Solutions to Assignment 10

10.1 Exercise 13.6 of Boulet's book;

Compute the inverse z-transform of $X(z)=\frac{z^2}{z+0.2}$, $0.2<|z|<\infty$ using the power series expansion method.

Answer:

$$X(z)=\frac{z^1}{(1+0.2z^{-1})}=\frac{z^2}{z+0.2},\quad 0.2<|z|<\infty$$

Long division yields:

$$\begin{array}{r l}
 & z-0.2+0.04z^{-1}-0.008z^{-2}\ldots \\
1+0.2z^{-1}\,) & z \\
 & \underline{z+0.2} \\
 & -0.2 \\
 & \underline{-0.2-0.04z^{-1}} \\
 & 0.04z^{-1} \\
 & \underline{0.04z^{-1}+0.008z^{-2}} \\
 & -0.008z^{-2}
\end{array}$$

Note that the resulting power series converges because the ROC implies $|0.2z^{-1}|<1$. The signal is:

$$\begin{aligned}
x[n] &= \delta[n+1]-0.2\delta[n]+(0.2)^2\,\delta[n-1]-(0.2)^3\,\delta[n-2]+\ldots \\
&= (-0.2)^{n+1}u[n+1]
\end{aligned}$$

10.2 Exercise 13.8 of Boulet's book.

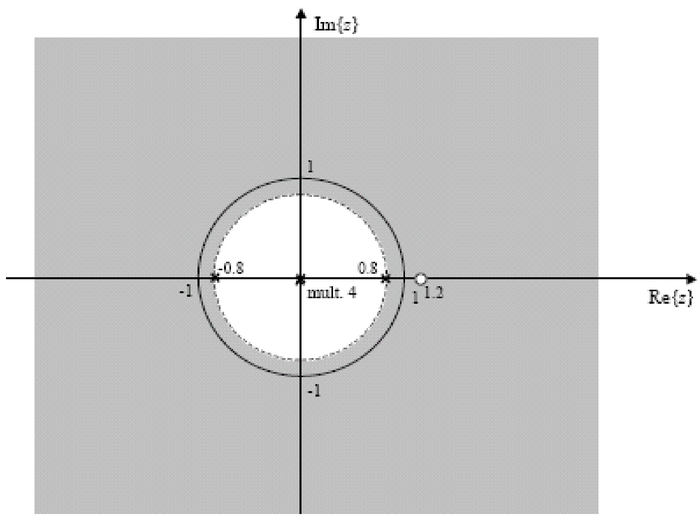
Consider a DLTI system with transfer function $H(z) = \dfrac{z^{-3} - 1.2z^{-4}}{(z-0.8)(z+0.8)}$.

(a) Sketch the pole-zero plot of the system.

Answer:

$$H(z) = \frac{z^{-5}(1-1.2z^{-1})}{(1-0.8z^{-1})(1+0.8z^{-1})} = \frac{z-1.2}{z^{4}(z-0.8)(z+0.8)}$$

poles at -0.8, 0.8 and 0 (multiplicity 4), zeros at 1.2 and at infinity.



(b) Find the ROC that makes this system stable.

Answer:

For stability, the ROC must include the unit circle. Hence, the ROC is $|z| > 0.8$, as shown on the above pole-zero plot.

(c) Is the system causal with the ROC that you found in (b)? Justify your answer.

Answer:

For the system to be causal, we must have $\lim_{z\to\infty} H(z) < \infty$, which is the case here (=0). Hence, the system is causal.

(d) Suppose that $H(e^{j\omega})$ is bounded for all frequencies. Find the response of the system $y[n]$ to the input $x[n] = u[n]$.

Answer:

This means that the ROC is chosen to be $|z| > 0.8$. We have $X(z) = \dfrac{1}{1-z^{-1}}$, $|z| > 1$, and

$$Y(z) = H(z)U(z) = \frac{z^{-5}(1-1.2z^{-1})}{(1-0.8z^{-1})(1+0.8z^{-1})(1-z^{-1})},\ 1 < |z|$$

We will treat the time delay later, so let's do a partial fraction expansion of:

$$\frac{(1-1.2z^{-1})}{(1-0.8z^{-1})(1+0.8z^{-1})(1-z^{-1})} = \frac{-0.56}{1-z^{-1}} + \frac{1}{1-0.8z^{-1}} + \frac{0.56}{1+0.8z^{-1}}$$

where the ROC of each term is taken to be the exterior of a circle of radius equal to the magnitude of the pole. This gives:

$$w[n] = (0.8)^n u[n] + 0.56(-0.8)^n u[n] - 0.56u[n]$$

and adding the time delay yields:

$$y[n] = (0.8)^{n-5} u[n-5] + 0.56(-0.8)^{n-5} u[n-5] - 0.56u[n-5]$$

10.3 Exercise 13.10 of Boulet's book
Also, for the H(z) in this problem, please determine the corresponding difference equation relating the output y[n] to the input x[n].

