Solutions to Assignment 9

9.1 Given a periodic signal x[n] and its Fourier series representation

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$$

we construct a signal y[n] using the FS coefficients of x[n]:

$$y[n]=a_n, \quad n=\dots, -2, -1, 0, 1, 2, \dots.$$

Prove that the Fourier series coefficients b_k of y[n] is x[-k]/N.

Answer:

$$x[k] = \sum_{n=0}^{N-1} a_n e^{jn\frac{2\pi}{N}k}$$

$$y[n] = \sum_{k=0}^{N-1} b_k e^{jk\frac{2\pi}{N}n}$$

$$b_k = \frac{1}{N}\sum_{n=0}^{N-1} y[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} a_n e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} x[-k]$$

9.2 Exercise 12.4 of Boulet's book

Given an LTI system with $h[n] = u[n]$, and an input $x[n] = (0.8)^n u[n]$, compute the DTFT of the output $Y(e^{j\omega})$ and its inverse DTFT $y[n]$.

Answer:

$$\begin{aligned} Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) &= \left(\frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)\right)\frac{1}{1-0.8e^{-j\omega}} \\ &= \frac{1}{(1-e^{-j\omega})(1-0.8e^{-j\omega})} + \pi\sum_{k=-\infty}^{\infty}\frac{1}{1-0.8e^{-j\omega}}\delta(\omega-2\pi k) \\ &= \frac{-4}{1-0.8e^{-j\omega}} + \frac{5}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\frac{1}{1-0.8e^{-j2\pi k}}\delta(\omega-2\pi k) \\ &= \frac{-4}{1-0.8e^{-j\omega}} + \frac{5}{1-e^{-j\omega}} + 5\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k) \end{aligned}$$

So that: $y[n] = 5u[n] - 4(0.8)^n u[n]$.

9.3 Exercise 12.6 of Boulet's book

Compute the Fourier transforms $X(e^{j\omega})$ of the following signals

(a) $x[n] = \alpha^n[\sin(\omega_0 n) + 2\cos(\omega_0 n)]u[n], \quad |\alpha| < 1.$

Answer:

$$\begin{aligned}
x[n] &= \left[\frac{1}{2j}\alpha^n\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) + \alpha^n\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)\right]u[n] \\
&= \frac{1}{2j}(\alpha e^{j\omega_0})^n u[n] - \frac{1}{2j}(\alpha e^{-j\omega_0})^n u[n] + (\alpha e^{j\omega_0})^n u[n] + (\alpha e^{-j\omega_0})^n u[n]
\end{aligned}$$

Using the table, we obtain the DTFT:

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{2j}\left[\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}} - \frac{1}{1-\alpha e^{-j(\omega+\omega_0)}}\right] + \left[\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}} + \frac{1}{1-\alpha e^{-j(\omega+\omega_0)}}\right] \\
&= \frac{1-0.5j}{1-\alpha e^{-j(\omega-\omega_0)}} + \frac{1+0.5j}{1-\alpha e^{-j(\omega+\omega_0)}} \\
&= \frac{(1-0.5j)(1-\alpha e^{-j(\omega+\omega_0)}) + (1+0.5j)(1-\alpha e^{-j(\omega-\omega_0)})}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} \\
&= \frac{2-\alpha e^{-j(\omega+\omega_0)} + 0.5j\alpha e^{-j(\omega+\omega_0)} - \alpha e^{-j(\omega-\omega_0)} - 0.5j\alpha e^{-j(\omega-\omega_0)}}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} \\
&= \frac{2-\alpha e^{-j\omega}(e^{-j\omega_0} + e^{j\omega_0}) + 0.5j\alpha e^{-j\omega}(e^{-j\omega_0} - e^{j\omega_0})}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} \\
&= \frac{2-2\alpha\cos\omega_0 e^{-j\omega} + \alpha\sin\omega_0 e^{-j\omega}}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} = \frac{2+\alpha(-2\cos\omega_0 + \sin\omega_0)e^{-j\omega}}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}}
\end{aligned}$$

(b) $x[n] = (u[n+2] - u[n-3]) * \sum_{k=-\infty}^{+\infty} e^{j\pi k}\delta[n-15k]$, where $*$ is the convolution operator.

