Assignment 8

Due Nov. 17, 2008 before class.

=======================Part 1 (no submission is required)======================

Practice makes perfect.

Do and understand all exercises in Chapter 8 of Benoit Boulet's book.

==================Part 2 (Handwritten and submission are required)================

8.1 In Boulet's book page 311-312, the second-order system is given by the transfer function:

$$H(s) = \frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where 0<ζ<1, i.e., the system is under damped.

(a) What are the poles of the system?
(b) Please show that the impulse response of the system is given by Eq. (8.60):

$$h(t) = A\frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_n\sqrt{1-\xi^2}\,t)u(t)$$

Hint: $e^{-at}\sin(\omega_n t)u(t) \leftrightarrow \dfrac{\omega_n}{(s+a)^2 + \omega_n{}^2}$

(c) Determine the time when h(t)=0.

(d) Show that the step response of the system is in the first line in Eq. (8.62):

$$s(t) = u(t) - e^{-\zeta\omega_n t}[\cos(\omega_n\sqrt{1-\zeta^2}\,t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t)]u(t)$$

Hint: Obtain the partial fraction expansion of H(s)/s, and do the inverse Lapalace transform using

$$e^{-at}\cos(\omega_n t)u(t) \leftrightarrow \frac{s+a}{(s+a)^2 + \omega_n{}^2}$$

$$e^{-at}\sin(\omega_n t)u(t) \leftrightarrow \frac{\omega_n}{(s+a)^2 + \omega_n{}^2}$$

Answer:

(a) Poles are

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad \zeta < 1$$

(b)

$$H(s)=\frac{A\omega_n{}^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{A\omega_n}{(s-p_1)(s-p_2)}=\frac{B}{s-p_1}+\frac{C}{s-p_2}$$

where

$$B=\frac{A\omega_n{}^2}{p_1-p_2}=\frac{A\omega_n}{j2\sqrt{1-\zeta^2}}$$

$$C=\frac{A\omega_n{}^2}{p_2-p_1}=-\frac{A\omega_n}{j2\sqrt{1-\zeta^2}}$$

then

$$H(s)=\frac{A\omega_n}{j2\sqrt{1-\zeta^2}}\frac{1}{s-p_1}-\frac{A\omega_n}{j2\sqrt{1-\zeta^2}}\frac{1}{s-p_2}$$

$$h(t)=[\frac{A\omega_n}{j2\sqrt{1-\zeta^2}}e^{p_1t}-\frac{A\omega_n}{j2\sqrt{1-\zeta^2}}e^{p_2t}]u(t)$$

$$=\frac{A\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t)u(t)$$

(c) h(t)=0 if

$$\omega_n\sqrt{1-\zeta^2}t=k\pi$$

$$t=\frac{k\pi}{\omega_n\sqrt{1-\zeta^2}},\qquad k=1,2,\ldots$$

(d)

The Laplace transform of the step response is

$$\frac{1}{s}H(s)=\frac{A\omega_n{}^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{A\omega_n}{s(s-p_1)(s-p_2)}=\frac{A_1}{s}+\frac{A_2 s+A_3}{s^2+2\zeta\omega_n s+\omega_n^2}$$

where

$$A_1=A$$

$$A_2=-A$$

$$A_3=-2\zeta\omega_n A$$

Then

$$\frac{1}{s}H(s)=\frac{A}{s}+\frac{-As-2\zeta\omega_n A}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}=\frac{A}{s}+\frac{-A(s+\zeta\omega_n)-\zeta\omega_n A}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$=\frac{A}{s}+\frac{-A(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}+\frac{-A\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

The step response is

$$s(t)=A[1-e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t)u(t)-\frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t)]u(t)$$

8.2 Exercise 8.4 of Boulet's book.

Compute the DC gain in dBs, the peak resonance in dBs, and the quality Q of the second-order causal filter with transfer function: $H(s) = \frac{1000}{s^2 + 2s + 100}$.

