Solution to Assignment 7

7.1 Exercise 7.4 of Boulet's book.

Use the unilateral Laplace transform to compute the output response $y(t)$ to the input $x(t) = \cos(10t)u(t)$ of the following causal LTI differential system with initial conditions $y(0^-) = 1,\ \frac{dy(0^-)}{dt} = 1$:

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t).$$

Answer:

Let us transform this differential equation:

$$s^2\mathcal{Y}(s) - sy(0^-) - \frac{dy(0^-)}{dt} + 5s\mathcal{Y}(s) - 5y(0^-) + 6\mathcal{Y}(s) = \mathcal{X}(s)$$

Using the initial conditions and simplifying, we obtain:

$$\begin{aligned}\mathcal{Y}(s) &= \frac{1}{s^2+5s+6}\mathcal{X}(s) + \frac{s+6}{s^2+5s+6} \\ &= \frac{1}{(s+3)(s+2)}\mathcal{X}(s) + \frac{s+6}{(s+3)(s+2)}\end{aligned}$$

But $\mathcal{X}(s) = \frac{s}{s^2+100},\ \mathrm{Re}\{s\} > 0$, thus

$$\begin{aligned}\mathcal{Y}(s) &= \underbrace{\frac{s}{(s+3)(s+2)(s^2+100)}}_{\mathrm{Re}\{s\}>0} + \underbrace{\frac{s+6}{(s+3)(s+2)}}_{\mathrm{Re}\{s\}>-2} \\ &= \frac{s^3+6s^2+101s+600}{(s+3)(s+2)(s^2+100)},\ \mathrm{Re}\{s\} > 0 \\ &= \frac{A}{s+2} + \frac{B}{s+3} + \frac{Cs+10D}{s^2+100} \\ &= \underbrace{\frac{3.9808}{s+2}}_{\mathrm{Re}\{s\}>-2} - \underbrace{\frac{2.9725}{s+3}}_{\mathrm{Re}\{s\}>-3} - \underbrace{\frac{0.0083s}{s^2+100}}_{\mathrm{Re}\{s\}>0} + \underbrace{\frac{10(0.00441)}{s^2+100}}_{\mathrm{Re}\{s\}>0}\end{aligned}$$

Finally, we use Table D.4 of Laplace transform pairs to get

$$y(t) = [3.9808e^{-2t} - 2.9725e^{-3t} - 0.0083\cos 10t + 0.00441\sin 10t]u(t).$$

7.2 Exercise 7.8 of Boulet's book

Consider the causal differential system described by:

$$\frac{1}{2}\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+2y(t)=-\frac{dx(t)}{dt}-x(t),$$

with initial conditions $\frac{dy(0^-)}{dt}=1, \quad y(0^-)=2$. Suppose that this system is subjected to the input signal $x(t)=u(t)$. Give the transfer function of the system and specify its ROC. Compute the steady-state response $y_{ss}(t)$ and the transient response $y_{tr}(t)$ for $t \geq 0$.

Answer:

Let's take the unilateral Laplace transform on both sides of the differential equation.

$$\left[s^2\mathcal{Y}(s)-sy(0^-)-\frac{dy(0^-)}{dt}\right]+2\left[s\mathcal{Y}(s)-y(0^-)\right]+4\mathcal{Y}(s)=-2s\mathcal{X}(s)-2\mathcal{X}(s)$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$.

$$\left(s^2+2s+4\right)\mathcal{Y}(s)=-2s\mathcal{X}(s)-2\mathcal{X}(s)+sy(0^-)+2y(0^-)+\frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s)=\underbrace{\frac{-2(s+1)\mathcal{X}(s)}{s^2+2s+4}}_{\text{zero-state resp.}}+\underbrace{\frac{(s+2)y(0^-)+\frac{dy(0^-)}{dt}}{s^2+2s+4}}_{\text{zero-input resp.}}$$

The transfer function is $\frac{-2(s+1)}{s^2+2s+4}$ and since the system is causal, the ROC is an open RHP to the right of the rightmost pole. The poles are $p_{1,2}=-1\pm j\sqrt{3}$. Therefore, the ROC is $\mathrm{Re}\{s\}>-1$. The unilateral LT of the input is given by

$$\mathcal{X}(s)=\frac{1}{s},\quad \mathrm{Re}\{s\}>0,$$

thus,

$$\mathcal{Y}(s)=\underbrace{\frac{-2(s+1)}{\left(s^2+2s+4\right)s}}_{\substack{\mathrm{Re}\{s\}>0\\ \text{zero-state resp.}}}+\underbrace{\frac{2s+5}{s^2+2s+4}}_{\substack{\mathrm{Re}\{s\}>-1\\ \text{zero-input resp.}}}=\frac{2s^2+3s-2}{\left(s^2+2s+4\right)s}$$

