. The zeros are $z_1=-2, z_2=1$. The system is stable, so its ROC must include the unit circle. With the fact that $H(\infty)$ is finite, we can conclude that the system is causal. The pole-zero plot is shown in Figure 13.2.

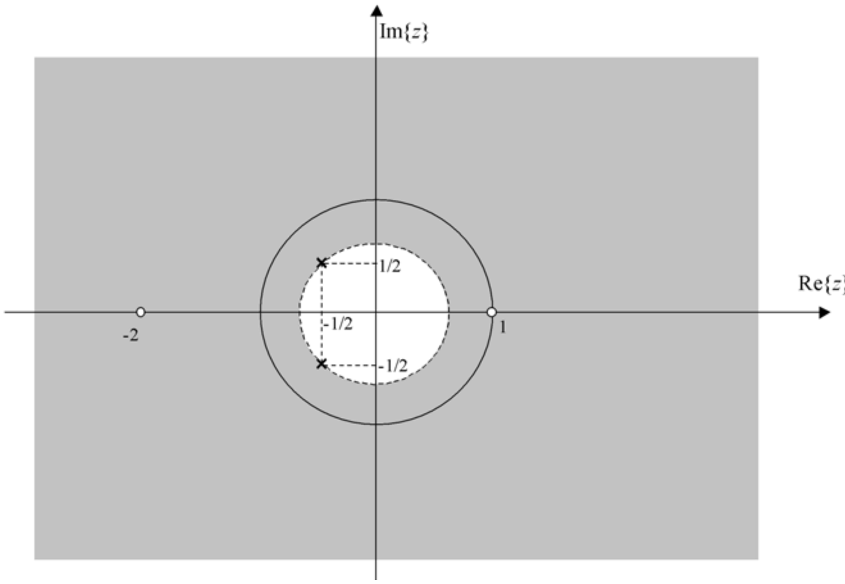


Figure 13.2: Pole-zero plot of transfer function in Problem 13.3(a).

(b) Find the corresponding difference system relating the output $y[n]$ to the input $x[n]$.

Answer:

We first write the transfer function as a function of z^{-1} of the form:

$$H(z) = \frac{1 + z^{-1} - 2z^{-2}}{1 + z^{-1} + 0.5z^{-2}}.$$

The corresponding difference system is

$$y[n] + y[n-1] + 0.5y[n-2] = x[n] + x[n-1] - 2x[n-2].$$

(c) Sketch the direct form realization of this system.

Answer:

The direct form realization of the system is given in Figure 13.3.

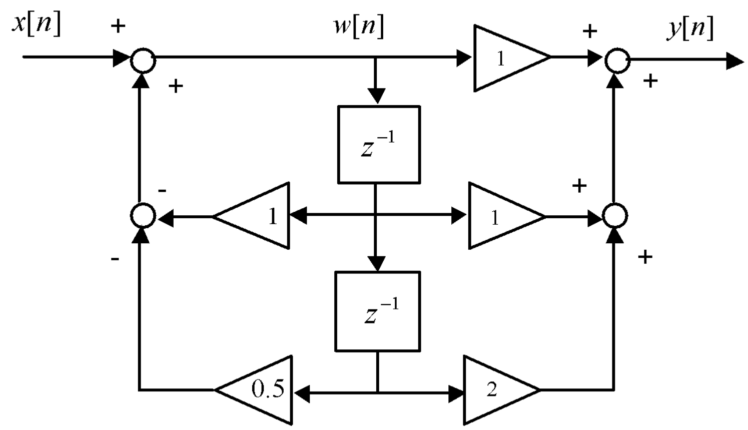


Figure 13.3: Direct form realization of transfer function in Problem 13.3.