Sketch the pole-zero plot and compute the impulse response $h[n]$ of the stable system with transfer function: $H(z)=\dfrac{2000z^3+1450z^2+135z}{(100z^2-81)(5z+4)}$.

Specify its ROC. Specify whether the system is causal or not.

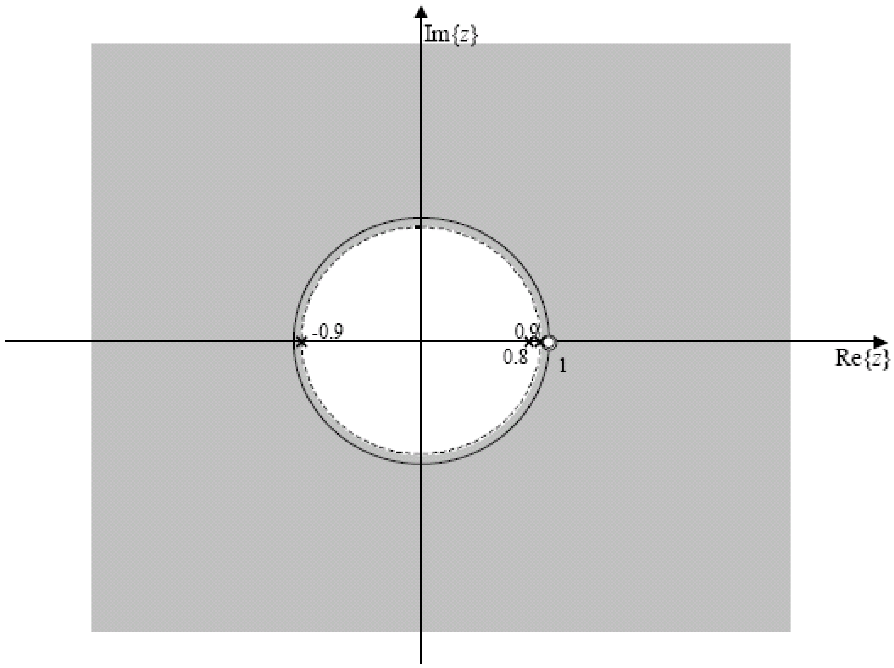
Answer:

$$
\begin{aligned}
H(z) &= \frac{2000z^3+1450z^2+135z}{(100z^2-81)(5z+4)}, \\
&= \frac{4+2.9z^{-1}+0.27z^{-2}}{(1+0.8z^{-1})(1+0.9z^{-1})(1-0.9z^{-1})}, \ |z|>0.9 \\
&= \left(\underbrace{\frac{-3}{(1+0.8z^{-1})}}_{|z|>0.8} + \underbrace{\frac{5}{(1+0.9z^{-1})}}_{|z|>0.9} + \underbrace{\frac{2}{(1-0.9z^{-1})}}_{|z|>0.9} \right)
\end{aligned}
$$

The inverse z-transform is obtained using Table D.10:

$$h[n]=-3(-0.8)^n u[n]+2(0.9)^n u[n]+5(-0.9)^n u[n].$$

The pole-zero plot is shown below.



The system is causal since ROC is the outside of a disk, including infinity.

For the H(z), the corresponding difference equation can be derived according to:

$$\frac{Y(z)}{X(z)} = \frac{2000 + 1450z^{-1} + 135z^{-2}}{500 + 400z^{-1} - 405z^{-2} - 324z^{-3}}$$

$$Y(z)(500 + 400z^{-1} - 405z^{-2} - 324z^{-3}) = X(z)(2000 + 1450z^{-1} + 135z^{-2})$$

Thus the difference equation is:

$$500y[n] + 400y[n-1] - 405y[n-2] - 324y[n-3] = 2000x[n] + 1450x[n-1] + 135x[n-2]$$

10.4 See Eq. (14.50) of Boulet's book. Now, the impulse response of a DT filter is defined by

$$h[n] = \begin{cases} 1/10, & n = 0,1,...,9 \\ 0, & otherwise \end{cases}$$

(a) Determine the transfer function of the system.
(b) Determine the frequency response of the filter.
(c) Sketch the poles and zeros of the system in the z-plane.