Answer:

First consider the DTFT of the pulse signal $y[n] := u[n+2] - u[n-3]$ which is:

$$Y(e^{j\omega}) = \frac{\sin(5/2\,\omega)}{\sin(\omega/2)}$$

Now ,

$$\begin{aligned} x[n] &= y[n] * \sum_{k=-\infty}^{+\infty} (-1)^k \delta[n-15k] \\ &= y[n] * \sum_{k=-\infty}^{+\infty} \delta[n-30k] - y[n] * \sum_{k=-\infty}^{+\infty} \delta[n-15-30k] \end{aligned}$$

In the frequency domain,

$$\begin{aligned} X(e^{j\omega}) &= Y(e^{j\omega})\frac{2\pi}{30}\left[\sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{30}) - e^{-j\omega 15}\sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{30})\right] \\ &= Y(e^{j\omega})(1 - e^{-j\omega 15})\frac{2\pi}{30}\sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{30}) \\ &= \frac{2\pi}{30}\sum_{k=-\infty}^{\infty} Y(e^{jk\frac{2\pi}{30}})(1 - e^{-jk\pi})\delta(\omega - k\frac{2\pi}{30}) \\ &= \frac{2\pi}{30}\sum_{k=-\infty}^{\infty} Y(e^{jk\frac{2\pi}{30}})(1 - e^{-jk\pi})\delta(\omega - k\frac{2\pi}{30}) \\ &= \frac{2\pi}{30}\sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{6})}{\sin(k\frac{\pi}{30})}(1 - (-1)^k)\delta(\omega - k\frac{2\pi}{30}) \end{aligned}$$

9.4 Exercise 12.8 of Boulet's book

Compute the Fourier transform $X(e^{j\omega})$ of the signal $x[n]$ shown in Figure 12.7 and sketch its magnitude and phase over the interval $\omega \in [-\pi, \pi]$.

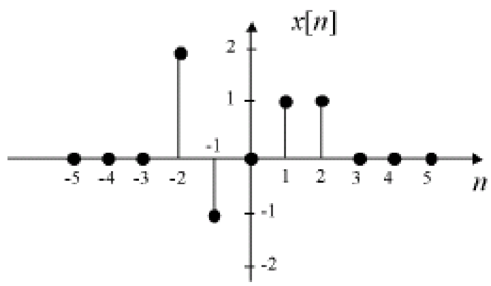


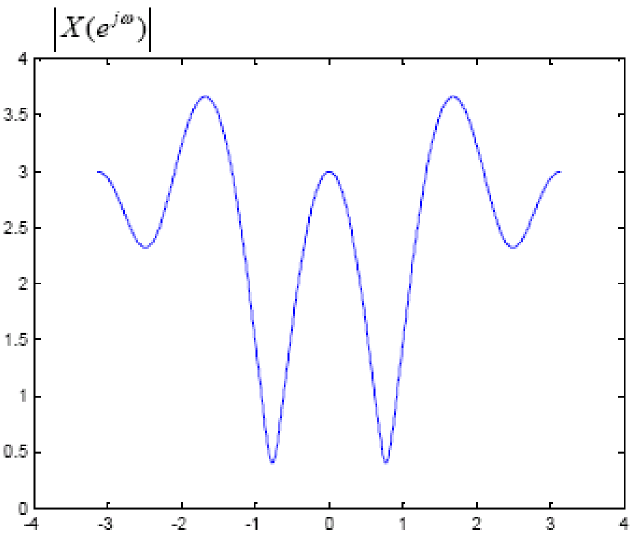
Figure 12.7: Signal in Exercise 12.8.

