Solutions to Problems in Chapter 1

Problems with Solutions

Problem 1.1

Write the following complex signals in polar form, i.e., in the form $x(t) = r(t)e^{j\theta(t)}$,

$r(t), \theta(t) \in \mathbb{R}, r(t) > 0$ for continuous-time signals, $x[n] = r[n]e^{j\theta[n]}$, $r[n], \theta[n] \in \mathbb{R}, r[n] > 0$ for discrete-time signals.

(a) $x(t) = \dfrac{t}{1+jt}$

Answer:

$$x(t) = \frac{t}{1+jt} = \frac{|t|}{\sqrt{t^2+1}} e^{j\theta(t)}$$

$$\Rightarrow \quad r(t) = \frac{|t|}{\sqrt{t^2+1}}, \quad \theta(t) = \begin{cases} \arctan\left(\dfrac{-t}{1}\right), & t \geq 0 \\ \arctan\left(\dfrac{-t}{1}\right) + \pi, & t < 0 \end{cases}$$

(b) $x[n] = nje^{n+j}, \quad n > 0$

Answer:

$$x[n] = nje^{n+j} = ne^{n} e^{j(1+\pi/2)}, \; n > 0$$

$$\Rightarrow \quad r[n] = ne^{n}, \quad \theta[n] = (1+\pi/2)$$

Problem 1.2

Determine if the following systems are: 1. Memoryless, 2. Time-invariant, 3. Linear, 4. Causal, 5. BIBO Stable. Justify your answers.

(a) $y[n] = x[1-n]$

1. Memoryless? No. For example, the output $y[0] = x[1]$ depends on a future value of the input.

2. Time-invariant? No.

$$\begin{aligned} y_1[n] &= Sx[n-N] = x[1-n-N] \\ &\neq x[1-(n-N)] = x[1-n+N)] = y[n-N] \end{aligned}$$

3. Linear? Yes. Let $y_1[n] := Sx_1[n] = x_1[1-n],\ y_2[n] := Sx_2[n] = x_2[1-n]$. Then, the output of the system with $x[n] := \alpha x_1[n] + \beta x_2[n]$ is given by:

$$\begin{aligned} y &= Sx: \\ y[n] &= x[1-n] = \alpha x_1[1-n] + \beta x_2[1-n] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

4. Causal? No. For example, the output $y[0] = x[1]$ depends on a future value of the input.

5. Stable? Yes. $|x[n]| < B, \forall n \Rightarrow |y[n]| = |x[1-n]| < B, \forall n$.

(b) $y(t) = \dfrac{x(t)}{1 + x(t-1)}$

1. Memoryless? No. The system has memory since at time t, it uses the past value of the input $x(t-1)$.

2. Time-invariant? Yes. $y_1(t) = Sx(t-T) = \dfrac{x(t-T)}{1+x(t-1-T)} = y(t-T)$

3. Linear? No. The system S is nonlinear since it does not have the superposition property:

For $x_1(t), x_2(t),$ let $y_1(t) = \dfrac{x_1(t)}{1+x_1(t-1)},\ y_2(t) = \dfrac{x_2(t)}{1+x_2(t-1)}$

Define $x(t) = ax_1(t) + bx_2(t)$.

Then $y(t) = \dfrac{ax_1(t)+bx_2(t)}{1+ax_1(t-1)+bx_2(t-1)} \neq \dfrac{ax_1(t)}{1+x_1(t-1)} + \dfrac{bx_2(t)}{1+x_2(t-1)} = ay_1(t) + by_2(t)$

4. Causal? Yes. The system is causal as the output is a function of the past and current values of the input $x(t-1)$ and $x(t)$ only.