Answer:

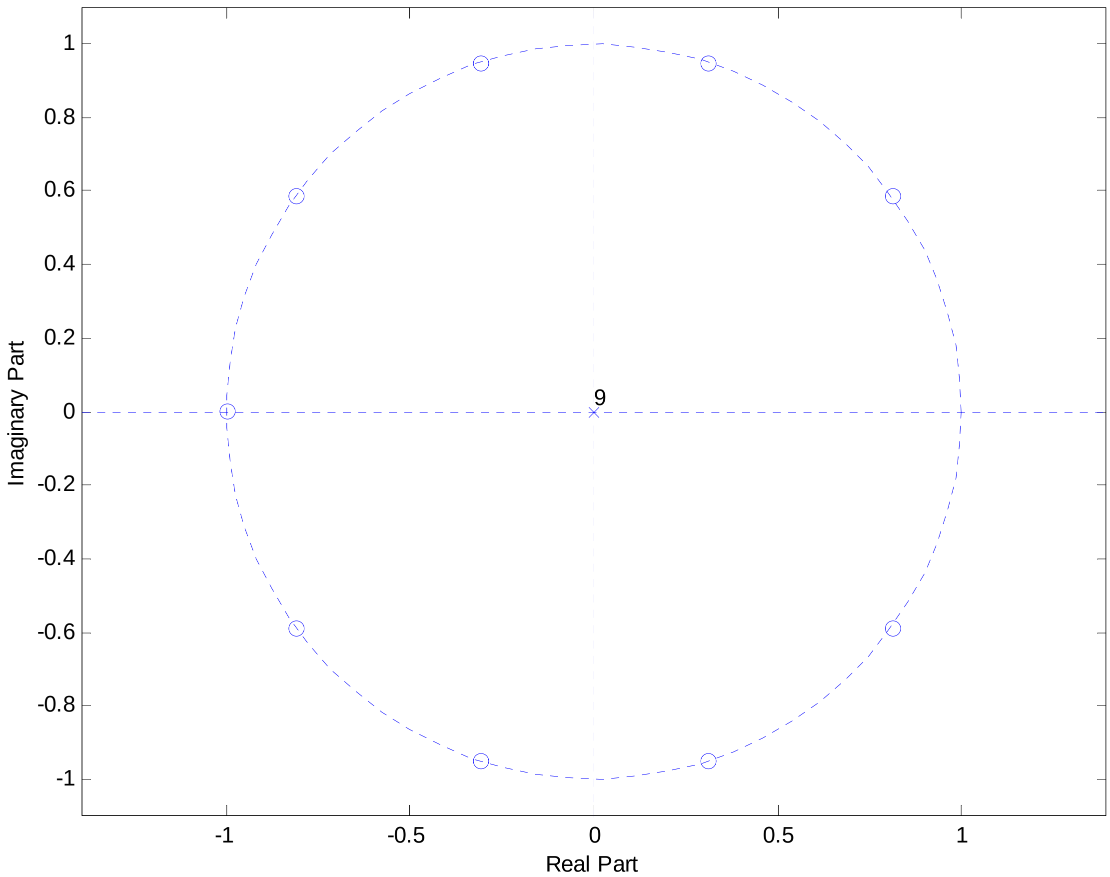
(a) Its transfer function can be obtained from the z-transform of the impulse response:

$$H(z) = \sum_{n=0}^{9} h[n]z^{-n} = \frac{1}{10}[1 + z^{-1} + ... + z^{-9}] = \frac{1}{10}\frac{1-z^{-10}}{1-z^{-1}} = \frac{1}{10}\frac{z^{10}-1}{z^{9}(z-1)}$$

(b) The frequency response of the filter is

$$H(z=e^{j\omega})=\frac{1}{10}\frac{1-e^{-j10\omega}}{1-e^{-j\omega}}=\frac{1}{10}\frac{e^{-j5\omega}(e^{j5\omega}-e^{-j5\omega})}{e^{-j\omega/2}(e^{j\omega/2}-e^{-j\omega/2})}=\frac{1}{10}e^{-j9\omega/2}\frac{\sin(5\omega)}{\sin(\omega/2)}$$

(c) The pole at z=1 is cancelled by the zero at z=1. Thus, there are 9 poles at z=0, and 9 zeros at $e^{j\frac{2\pi}{10}k}$, $k=1,2,...9$ in the z-plane, as shown in the figure below.