Problem 13.4

(*a bit of actuarial mathematics using the unilateral z-transform…*)

The balance of a bank account after each year with interest compounded annually may be described by the difference equation

$$y[n] = \left(1 + r\right) y[n-1] + x[n],$$

where r is the annual interest rate, $y[n]$ is the account balance at the beginning of the $(n+1)^{\text{st}}$ year, the input "signal" $x[n] = M(u[n] - u[n-L])$ is composed of L consecutive annual deposits of M dollars, and the initial condition $y[-1] = y_{-1}$ is the initial amount in the account before the first deposit.

(a) Is this system stable?

Answer:

The system is unstable since the system is causal and the pole $1+r$ is larger than one, i.e., outside of the unit circle.

(b) Use the unilateral z-transform to compute $\mathcal{Y}(z)$

Answer:

Taking the unilateral z-transform on both sides, we obtain:

$$\mathcal{Y}(z)-(1+r)\left(z^{-1}\mathcal{Y}(z)+y[-1]\right)=\mathcal{X}(z)$$

$$\mathcal{Y}(z)=\frac{(1+r)\,y[-1]}{1-(1+r)\,z^{-1}}+\frac{\mathcal{X}(z)}{1-(1+r)\,z^{-1}}$$

$$\mathcal{Y}(z)=\frac{(1+r)\,y[-1]}{1-(1+r)\,z^{-1}}+\frac{M(1-z^{-L})}{\left[1-(1+r)\,z^{-1}\right](1-z^{-1})}$$

$$=\frac{(1+r)\,y[-1](1-z^{-1})+M(1-z^{-L})}{\left[1-(1+r)\,z^{-1}\right](1-z^{-1})}$$

more useful form:

$$=\frac{(1+r)\,y[-1]+M\sum_{k=0}^{L-1}z^{-k}}{\left[1-(1+r)\,z^{-1}\right]}$$

(c) Find the annual deposit M as a function of the final balance in the account after L years $y[L-1]$ and the interest rate r. Compute the annual deposit M if you want to accrue \$100,000 after 20 years at 5% annual interest rate with an initial balance of $y_{-1}=\$1000$.

Answer:

The account balance at the beginning of the $(n+1)^{\text{st}}$ year is

$$
\begin{aligned}
y[n] &= \mathcal{Z}^{-1}\{\boldsymbol{Y}(z)\} \\
&= \mathcal{Z}^{-1}\left\{\frac{(1+r)\,y[-1]+M\sum_{k=0}^{L-1} z^{-k}}{\left[1-(1+r)\,z^{-1}\right]}\right\} \\
&= \left[(1+r)\,y[-1]\right](1+r)^{n}\,u[n] + M\sum_{k=0}^{L-1}(1+r)^{n-k}\,u[n-k]
\end{aligned}
$$

When the last payment is made at $n = L-1$, we get

$$
\begin{aligned}
y[L-1] &= \left[(1+r)\,y[-1]\right](1+r)^{L-1} + M\sum_{k=0}^{L-1}(1+r)^{L-1-k} \\
&= (1+r)^{L}\,y[-1] + M\sum_{m=0}^{L-1}(1+r)^{m}
\end{aligned}
$$

which yields:

$$
M = \frac{y[L-1]-(1+r)^{L}\,y[-1]}{\sum_{m=0}^{L-1}(1+r)^{m}} = \frac{y[L-1]-(1+r)^{L}\,y[-1]}{\dfrac{1-(1+r)^{L}}{1-(1+r)}} = \frac{r\left[y[L-1]-(1+r)^{L}\,y[-1]\right]}{(1+r)^{L}-1}
$$

$$
\begin{aligned}
M &= \frac{r\left[y[L-1]-(1+r)^{L}\,y[-1]\right]}{(1+r)^{L}-1} \\
&= \frac{0.05\left[\$100000-(1.05)^{20}\cdot\$1000\right]}{(1.05)^{20}-1} \\
&= \$2944
\end{aligned}
$$

Exercises

Problem 13.5

Compute the z-transform of each of the following signals and sketch its pole-zero plot, indicating the ROC.