Answer:

The DC gain is $20\log_{10}|H(j0)| = 20\log_{10} 10 = 20dB$. First, we find the damping ratio and the natural undamped frequency of the filter: $\zeta = 0.1$, $\omega_n = 10$. The resonant frequency can be computed $\omega_{\max} := \omega_n\sqrt{1-2\zeta^2} = 10\sqrt{1-0.02} = 9.8995$. At the resonant frequency, the magnitude of the peak resonance is given by the DC gain plus the peak gain:

$$\begin{aligned} 20\log_{10}|H(j\omega_{\max})| &= 20 - 20\log_{10}\left\{2\zeta\sqrt{1-\zeta^2}\right\} \\ &= 20 - 20\log_{10}\left\{0.2\sqrt{0.99}\right\} \\ &= 34.02dB \end{aligned}$$

Quality: $Q = \frac{1}{2\zeta} = \frac{1}{0.2} = 5$.

8.3 Exercise 8.6 of Boulet's book.

Compute the group delay of a communication channel represented by the causal first-order system $H(s) = \frac{1}{0.01s+1}$, $\mathrm{Re}\{s\} > -100$. Compute the approximate value of the channel's delay at very low frequencies.

Answer:

The group delay is given by $\tau(\omega) = -\frac{d}{d\omega}\angle H(j\omega)$. We need to compute the phase of the frequency response first: $\angle H(j\omega) = \angle \frac{1}{0.01j\omega+1} = \arctan\left(\frac{-0.01\omega}{1}\right)$. Group delay:

$$\begin{aligned}
\tau(\omega) &= -\frac{d}{d\omega}\angle H(j\omega) \\
&= -\frac{d}{d\omega}\arctan\left(\frac{-0.01\omega}{1}\right) \\
&= -\frac{1}{1+(0.01\omega)^2}(-0.01) \\
&= \frac{0.01}{1+(0.01\omega)^2}
\end{aligned}$$

At very low frequencies: $\tau(\omega) \approx \frac{0.01}{1+0} = 0.01s$, so the channel introduces a delay of approxiamtely 10ms, which is equal to the time constant.

8.4 Exercise 8.8 of Boulet’s book.

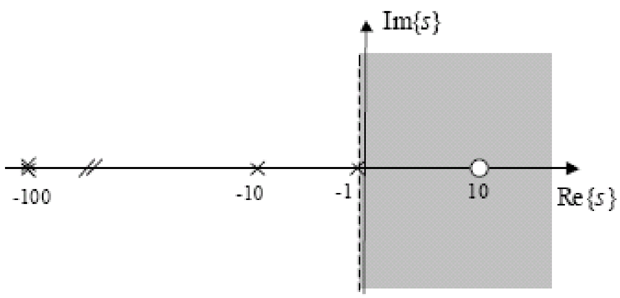
Sketch the pole-zero plots in the s-plane and the Bode plots (magnitude and phase) for the following systems. Specify if the transfer functions have poles or zeros at infinity.

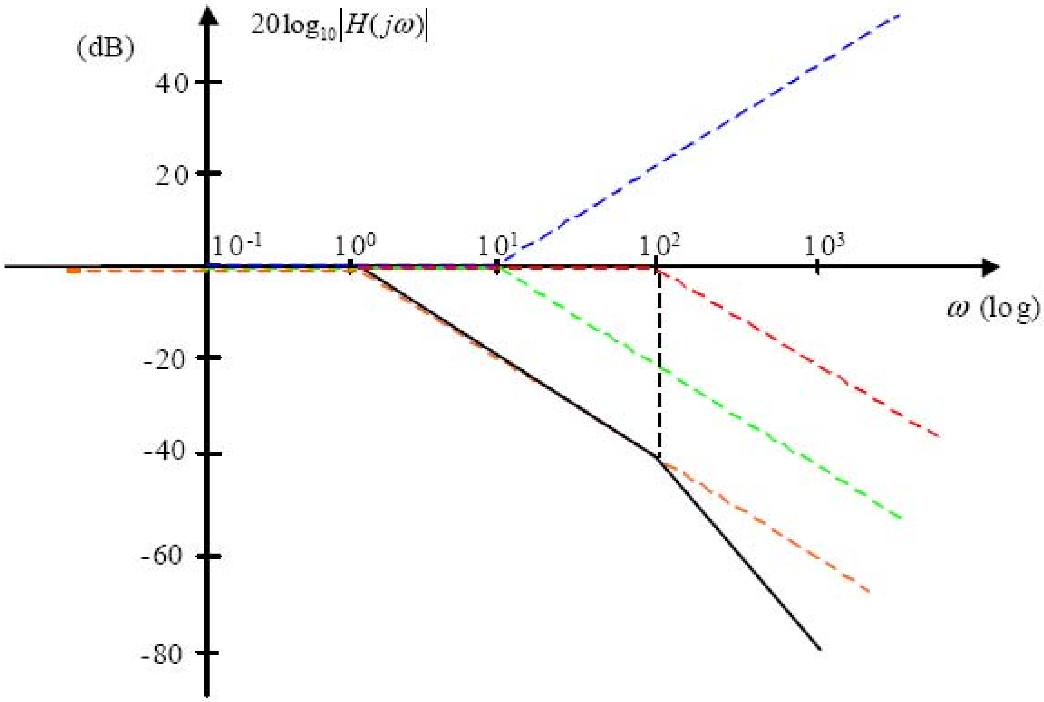
(a) $H(s)=\dfrac{100(s-10)}{(s+1)(s+10)(s+100)}$, $\mathrm{Re}\{s\}>-1$.