Answer:

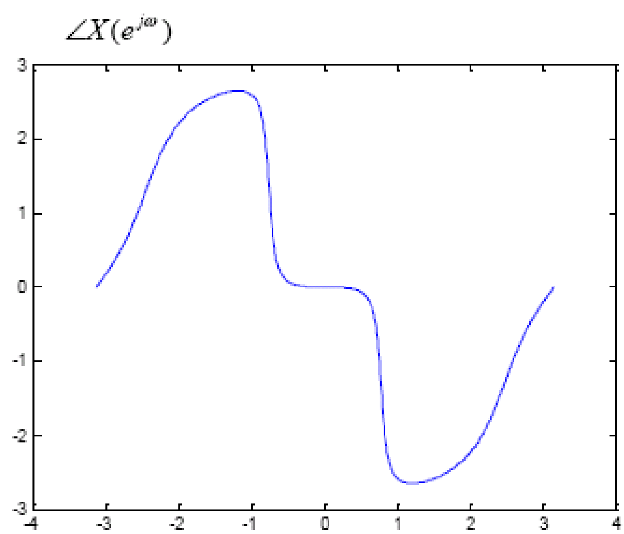
$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= 2e^{j\omega 2} - e^{j\omega} + e^{-j\omega} + e^{-j\omega 2} \\
&= [2\cos(2\omega) - \cos\omega + \cos\omega + \cos(2\omega)] + j[2\sin(2\omega) - \sin\omega - \sin\omega - \sin(2\omega)] \\
&= [3\cos(2\omega)] + j[\sin(2\omega) - 2\sin\omega]
\end{aligned}
$$

Magnitude is

$$
\begin{aligned}
\left|X(e^{j\omega})\right| &= \sqrt{[3\cos(2\omega)]^2 + [\sin(2\omega) - 2\sin\omega]^2} \\
&= \sqrt{9\cos^2(2\omega) + \sin^2(2\omega) - 4\sin(2\omega)\sin\omega + 4\sin^2(\omega)} \\
&= \sqrt{8\cos^2(2\omega) + 1 - 4\sin(2\omega)\sin\omega + 4\sin^2(\omega)}
\end{aligned}
$$



Phase: $$\angle X(e^{j\omega}) = \arctan\frac{\sin(2\omega) - 2\sin\omega}{3\cos(2\omega)}$$



9.5 Exercise 12.10 of Boulet's book

Consider a DLTI system with impulse response $h[n] = (-0.4)^n u[n] - (0.5)^{n-2} u[n-2]$. Compute the output signal $y[n]$ for the input $x[n] = (0.2)^n u[n]$. Use the DTFT.

Answer:

$$H(e^{j\omega}) = \frac{1}{1+0.4e^{-j\omega}} - \frac{e^{-j2\omega}}{1-0.5e^{-j\omega}}$$
$$X(e^{j\omega}) = \frac{1}{1-0.2e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left(\frac{1}{(1+0.4e^{-j\omega})(1-0.2e^{-j\omega})} + \frac{e^{-j2\omega}}{(1-0.5e^{-j\omega})(1-0.2e^{-j\omega})}\right)$$
$$= \frac{2/3}{(1+0.4e^{-j\omega})} + \frac{1/3}{(1-0.2e^{-j\omega})} + \frac{5/3e^{-j2\omega}}{(1-0.5e^{j\omega})} + \frac{-2/3e^{-j2\omega}}{(1-0.2e^{-j\omega})}$$

Using Table D.7, we obtain: $y[n] = \frac{2}{3}(-0.4)^n u[n] + \frac{5}{3}(0.5)^n u[n] - \frac{1}{3}(0.2)^n u[n]$.

9.6 Determine the inverse Fourier transforms of

(a) $X_1(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\{2\pi\delta(\omega-2\pi k) + \pi\delta(\omega-\frac{\pi}{2}-2\pi k) + \pi\delta(\omega+\frac{\pi}{2}-2\pi k)$

(b) $X_2(e^{j\omega}) = \begin{cases} 2j, & 0 < \omega \le \pi \\ -2j, & -\pi < \omega \le 0 \end{cases}$

Answer:

(a) Method 1: apply the inverse FT equation:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}[2\pi\delta(\omega) + \pi\delta(\omega-\frac{\pi}{2}) + \pi\delta(\omega+\frac{\pi}{2})]e^{j\omega n}d\omega$$
$$= e^{j\omega 0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n} = 1+\cos(\pi n/2)$$