10.5 Find the inverse z-transform of

$$X(z)=\frac{1-\frac{1}{1024}z^{-10}}{(1-\frac{1}{2}z^{-1})},\qquad |z|>0$$

Answer:

Since the ROC includes the entire z-plane, we know that the signal must be finite length. From the finite sum formula, we have

$$\left[\frac{1-\frac{1}{1024}z^{-10}}{1-\frac{1}{2}z^{-1}}\right]=\sum_{n=0}^{9}(\frac{1}{2})^n z^{-n}$$

Comparing this with the definition of z-transform, we have

$$x[n] = \begin{cases} (\tfrac{1}{2})^n, & 0 \le n \le 9 \\ 0, & otherwise \end{cases}$$

10. 6 Consider a system whose input x[n] and output y[n] are related by

y[n-1]+2y[n]=x[n]

(a) Determine the zero-input response of the system if y[-1]=2.
(b) Determine the zero-state response of the system to the input $x[n]=(1/4)^n u[n]$.
(c) Determine the output of the system for n>=0 when the initial condition and the input are given as in (a) and (b).

Answer:

Applying the unilateral z-transform to the difference equation, we have:

$$z^{-1}\mathcal{Y}[z] + y[-1] + 2\mathcal{Y}[z] = \mathcal{X}[z]$$

(a) For the zero-input response, assume that x[n]=0. Since we have given that y[-1]=2, we have

$$z^{-1}\mathcal{Y}[z] + y[-1] + 2\mathcal{Y}[z] = 0$$

$$\mathcal{Y}[z] = \frac{-2}{2+z^{-1}} = \frac{-1}{1+\frac{1}{2}z^{-1}}$$

Taking the inverse z-transform, we have the zero-input response:

$$y[n] = -(-\frac{1}{2})^n u[n]$$

(b) For the zero-state response, set y[-1]=0. Also we have

$$\mathcal{X}(z) = \frac{1}{1-\frac{1}{4}z^{-1}}, \qquad |z| > 1/4$$

Therefore,

$$\mathcal{Y}[z] = (\frac{1}{1-\frac{1}{4}z^{-1}})(\frac{1/2}{1+\frac{1}{2}z^{-1}})$$

By doing partial fraction expansion, we get:

$$\mathcal{Y}[z] = \frac{1/6}{1-\frac{1}{4}z^{-1}} + \frac{1/3}{1+\frac{1}{2}z^{-1}}$$

Taking inverse z-transform, we get the zero-state response:

$$y[n] = \frac{1}{6}(\frac{1}{4})^n u[n] + \frac{1}{3}(-\frac{1}{2})^n u[n]$$

(c) The total response is the sum of zero-input response and the zero-state response:

$$y[n] = \frac{1}{6}(\frac{1}{4})^n u[n] - \frac{2}{3}(-\frac{1}{2})^n u[n]$$

10.7 Consider a signal $x(t)=\cos(200\pi t)$.
(a) Determine the sampling frequency F_s such that x(t) can be recovered via low-pass filtering of the sampled signal $x(nT_s)$, where $T_s=1/F_s$.
(b) If $F_s=120$ Hz, what frequency components are contained in the sampled signal $x(nT_s)$?

Answer:

(a) The frequency of the signal x(t) is 100 Hz.
According to the sampling theory, the sampling frequency F_s must be greater than $2f_M$, where $f_M=100$ Hz. Thus, F_s must be greater than 200 Hz.
(b) If $F_s=120$ Hz, then the signal $x[nT_s]$ contain frequency components at $\pm f_M+kF_s$, k=…,-3, -2, -1, 0, 1, 2, 3, …
Considering positive frequencies, we know the frequency components are located at:
100+120=220 Hz,
-100+120=20 Hz,
100+240=340 Hz,
-100+240 =140 Hz,
100+360=460 Hz,
-100+360=260 Hz,
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