Answer:

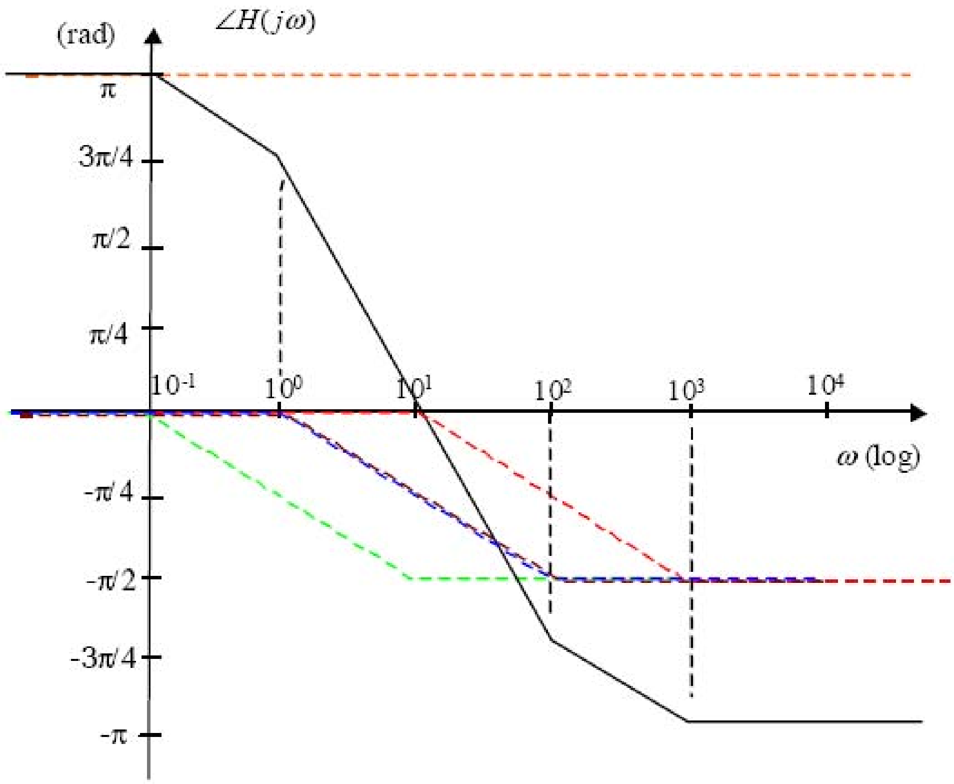
$$H(s)=\frac{100(s-10)}{(s+1)(s+10)(s+100)}=\frac{-(-s/10+1)}{(s+1)(s/10+1)(s/100+1)}$$

Break frequencies at $\omega_1=10$ (zero), $\omega_2=1$, $\omega_3=10$, $\omega_4=100$ (poles), two zeros at ∞.