5. Stable? No. For the bounded input $x(t) = -1, \forall t \Rightarrow |y(t)| = \infty$, i.e. the output is unbounded.

(c) $y(t) = tx(t)$

1. Memoryless? Yes. The output at time t depends only on the current value of the input $x(t)$.

2. Time-invariant? No. $y_1(t) = Sx(t-T) = tx(t-T) \neq (t-T)x(t-T) = y(t-T)$

3. Linear? Yes. Let $y_1(t) := Sx_1(t) = tx_1(t),\ \ y_2(t) := Sx_2(t) = tx_2(t)$. Then,

$$\begin{aligned} y(t) &= S[ax_1(t)+bx_2(t)] = t[ax_1(t)+bx_2(t)] \\ &= atx_1(t) + btx_2(t) = ay_1(t) + by_2(t) \end{aligned}$$

4. Causal? Yes. The output at time t depends on the present value of the input only.

5. Stable? No. Consider the constant input $x(t) = B \Rightarrow$ for any K, $\exists T$ such that $|y(T)| = |TB| > K$,

namely, $T > \frac{K}{|B|}$, i.e., the output is unbounded.

(d) $y[n] = \sum_{k=-\infty}^{0} x[n-k]$

1. Memoryless? No. The output $y[n]$ is computed using all future values of the input.

2. Time-invariant? Yes. $y_1[n] = Sx[n-N] = \sum_{k=-\infty}^{0} x[n-N-k] = y[n-N]$

3. Linear? Yes. Let $y_1[n] := Sx_1[n] = \sum_{k=-\infty}^{0} x_1[n-k]$, $y_2[n] := Sx_2[n] = \sum_{k=-\infty}^{0} x_2[n-k]$. Then, the output of the system with $x[n] := \alpha x_1[n] + \beta x_2[n]$ is given by:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{0} x[n-k] = \sum_{k=-\infty}^{0} \alpha x_1[n-k] + \beta x_2[n-k] = \alpha \sum_{k=-\infty}^{0} x_1[n-k] + \beta \sum_{k=-\infty}^{0} x_2[n-k] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

4. Causal? No. The output $y[n]$ depends on future values of the input $x[n+|k|]$.

5. Stable? No. For the input signal $x[n] = B, \forall n \Rightarrow |y[n]| = \left| \sum_{k=-\infty}^{0} x[n-k] \right| = \left| \sum_{k=-\infty}^{0} B \right| = +\infty$, i.e., the output is unbounded.

Problem 1.3

Find the fundamental periods (T for continuous-time signals, N for discrete-time signals) of the following periodic signals.

(a) $x(t) = \cos(13\pi t) + 2\sin(4\pi t)$

Answer:

$x(t+T) = \cos(13\pi t + 13\pi T) + 2\sin(4\pi t + 4\pi T)$

will equal $x(t)$ if $\exists k, p \in \mathbb{Z}$ such that $13\pi T = 2\pi k,\ 4\pi T = 2\pi p$,

which yields $T = \frac{2k}{13} = \frac{p}{2} \Rightarrow \frac{p}{k} = \frac{4}{13}$. The numerator and denominator are coprime (no common divisor except 1), thus we take $p = 4,\ k = 13$ and the fundamental period is $T = \frac{p}{2} = 2$.

(b) $x[n] = e^{j7.351\pi n}$

Answer:

$x[n] = e^{j7.351\pi n} = e^{j\frac{7351}{1000}\pi n}$, thus the frequency is $\omega_0 = \frac{7351}{1000}\pi = \frac{7351}{2000}2\pi$ and the number 7351 is prime, so the fundamental period is $N = 2000$.

Problem 1.4

Sketch the signals $x[n] = u[n+3] - u[n] + 0.5^n u[n] - 0.5^{n-4} u[n-4]$, and

$y[n] = nu[-n] - \delta[n-1] - nu[n-3] + (n-4)u[n-6]$.