(a) $x[n]=\alpha^n\cos(\omega_0 n+\theta)u[n],\quad |\alpha|>1$. Use the values $\alpha=1.5,\ \omega_0=\frac{3\pi}{4},\ \theta=-\frac{\pi}{4}$ for the pole-zero plot.

Answer:

$$\begin{aligned}x[n]&=\frac{1}{2}\alpha^n\left(e^{j(\omega_0 n+\theta)}+e^{-j(\omega_0 n+\theta)}\right)u[n]\\&=\frac{1}{2}e^{j\theta}(\alpha e^{j\omega_0})^n u[n]+\frac{1}{2}e^{-j\theta}(\alpha e^{-j\omega_0})^n u[n]\end{aligned}$$

The z-transform:

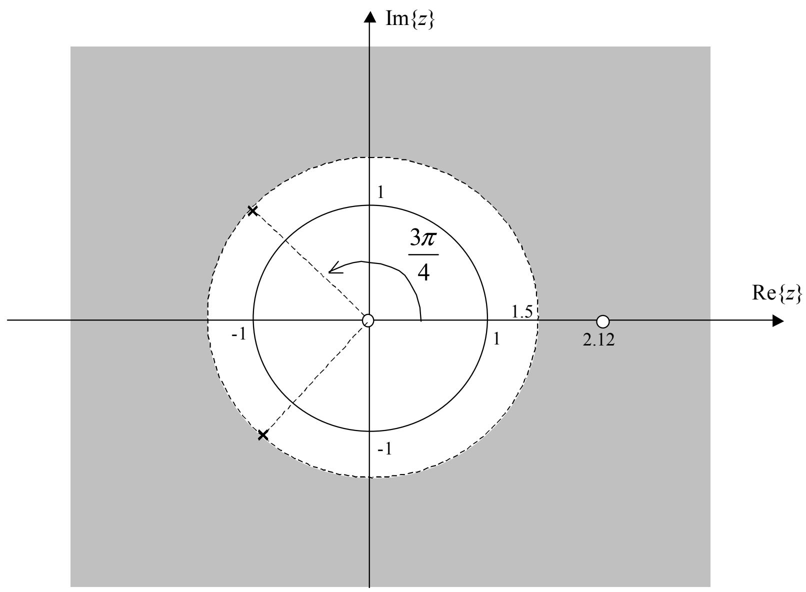
$$\begin{aligned}X(z)&=\sum_{n=-\infty}^{\infty}x[n]z^{-n}\\&=\frac{1}{2}e^{j\theta}\sum_{n=0}^{\infty}(\alpha e^{j\omega_0}z^{-1})^n+\frac{1}{2}e^{-j\theta}\sum_{n=0}^{\infty}(\alpha e^{-j\omega_0}z^{-1})^n,\ \left|\alpha e^{j\omega_0}z^{-1}\right|<1\\&=\frac{1}{2}\left[\frac{e^{j\theta}}{1-\alpha e^{j\omega_0}z^{-1}}+\frac{e^{-j\theta}}{1-\alpha e^{-j\omega_0}z^{-1}}\right],\ |z|>|\alpha|\\&=\frac{1}{2}\left[\frac{e^{j\theta}-\alpha e^{j\theta}e^{-j\omega_0}z^{-1}+e^{-j\theta}-\alpha e^{-j\theta}e^{j\omega_0}z^{-1}}{(1-\alpha e^{j\omega_0}z^{-1})(1-\alpha e^{-j\omega_0}z^{-1})}\right],\ |z|>|\alpha|\\&=\frac{\cos\theta-\alpha\cos(\theta-\omega_0)z^{-1}}{1-2\alpha\cos\omega_0 z^{-1}+\alpha^2z^{-2}},\ |z|>|\alpha|\end{aligned}$$