(dB)
$20\log_{10}|H(j\omega)|$
40
20
-20
-40
-60
-80
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
ω (log)

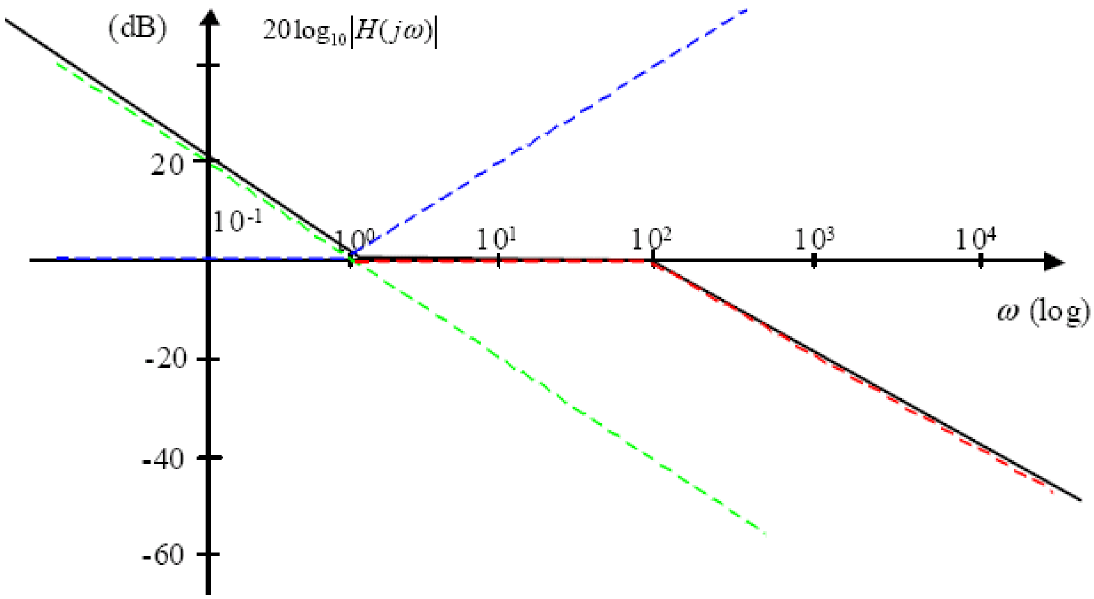


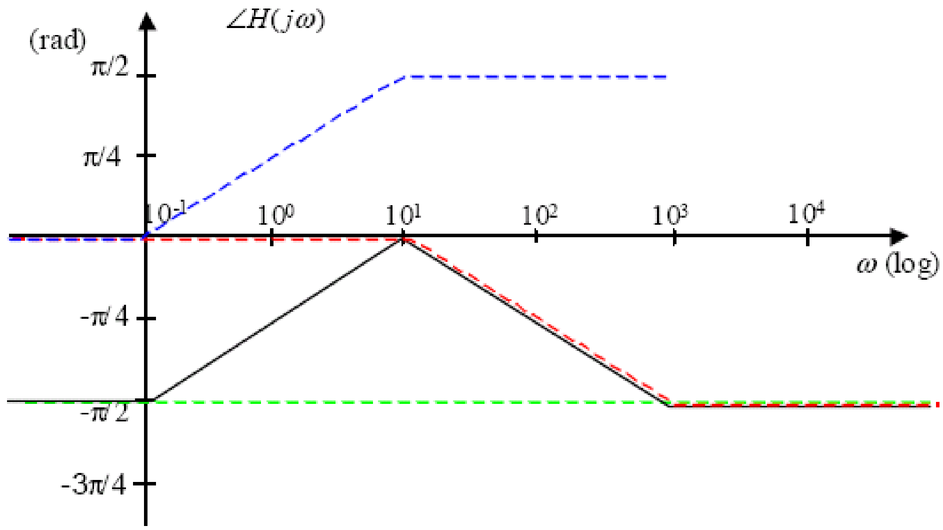
(rad)
$\angle H(j\omega)$
π
3π/4
π/2
π/4
-π/4
-π/2
-3π/4
-π
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
ω (log)

(b) $H(s)=\dfrac{s+1}{s(0.01s+1)}$, $\mathrm{Re}\{s\}>0$.