Answer:

Signals $x[n]$ and $y[n]$ are sketched in Figure 1.1 and Figure 1.2

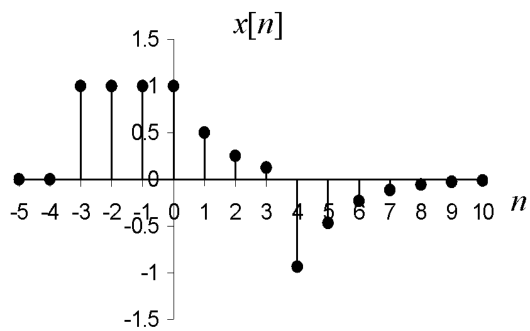

Figure 1.1: Signal $x[n]$

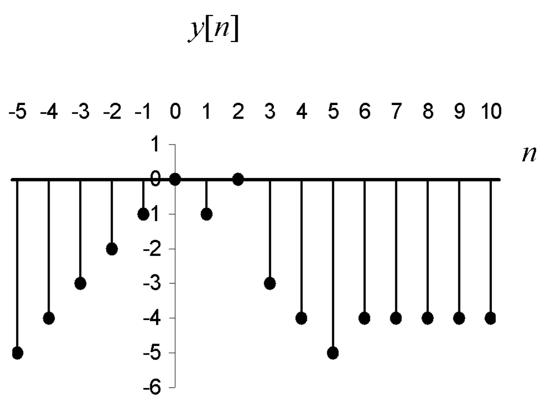

Figure 1.2: Signal $y[n]$

(b) Find expressions for the signals shown in Figure 1.3.

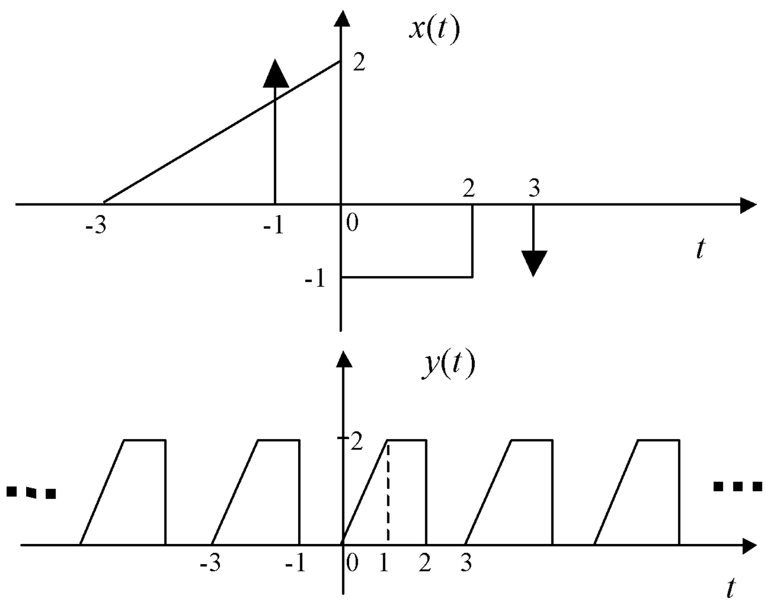

Figure 1.3: Plots of continuous-time signals $x(t)$ and $y(t)$

Answer:

$$x(t)=\frac{2}{3}(t+3)u(t+3)-\frac{2}{3}(t+3)u(t)-u(t)+u(t-2)+2\delta(t+1)-\delta(t-3)$$

$$y(t)=\sum_{k=-\infty}^{\infty} 2(t-3k)u(t-3k)-2(t-3k-1)u(t-3k-1)-2u(t-2-3k)$$

Problem 1.5

Properties of even and odd signals.

(a) Show that if x[n] is an odd signal, then $\sum_{n=-\infty}^{+\infty} x[n]=0$

Answer:

For an odd signal,

$$x[n]=-x[n]\Rightarrow x[0]=0 \text{ and } \sum_{n=-\infty}^{+\infty} x[n]=x[0]+\sum_{n=1}^{+\infty}(x[n]-x[n])=0$$

(b) Show that if $x_1[n]$ is odd and $x_2[n]$ is even, then their product is odd.

Answer:

$$\begin{aligned} &x_1[n]=x_1[-n], \quad x_2[n]=-x_2[-n] \\ \Rightarrow \quad &x_1[-n]x_2[-n]=-x_1[n]x_2[n] \end{aligned}$$

(c) Let $x[n]$ be an arbitrary signal with even and odd parts $x_e[n]=Ev\{x[n]\}, \quad x_o[n]=Od\{x[n]\}$.