For the pole-zero plot:

$$X(z) = \frac{\cos\theta - \alpha\cos(\theta-\omega_0)z^{-1}}{1-2\alpha\cos\omega_0 z^{-1} + \alpha^2 z^{-2}}, \ |z| > |\alpha|$$

$$= \frac{\frac{1}{\sqrt{2}} - 1.5z^{-1}}{1+2(1.5)\frac{1}{\sqrt{2}}z^{-1} + (1.5)^2 z^{-2}}, \ |z| > 1.5$$

$$= \frac{z(\frac{1}{\sqrt{2}}z - 1.5)}{z^2 + \frac{3}{\sqrt{2}}z + 2.25} = \frac{\frac{1}{\sqrt{2}}z(z-\frac{3}{\sqrt{2}})}{(z-1.5e^{j\frac{3\pi}{4}})(z-1.5e^{-j\frac{3\pi}{4}})}, \ |z| > 1.5$$



(b) $x[n] = u[n+4] - u[n]$

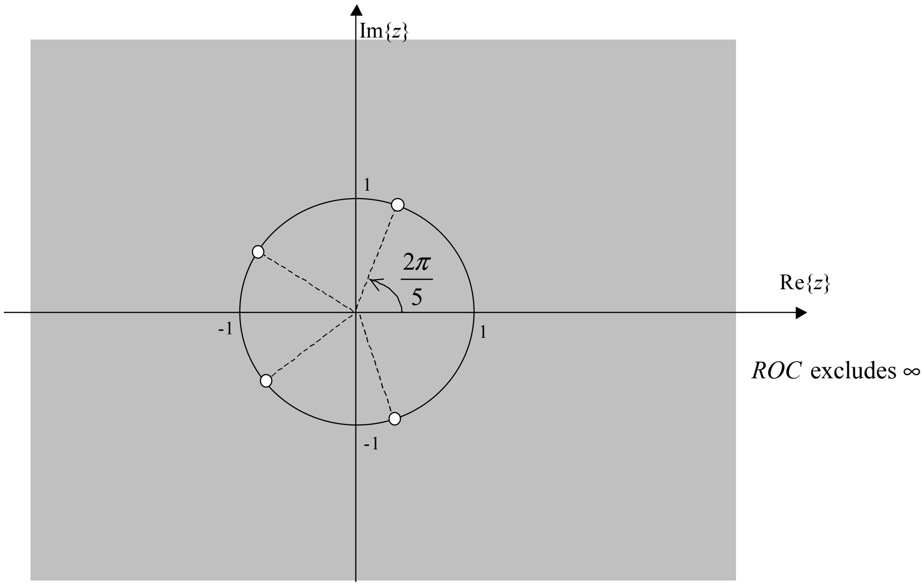
Answer:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= z^4 + z^3 + z^2 + z^1, \quad |z| < \infty$$

For pole-zero plot:

$$X(z) = (z - e^{j\frac{2\pi}{5}})(z - e^{j\frac{4\pi}{5}})(z - e^{j\frac{6\pi}{5}})(z - e^{j\frac{8\pi}{5}}), \quad |z| < \infty$$