Let's compute the overall response:

$$\mathcal{Y}(s)=\frac{2s^2+3s-2}{(s^2+2s+4)s}, \quad \mathrm{Re}\{s\}>0$$

$$=\underbrace{\frac{A\sqrt{3}+B(s+1)}{(s+1)^2+3}}_{\mathrm{Re}\{s\}>-1}+\underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0}$$

$$=\underbrace{\frac{A\sqrt{3}+B(s+1)}{(s+1)^2+3}}_{\mathrm{Re}\{s\}>-1}-\underbrace{\frac{0.5}{s}}_{\mathrm{Re}\{s\}>0}$$

Let $s=-1$ to compute $\frac{-3}{-3}=\frac{1}{\sqrt{3}}A+\frac{1}{2}\Rightarrow A=\frac{\sqrt{3}}{2}$, then multiply both sides by s and let $s\to\infty$ to get $B=2.5$:

$$\mathcal{Y}(s)=\underbrace{\frac{\frac{\sqrt{3}}{2}(\sqrt{3})+2.5(s+1)}{(s+1)^2+3}}_{\mathrm{Re}\{s\}>-1}-\underbrace{\frac{0.5}{s}}_{\mathrm{Re}\{s\}>0}$$

Notice that the second term $-\frac{0.5}{s}$ is the steady-state response, and thus $y_{ss}(t)=-0.5u(t)$.

Taking the inverse Laplace transform using the table yields

$$y_{tr}(t)=\left[\frac{\sqrt{3}}{2}e^{-t}\sin(\sqrt{3}t)+\frac{5}{2}e^{-t}\cos(\sqrt{3}t)\right]u(t).$$

7.3 Consider the system characterized by the differential Eq.

$$\frac{d^3y(t)}{dt^3}+6\frac{d^2y(t)}{dt^2}+11\frac{dy(t)}{dt}+6y(t)=x(t)$$

(a) Determine the zero-state response of the system for the input $x(t)=e^{-4t}u(t)$.

(b) Determine the zero-input response of the system for $t>0^-$, given that

$$y(0^-)=1,$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0^-}=-1$$

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=0^-}=1$$

(c) Determine the output of the system when the input and the initial conditions are the same as given in (a) and (b).

(d) Indicate the transient response and steady-state response of the output obtained in (c)

Answer:
Taking the unilateral LT on both sides of the Eq, we get:

$$\begin{aligned}&s^3Y(s)-s^2y(0^-)-sy'(0^-)-y''(0^-)+6s^2Y(s)-6sy(0^-)-6y'(0^-)\\&+11sY(s)-11y(0^-)+6Y(s)=X(s)\end{aligned} \qquad \text{(A 7.3)}$$

(a) For the zero-state response, all initial conditions $y^{(n)}(0^-)=0$, n=0, 1,2.
Taking the unilateral LT of x(t):

$$X(s)=\frac{1}{s+4} \quad \mathrm{Re}\{s\}>-4$$

Then, $Y(s)[s^3+6s^2+11s+6]=\frac{1}{s+4}$

$$\begin{aligned}Y(s)&=\frac{1}{(s^3+6s^2+11s+6)(s+4)}=\frac{1}{(s+1)(s+2)(s+3)(s+4)}\\&=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3}+\frac{D}{s+4}\\&=\frac{\frac{1}{6}}{s+1}+\frac{-\frac{1}{2}}{s+2}+\frac{\frac{1}{2}}{s+3}+\frac{\frac{-1}{6}}{s+4}\end{aligned}$$

Taking the inverse unilateral Laplace transform of the partial fraction expansion, we get

$$y(t)=\frac{1}{6}e^{-t}u(t)-\frac{1}{2}e^{-2t}u(t)+\frac{1}{2}e^{-3t}u(t)-\frac{1}{6}e^{-4t}u(t)$$