Answer:

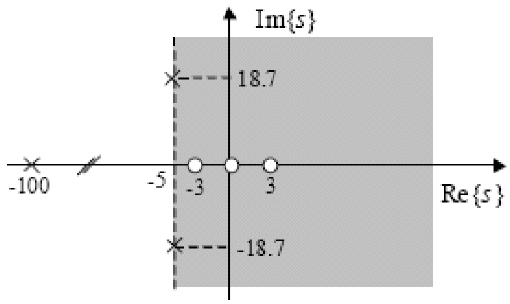
Break frequencies at $\omega_1=1$ (zero); $\omega_2=0$, $\omega_3=100$ (poles), one zero at ∞

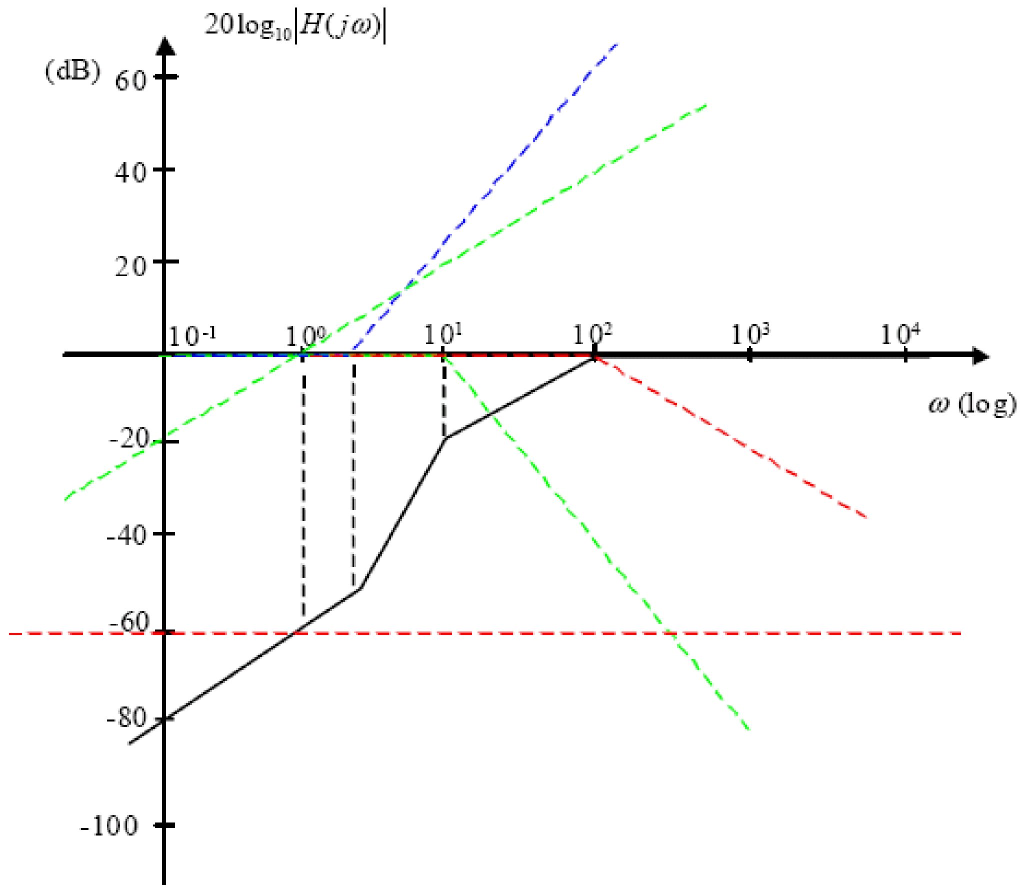


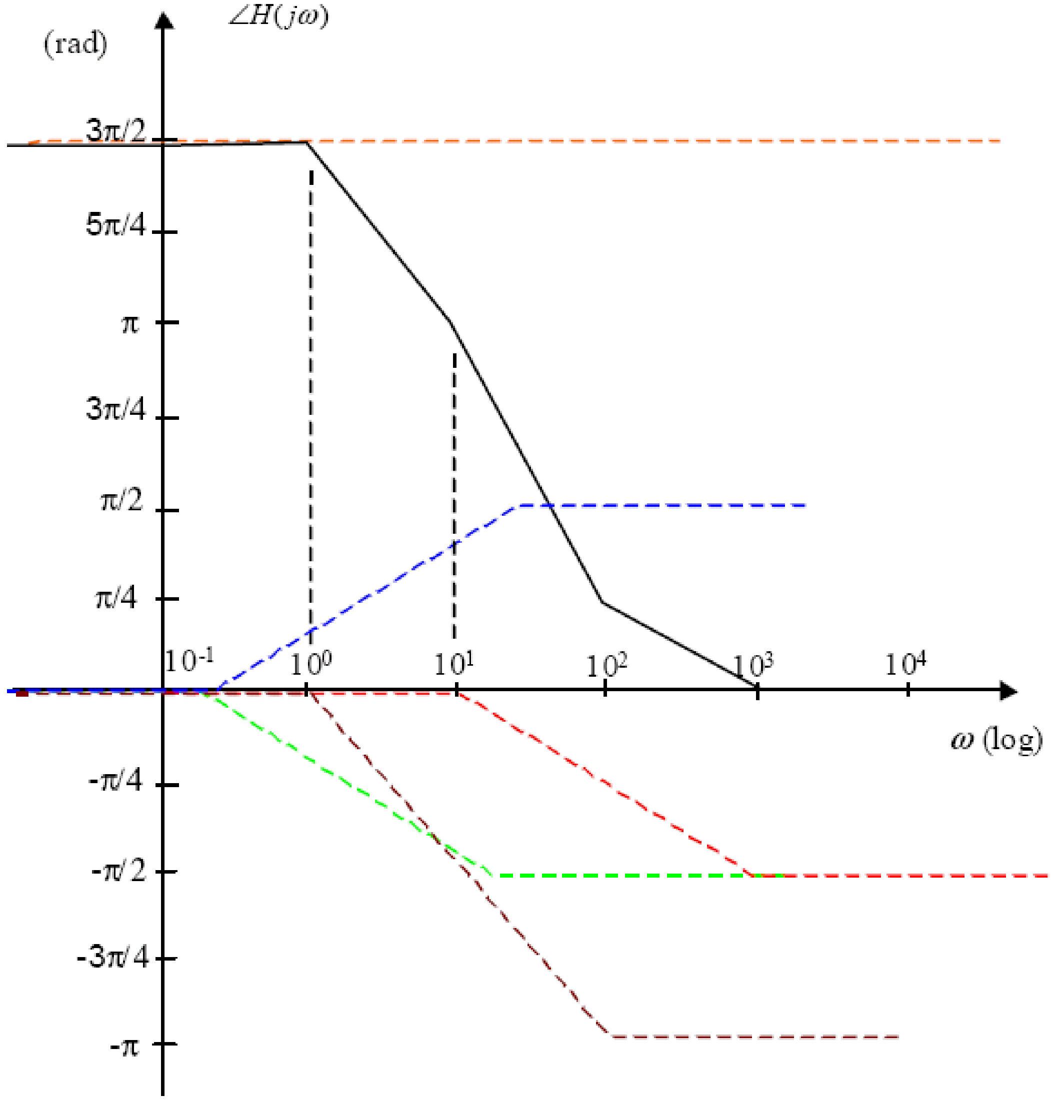


(c) $H(s)=\dfrac{s(s^2-9)}{(s+100)(s^2+10s+100)}$, $\mathrm{Re}\{s\}>-5$

We have: $H(s)=\dfrac{s(s-3)(s+3)}{(s+100)(s^2+10s+100)}=\dfrac{-0.0009s(-s/3+1)(s/3+1)}{(s/100+1)(s^2/100+s/10+1)}$







(rad)
∠H(jω)
3π/2
5π/4
π
3π/4
π/2
π/4
10-1
10⁰
10¹
10²
10³
10⁴
ω (log)
-π/4
-π/2
-3π/4
-π

8.5 Exercise 8.10 of Boulet's book.

Consider the causal differential system described by its direct form realization shown in Figure 8.9.

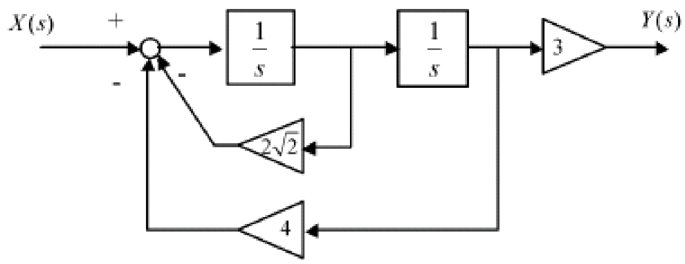


Figure 8.9: System of Exercise 8.10.

This system has initial conditions $\frac{dy(0^-)}{dt} = -1, \quad y(0^-) = 2$. Suppose that the system is subjected to the unit step input signal $x(t) = u(t)$.