(c) $x[n] = (-2)^n u[-n+3]$

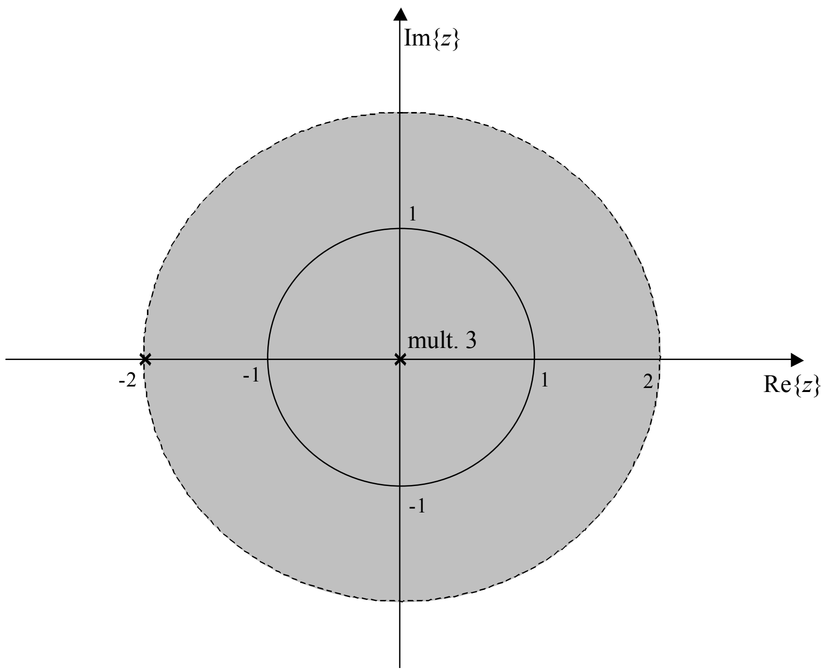
Answer:

$$x[n] = (-2)^n u[-n+3] = (-2)^3 (-2)^{n-3} u[-(n-3)]$$

$$\begin{aligned}
X(z) &= z^{-3} \sum_{n=-\infty}^{\infty} w[n] z^{-n} = -8z^{-3} \sum_{n=-\infty}^{0} (-2z^{-1})^n \\
&= -8z^{-3} \sum_{m=0}^{+\infty} (-2z^{-1})^{-m} = -8z^{-3} \sum_{m=0}^{+\infty} (-0.5z)^m \\
&= \frac{-8z^{-3}}{1-(-0.5)z}, \ |(-0.5)z| < 1 = \frac{-8z^{-3}}{1+0.5z}, \ |z| < 2 \\
&= \frac{-16z^{-4}}{1+2z^{-1}}, \ |z| < 2
\end{aligned}$$

Pole zero-plot:

$$X(z) = \frac{-16}{z^3(z+2)}, \ |z| < 2$$



Problem 13.6

Compute the inverse z-transform of $X(z) = \frac{z^2}{z+0.2}, \quad 0.2 < |z| < \infty$ using the power series expansion method.

Problem 13.7

Consider the following so-called auto-regressive moving-average causal filter S initially at rest:

$$S: \qquad y[n] - 0.9y[n-1] + 0.81y[n-2] = x[n] - x[n-2].$$

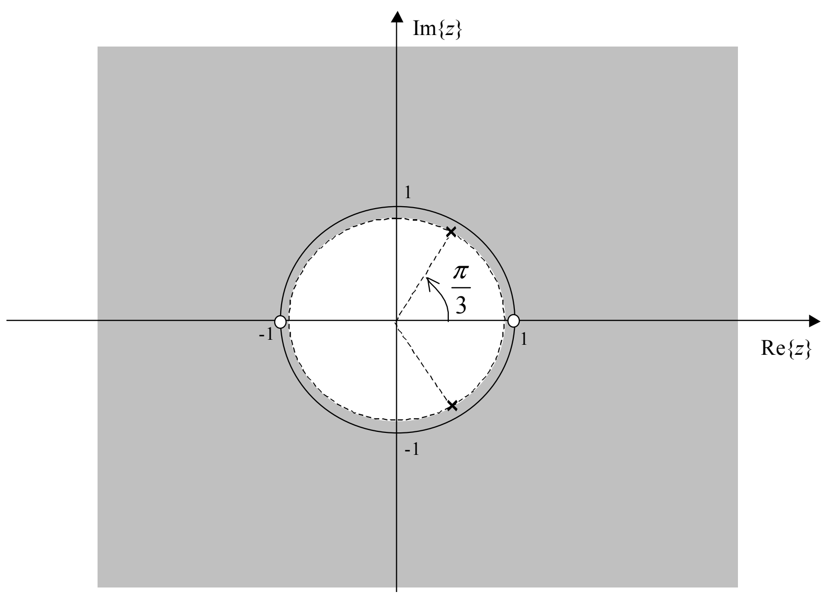
(a) Compute the z-transform of the impulse response of the filter $H(z)$ (the transfer function) and give its region of convergence. Sketch the pole-zero plot.