Method 2: apply the property of DTFT:

Look up the Table D.7 in Boulet's book, we have the following DTFT pair:

$$1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-2\pi k)$$

According to the frequency-shifting property of the DTFT in Eq. (12.42), we have:

$$e^{j\frac{\pi}{2}n}1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\frac{\pi}{2}-2\pi k)$$

$$e^{-j\frac{\pi}{2}n}1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+\frac{\pi}{2}-2\pi k)$$

or:

$$e^{j\frac{\pi}{2}n}1/2 \leftrightarrow \sum_{k=-\infty}^{\infty} \pi\delta(\omega-\frac{\pi}{2}-2\pi k)$$

$$e^{-j\frac{\pi}{2}n}1/2 \leftrightarrow \sum_{k=-\infty}^{\infty} \pi\delta(\omega+\frac{\pi}{2}-2\pi k)$$

Thus,

$$x[n] = 1 + (e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n})/2 = 1 + \cos(\pi n/2)$$

(b) Applying the inverse FT equation, we have

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\omega})e^{j\omega n}d\omega = -\frac{1}{2\pi}\int_{-\pi}^{0} 2je^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi} 2je^{j\omega n}d\omega$$
$$= \frac{j}{\pi}[-\frac{1-e^{-jn\pi}}{jn} + \frac{e^{jn\pi}-1}{jn}] = -(4/n\pi)\sin^2(n\pi/2)$$

9.7 Consider a causal and stable LTI system S whose input is x[n] and output y[n] are related through the second-order difference system:

y[n]-(1/6)y[n-1]-(1/6)y[n-2]=x[n]

(a) Determine the frequency response of the system $H(e^{j\omega})$;
(b) determine the impulse response h[n] of the system

Answer:

(a) Taking the FT of both sides of the difference equation, we have:

$$Y(e^{j\omega})[1-(1/6)e^{-j\omega} - (1/6)e^{-j\omega 2}] = X(e^{j\omega})$$

Therefore,

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-(1/6)e^{-j\omega} - (1/6)e^{-j\omega 2}} = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}$$

(b) Using partial fraction expansion, we have

$$H(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})} = \frac{3/5}{1-\frac{1}{2}e^{-j\omega}} + \frac{2/5}{1+\frac{1}{3}e^{-j\omega}}$$

Taking the inverse FT according to the FT pairs in the Table D.7, we have

$$h[n] = (3/5)(\frac{1}{2})^n u[n] + (2/5)(-\frac{1}{3})^n u[n]$$

9.8 A causal and stable system has the input and output signals as follows

$$\left(\frac{4}{5}\right)^n u[n] \rightarrow n\left(\frac{4}{5}\right)^n u[n]$$

(a) Determine the frequency response of the system $H(e^{j\omega})$;
(b) A difference equation can be used to characterize the input-output relationship of the system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Determine the coefficients of the difference equation.

Answer:

(a) Since the LTI system is causal and stable, a single input-output pair is sufficient to determine the frequency response of the system. The frequency response is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$x[n] = \left(\frac{4}{5}\right)^n u[n] \leftrightarrow \frac{1}{1-(4/5)e^{-j\omega}}$$

$$y[n] = n\left(\frac{4}{5}\right)^n u[n] \leftrightarrow j\frac{d}{\omega}[\frac{1}{1-(4/5)e^{-j\omega}}] = \frac{(4/5)e^{-j\omega}}{(1-(4/5)e^{-j\omega})^2}$$

Therefore,

$$H(e^{j\omega}) = \frac{(4/5)e^{-j\omega}}{1-(4/5)e^{-j\omega}}$$

(b)

$$H(e^{j\omega}) = \frac{(4/5)e^{-j\omega}}{1-(4/5)e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$[1-(4/5)e^{-j\omega}]Y(e^{j\omega}) = (4/5)e^{-j\omega}X(e^{j\omega})$$

$$y[n]-(4/5)y[n-1] = (4/5)x[n-1]$$