Show that $\sum_{n=-\infty}^{+\infty} x^2[n]=\sum_{n=-\infty}^{+\infty} x_e{}^2[n]+\sum_{n=-\infty}^{+\infty} x_o{}^2[n]$.

Answer:

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} x^2[n] &= \sum_{n=-\infty}^{+\infty}(x_e[n]+x_o[n])^2 = \sum_{n=-\infty}^{+\infty} x_e{}^2[n]+2\underbrace{\sum_{n=-\infty}^{+\infty} x_e[n]x_o[n]}_{=0}+\sum_{n=-\infty}^{+\infty} x_o{}^2[n] \\ &= \sum_{n=-\infty}^{+\infty} x_e{}^2[n]+\sum_{n=-\infty}^{+\infty} x_o{}^2[n] \end{aligned}$$

(d) Similarly, show that $\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} x_e^{\,2}(t)dt + \int_{-\infty}^{+\infty} x_o^{\,2}(t)dt$.

Answer:

$$\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} (x_e(t)+x_o(t))^2\, dt = \int_{-\infty}^{+\infty} x_e^{\,2}(t)dt + 2\underbrace{\int_{-\infty}^{+\infty} x_e(t)x_o(t)dt}_{0} + \int_{-\infty}^{+\infty} x_o^{\,2}(t)dt$$

$$= \int_{-\infty}^{+\infty} x_e^{\,2}(t)dt + \int_{-\infty}^{+\infty} x_o^{\,2}(t)dt$$

Exercises

Problem 1.6

Write the following complex signals in rectangular form: $x(t) = a(t) + jb(t),\ a(t), b(t) \in \mathbb{R}$ for continuous-time signals, $x[n] = a[n] + jb[n],\ a[n], b[n] \in \mathbb{R}$ for discrete-time signals.

(a) $x(t) = e^{(-2+j3)t}$

(b) $x(t) = e^{-j\pi t}u(t) + e^{(2+j\pi)t}u(-t)$

Problem 1.7

Use the sampling property of the impulse to simplify the following expressions.

(a) $x(t) = e^{-t}\cos(10t)\delta(t)$

Answer:

$$x(t) = e^{-t}\cos(10t)\delta(t) = e^{0}\cos(10\cdot 0)\delta(t) = \delta(t)$$

(b) $x(t) = \sin(2\pi t)\sum_{k=0}^{\infty}\delta(t-k)$

Answer:

$$\begin{aligned} x(t) &= \sin(2\pi t)\sum_{k=0}^{\infty}\delta(t-k) = \sum_{k=0}^{\infty}\sin(2\pi t)\delta(t-k) \\ &= \sum_{k=0}^{\infty}\sin(2\pi k)\delta(t-k) = \sum_{k=0}^{\infty}0\delta(t-k) = 0 \end{aligned}$$

(c) $x[n] = \cos(0.2\pi n)\sum_{k=-\infty}^{0}\delta[n-10k]$

Answer:

$$\begin{aligned} x[n] &= \cos(0.2\pi n)\sum_{k=-\infty}^{0}\delta[n-10k] = \sum_{k=-\infty}^{0}\cos(0.2\pi n)\delta[n-10k] \\ &= \sum_{k=-\infty}^{0}\cos(2\pi k)\delta[n-10k] = \sum_{k=-\infty}^{0}\delta[n-10k] \end{aligned}$$

Problem 1.8

Compute the convolution: $\delta(t-T) * e^{-2t}u(t) = \int_{-\infty}^{\infty}\delta(\tau-T)e^{-2(t-\tau)}u(t-\tau)d\tau$.

Problem 1.9

Write the following complex signals in (i) polar form and (ii) rectangular form.

Polar form: $x(t) = r(t)e^{j\theta(t)}$, $r(t), \theta(t) \in \mathbb{R}$ for continuous-time signals,

$x[n] = r[n]e^{j\theta[n]}$, $r[n], \theta[n] \in \mathbb{R}$ for discrete-time signals.