Answer:

The z-transform of this filter is given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}} = \frac{(1 - z^{-1})(1 + z^{-1})}{(1 - 0.9e^{j\frac{\pi}{3}}z^{-1})(1 - 0.9e^{-j\frac{\pi}{3}}z^{-1})}, \quad |z| > 0.9$$

Pole zero plot:



(b) Compute the impulse response $h[n]$ of the filter.

Answer:

Using the table, we identify the angles and magnitude of the poles: $r = 0.9$, $\omega_0 = \frac{\pi}{3}$, thus

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}} = A + \frac{B(1 - 0.45z^{-1}) + 0.7794Cz^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}, \quad |z| > 0.9$$

Letting *z* tend to infinity, we find that $A + B = 1$. We find the remaining two equations by multiplying on both sides by the denominator:

$$\begin{aligned} &1 - z^{-2} = A - 0.9Az^{-1} + 0.81Az^{-2} + B(1 - 0.45z^{-1}) + 0.7794Cz^{-1} \\ &\Rightarrow -0.9A - 0.45B + 0.7794C = 0, \text{ and } 0.81A = -1 \end{aligned}.$$

We obtain:

$$A = -1.235,\ B = 2.235,\ C = -0.1354,$$

so that

$$H(z) = -1.235 + \frac{2.235(1 - 0.45z^{-1}) + (-0.1354)0.7794z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}, \quad |z| > 0.9.$$

From the table:

$$h[n] = -1.235\delta[n] + 2.235(0.9)^n \cos\left(\frac{\pi}{3}n\right)u[n] - 0.1354(0.9)^n \sin\left(\frac{\pi}{3}n\right)u[n]$$

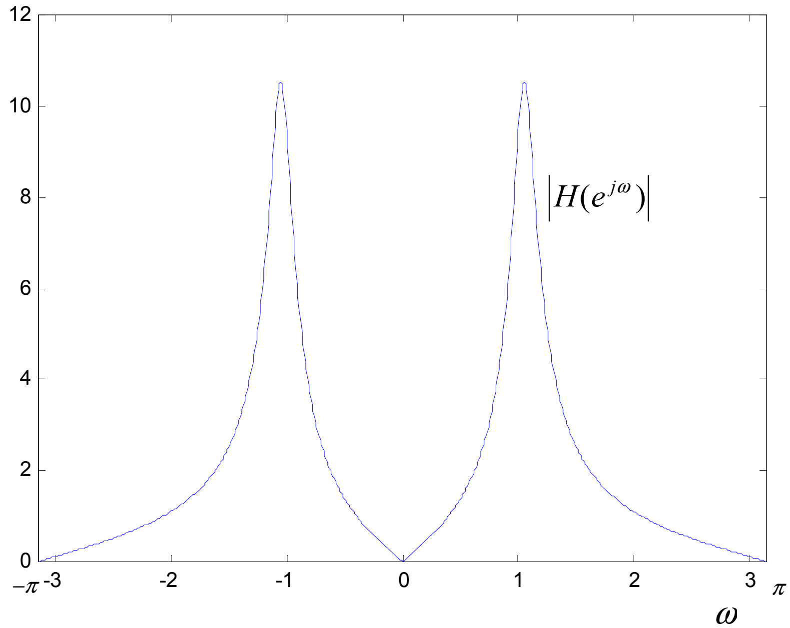
(c) Compute the frequency response of the filter. Plot its magnitude. What type of filter is it? Low-pass, high-pass or band-pass?

Answer:

$$H(e^{j\omega}) = \frac{1 - e^{-2j\omega}}{1 - 0.9e^{-j\omega} + 0.81e^{-2j\omega}}$$

$$\left|H(e^{j\omega})\right| = \left|\frac{1 - e^{-2j\omega}}{1 - 0.9e^{-j\omega} + 0.81e^{-2j\omega}}\right|$$

Magnitude:



This is clearly a bandpass filter.