(a) Write the differential equation of the system. Find the system's damping ratio ζ and undamped natural frequency ω_n. Give the transfer function of the system and specify its ROC. Sketch its pole-zero plot. Is the system stable? Justify.

Answer:

Differential equation: $\frac{d^2y(t)}{dt^2} + 2\sqrt{2}\frac{dy(t)}{dt} + 4y(t) = 3x(t)$.

Let's take the unilateral Laplace transform on both sides of the differential equation.

$$\left[s^2\mathcal{Y}(s)-sy(0^-)-\frac{dy(0^-)}{dt}\right]+2\sqrt{2}\left[s\mathcal{Y}(s)-y(0^-)\right]+4\mathcal{Y}(s)=3\mathcal{X}(s)$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$.

$$\left(s^2+2\sqrt{2}s+4\right)\mathcal{Y}(s)=3\mathcal{X}(s)+sy(0^-)+2\sqrt{2}y(0^-)+\frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s)=\underbrace{\frac{3\mathcal{X}(s)}{s^2+2\sqrt{2}s+4}}_{\text{zero-state resp.}}+\underbrace{\frac{(s+2\sqrt{2})y(0^-)+\frac{dy(0^-)}{dt}}{s^2+2\sqrt{2}s+4}}_{\text{zero-input resp.}}$$

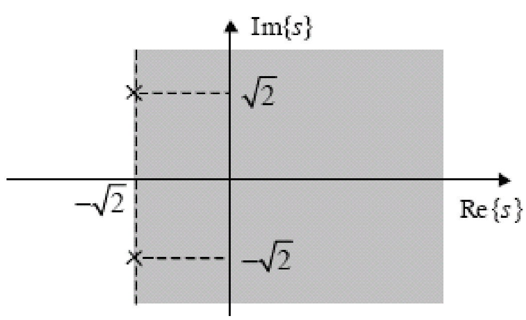
The transfer function is $\mathcal{H}(s)=\dfrac{3}{s^2+2\sqrt{2}s+4}$,

and since the system is causal, the ROC is an open RHP to the right of the rightmost pole.

The undamped natural frequency is $\omega_n=2$ and the damping ratio is $\zeta=\dfrac{1}{\sqrt{2}}$. The poles are

$$p_{1,2}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}=-\sqrt{2}\pm j2\sqrt{1-\frac{1}{2}}=-\sqrt{2}\pm j\sqrt{2}\,.$$

Therefore the ROC is $\mathrm{Re}\{s\}>-\sqrt{2}$. System is *stable* as $j\omega$-axis is contained in ROC. Pole-zero plot:



(b) Compute the step response of the system (including the effect of initial conditions), its steady-state response $y_{ss}(t)$ and its transient response $y_{tr}(t)$ for $t \geq 0$. Identify the zero-state response and the zero-input response in the Laplace domain.

Answer:

The unilateral LT of the input is given by

$$\mathcal{X}(s) = \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0,$$

thus

$$\mathcal{Y}(s) = \underbrace{\frac{3}{(s^2 + 2\sqrt{2}s + 4)s}}_{\substack{\mathrm{Re}\{s\}>0 \\ \text{zero-state resp.}}} + \underbrace{\frac{2(s + 2\sqrt{2}) - 1}{s^2 + 2\sqrt{2}s + 4}}_{\substack{\mathrm{Re}\{s\}>-1 \\ \text{zero-input resp.}}} = \frac{2s^2 + (4\sqrt{2} - 1)s + 3}{\left(s^2 + 2\sqrt{2}s + 4\right)s}$$

Let's compute the overall response:

$$\begin{aligned}
\mathcal{Y}(s) &= \frac{2s^2 + (4\sqrt{2} - 1)s + 3}{\left(s^2 + 2\sqrt{2}s + 4\right)s}, \quad \mathrm{Re}\{s\} > 0 \\
&= \underbrace{\frac{A\sqrt{2} + B(s + \sqrt{2})}{\left(s + \sqrt{2}\right)^2 + 2}}_{\mathrm{Re}\{s\}>-\sqrt{2}} + \underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0} \\
&= \underbrace{\frac{A\sqrt{2} + B(s + \sqrt{2})}{\left(s + \sqrt{2}\right)^2 + 2}}_{\mathrm{Re}\{s\}>-\sqrt{2}} + \underbrace{\frac{0.75}{s}}_{\mathrm{Re}\{s\}>0}
\end{aligned}$$