Rectangular form: $x(t) = a(t) + jb(t),\ a(t), b(t) \in \mathbb{R}$ for continuous-time signals,

$x[n] = a[n] + jb[n],\ a[n], b[n] \in \mathbb{R}$ for discrete-time signals.

(a) $x_1(t) = j + \dfrac{t}{1-j}$

Answer:

$$x_1(t) = j + \frac{t}{1-j} = j + \frac{(1+j)t}{2} = 0.5t + j(1+0.5t)$$
$$\Rightarrow a(t) = 0.5t, \quad b(t) = (1+0.5t)$$
$$x_1(t) = \sqrt{0.25t^2 + (1+0.5t)^2}\, e^{j\arctan\left[(1+0.5t)/0.5t\right]}$$
$$\Rightarrow \quad r_1(t) = \sqrt{0.5t^2 + t + 1}, \ \theta_1(t) = \arctan\left[(1+0.5t)/0.5t\right]$$

(b) $x_2[n] = jn + e^{j2n}$

Answer:

$$x_2[n] = jn + e^{j2n} = \cos(2n) + j[n + \sin(2n)]$$
$$\Rightarrow a_2[n] = \cos(2n),\ b_2[n] = n + \sin(2n)$$
$$= \sqrt{\cos(2n)^2 + [n+\sin(2n)]^2}\, e^{j\arctan[(n+\sin(2n))/\cos(2n)]}$$
$$= \sqrt{1 + n^2 + 2n\sin(2n)}\, e^{j\arctan[(n+\sin(2n))/\cos(2n)]}$$
$$\Rightarrow \quad r_2[n] = \sqrt{1+n^2+2n\sin(2n)},\ \theta_2[n] = j\arctan[(n+\sin(2n))/\cos(2n)]$$

Problem 1.10

Determine whether the following systems are: 1. Memoryless, 2. Time-invariant, 3. Linear, 4. Causal, 5. BIBO Stable. Justify your answers.

(a) $y(t)=\frac{d}{dt}x(t)$, where the time derivative of $x(t)$ is defined as $\frac{d}{dt}x(t):=\lim_{\Delta t\to 0}\frac{x(t)-x(t-\Delta t)}{\Delta t}$.

(b) $y(t)=\frac{t}{1+x(t-1)}$

(c) $y(t)=2tx(2t)$

(d) $y[n]=\sum_{k=-\infty}^{n}x[k-n]$

(e) $y[n]=x[n]+nx[n+1]$

(f) $y[n]=x[n]+x[n-2]$

Problem 1.11

Find the fundamental periods and fundamental frequencies of the following periodic signal.

(a) $x[n]=\cos(0.01\pi n)e^{j0.13\pi n}$

Answer:

$$x[n]=\cos(0.01\pi n)e^{j0.13\pi n}=0.5e^{j0.14\pi n}+0.5e^{-j0.12\pi n}=0.5e^{j\frac{7}{100}(2\pi)n}+0.5e^{-j\frac{6}{100}(2\pi)n},$$

thus $\frac{m_1}{N_1}=\frac{7}{100}$ for the first term, and $\frac{m_2}{N_2}=\frac{6}{100}=\frac{3}{50}$ for the second. The fundamental period (common to both terms) is $N=100$, and the fundamental frequency is $\Omega_0=\frac{2\pi}{100}=\frac{\pi}{50}$.

(b) $x(t)=\sum_{k=-\infty}^{\infty}e^{-(t-2k)}\cos\big(4\pi(t-2k)\big)\big[u(t-2k)-u(t-2k-1)\big]$. Sketch this signal.

Answer:

This signal is constructed from a damped cosine term which lasts one second ($k=0$ in the summation) and is repeated every two seconds. Therefore, its fundamental period is $T=2$ seconds and its fundamental frequency is $\omega_0 = \frac{2\pi}{T} = \pi$ radians/s. The signal is sketched below.