(d) Compute and sketch the unit step response $s[n]$ of the filter.

Answer:

$$S(z)=\frac{1}{1-z^{-1}}H(z)=\frac{1+z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}=\frac{A(1-0.45z^{-1})+0.7794Bz^{-1}}{1-0.9z^{-1}+0.81z^{-2}},\quad |z|>0.9$$
$$=\frac{(1-0.45z^{-1})+0.7794(1.86)z^{-1}}{1-0.9z^{-1}+0.81z^{-2}},\quad |z|>0.9$$

From the table:

$$s[n]=(0.9)^n\cos\left(\frac{\pi}{3}n\right)u[n]+1.86(0.9)^n\sin\left(\frac{\pi}{3}n\right)u[n]$$

Problem 13.8

Consider a DLTI system with transfer function $H(z)=\dfrac{z^{-3}-1.2z^{-4}}{(z-0.8)(z+0.8)}$.

(a) Sketch the pole-zero plot of the system.

(b) Find the ROC that makes this system stable.

(c) Is the system causal with the ROC that you found in (b)? Justify your answer.

(d) Suppose that $H(e^{j\omega})$ is bounded for all frequencies. Find the response of the system $y[n]$ to the input $x[n]=u[n]$.

Problem 13.9

Compute the inverse z-transform $x[n]$ of $X(z)=\dfrac{1}{z(z+0.4)},\quad |z|>0.4$ using the method of long division and sketch it.

Answer:

$$X(z) = \frac{1}{z(z+0.4)} = \frac{z^{-2}}{(1+0.4z^{-1})}, \quad 0.4 < |z|$$

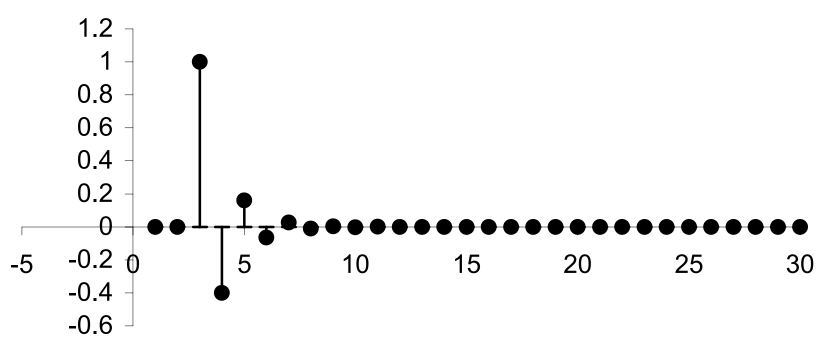
Long division yields:

$$
\begin{array}{r l}
 & z^{-2} - 0.4z^{-3} + 0.16z^{-4} - 0.064z^{-5} \ldots \\
1+0.4z^{-1} \big) & z^{-2} \\
 & \underline{-(z^{-2} + 0.4z^{-3})} \\
 & \quad -0.4z^{-3} \\
 & \quad \underline{-(-0.4z^{-3} - 0.16z^{-4})} \\
 & \qquad 0.16z^{-4} \\
 & \qquad \underline{-(0.16z^{-4} + 0.064z^{-5})} \\
 & \qquad\quad -0.064z^{-5}
\end{array}
$$

Note that the resulting power series converges because the ROC implies $|0.4z^{-1}| < 1$. The time-domain signal is:

$$
\begin{aligned}
x[n] &= \delta[n-2] - 0.4\delta[n-3] + (0.4)^2\,\delta[n-4] - (0.4)^3\,\delta[n-5] + \ldots \\
&= (-0.4)^{n-2}\,u[n-2]
\end{aligned}
$$

This signal is plotted below.