Let $s = -\sqrt{2}$ to compute:

$$\frac{2(2)+(4\sqrt{2}-1)(-\sqrt{2})+3}{2(-\sqrt{2})}=\frac{1}{\sqrt{2}}A+\frac{0.75}{-\sqrt{2}}$$

$$\frac{-1+\sqrt{2}}{-2\sqrt{2}}=\frac{1}{\sqrt{2}}A+\frac{0.75}{-\sqrt{2}},$$

$$\Rightarrow A=\frac{1-\sqrt{2}}{2}+0.75=0.5429$$

then multiply both sides by s and let $s\to\infty$ to get $2=B+0.75\Rightarrow B=1.25$:

$$\mathcal{Y}(s)=\underbrace{\frac{0.5429\sqrt{2}}{\left(s+\sqrt{2}\right)^2+2}}_{\operatorname{Re}\{s\}>-\sqrt{2}}+\underbrace{\frac{1.25(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\operatorname{Re}\{s\}>-\sqrt{2}}+\underbrace{\frac{0.75}{s}}_{\operatorname{Re}\{s\}>0}$$

Notice that the second term $\frac{1}{s}$ is the steady-state response, and thus $y_{ss}(t)=0.75u(t)$.

Taking the inverse Laplace transform using the table yields:

$$y(t)=0.5429e^{-\sqrt{2}t}\sin(\sqrt{2}t)u(t)+1.25e^{-\sqrt{2}t}\cos(\sqrt{2}t)u(t)+0.75u(t).$$

Thus, the transient response is $y_{tr}(t)=0.5429e^{-\sqrt{2}t}\sin(\sqrt{2}t)u(t)+1.25e^{-\sqrt{2}t}\cos(\sqrt{2}t)u(t)$.

(c) Compute the percentage of first overshoot in the step response of the system assumed this time to be initially at rest.

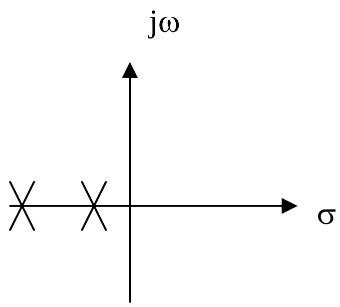
Answer:

Transfer function is $\mathcal{H}(s)=\frac{3}{s^2+2\sqrt{2}s+4},\operatorname{Re}\{s\}>\sqrt{2}$ with damping ratio $\zeta=\frac{1}{\sqrt{2}}$:

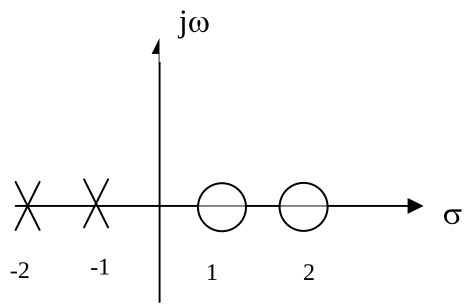
$$OS=100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\%=100e^{-\frac{0.707\pi}{0.707}}\%=100e^{-\pi}\%=4.3\%.$$

8.6 Given the poles and zeros of 4 systems as shown in the following s-planes, is each system LPF, HPF,BPF, all pass, or/and minimum phase system?

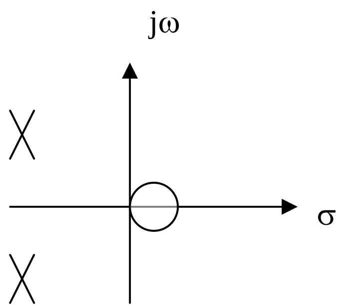
(a)



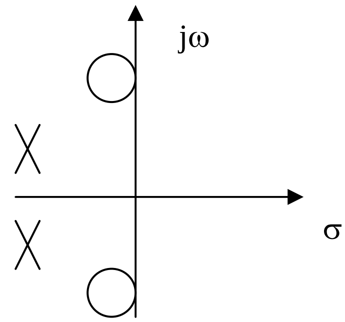
(b)



(c)



(d)



Answer:

(a) LPF and minimum phase.
(b) All-pass and non-minimum phase.
(c) High-pass and non-minimum phase.
(d) Low-pass and minimum phase.