Problem 13.10

Sketch the pole-zero plot and compute the impulse response $h[n]$ of the stable system with transfer function: $H(z) = \dfrac{2000z^3 + 1450z^2 + 135z}{(100z^2 - 81)(5z + 4)}$.

Specify its ROC. Specify whether the system is causal or not.

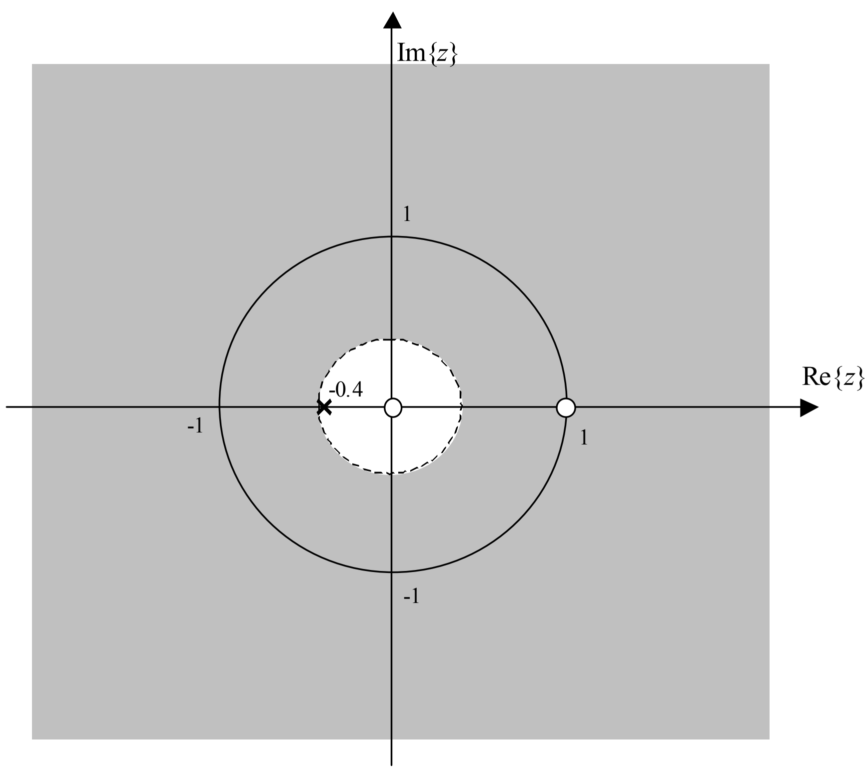
Problem 13.11

Consider the DLTI system with transfer function $H(z) = \dfrac{z-1}{z^{-1}(z+0.4)}$.

(a) Sketch the pole-zero plot of the system.

Answer: $H(z) = \dfrac{z-1}{z^{-1}(z+0.4)} = \dfrac{z^2 - z}{(z+0.4)}$

poles at –0.4 and at infinity, zeros at 0 and 1.



(b) Find the ROC that makes this system stable.

Answer:

For stability, the ROC must include the unit circle, so it is $|z| > 0.4$, excluding infinity.

(c) Is the system causal with the ROC that you found in (b)? Justify your answer.

Answer:

For the system to be causal, we must have $\lim_{z\to\infty} H(z) < \infty$, which is not the case here. Hence the system is not causal.

(d) Suppose that $H(e^{j\omega})$ is bounded for all frequencies. Calculate the response of the system $y[n]$ to the input $x[n] = u[n]$.

Answer:

This means that the ROC is chosen as $|z| > 0.4$. We have $X(z) = \dfrac{1}{1-z^{-1}}$, $|z| > 1$, and

$$Y(z) = H(z)U(z) = \frac{z^2(z-1)}{(z+0.4)(z-1)} = \frac{z^2}{(z+0.4)} = \frac{z}{(1+0.4z^{-1})},\ 0.4 < |z| < \infty$$

thus, $y[n] = (-0.4)^{n+1}u[n+1]$.