(b) For the zero-input response, X(s)=0. From Eq. (A7.3), we get

$$Y(s)=\frac{s^2+5s+6}{[s^3+6s^2+11s+6]}=\frac{1}{s+1}$$

Taking the inverse unilateral Laplace transform of the above Eq., we get

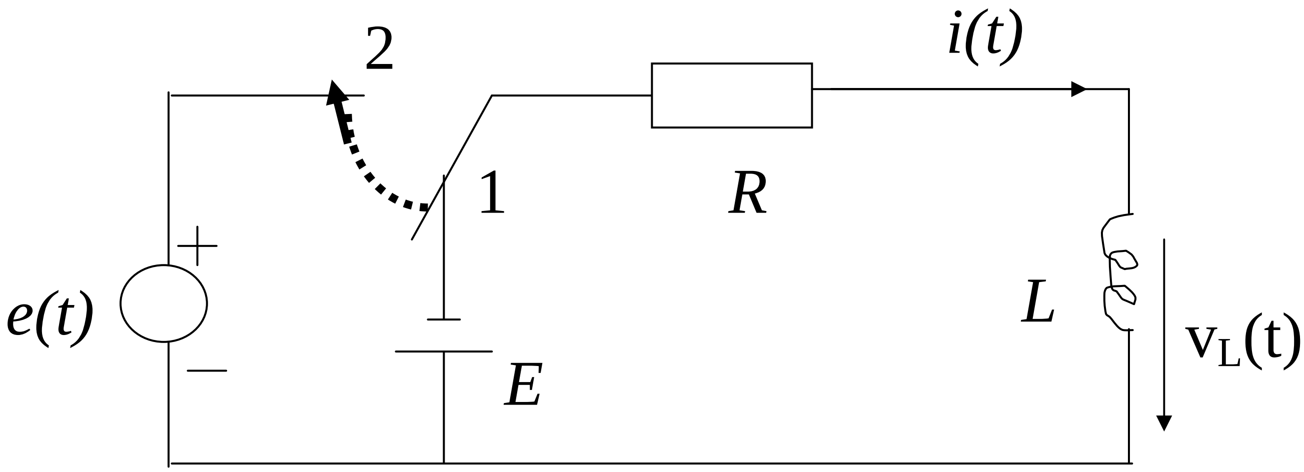
$$y(t)=e^{-t}u(t)$$

(c) The total response is

$$y(t)=\frac{7}{6}e^{-t}u(t)-\frac{1}{6}e^{-4t}u(t)+\frac{1}{2}e^{-2t}u(t)-\frac{1}{2}e^{-3t}u(t)$$

(d) According to Boulet's book, the response determined by the input pole is the steady response. In this case, however, the response corresponding to the input pole(s= - 4) is (-1/6)e^{-4t}u(t), which is transient. Another book mentioned that if the input pole is in the left half s-plane, the corresponding response is transient. The total response is transient.

7.4 Determine the signal *i(t)*, the current in the following circuit, using the unilateral Laplace transform. At time t=0, the switch is turned on 2 from 1, and the voltage source $e(t) = Ae^{i\omega_0 t} u(t)$. Please indicate the transient response, steady-state response, zero-input response, zero-state responses of *i(t)*.



Answer:

Write the differential Eq. about i(t):

$$L\frac{di(t)}{dt} + Ri(t) = Ae^{j\omega_0 t}u(t)$$

and taking unilateral LT on bouth sides:

$$L[sI(s) - i(0^-)] + RI(s) = \frac{A}{s - j\omega_0}, \qquad ROC \in \text{Re}\{s\} > 0$$

Then

$$I(s)[Ls + R] = \frac{A}{s - j\omega_0} + Li(0^-)$$

$$I(s) = \frac{1/L}{s + R/L}\{\frac{A\omega_0}{s - j\omega_0} + Li(0^-)\}$$

$$I(s) = \frac{A\omega_0}{L}[\frac{C_1}{s + R/L} + \frac{C_2}{s - j\omega_0}] + \frac{i(0^-)}{s + R/L}, \qquad ROC \in \text{Re}\{s\} > 0$$

where

$$C_1 = \frac{1}{s+R/L}\frac{1}{s-j\omega_0}(s+R/L)\Big|_{s=-R/L} = \frac{1}{(\frac{-R}{L})-j\omega_0} = \frac{L}{-R-j\omega_0 L}$$

$$C_2 = \frac{1}{s+R/L}\frac{1}{s-j\omega_0}(s-j\omega_0)\Big|_{s=j\omega_0} = \frac{1}{j\omega_0 + R/L} = \frac{L}{j\omega_0 L + R}$$

Take inverse LT of I(s):

$$i(t) = \frac{A\omega_0}{-R-j\omega_0 L}e^{-Rt/L}u(t) + \frac{A\omega_0}{R+j\omega_0 L}e^{j\omega_0 t}u(t) + i(0^-)e^{-Rt/L}u(t)$$

Where $i(0^-)$=-E/R.

The second term is the steady-state response (determined by the input pole).
The last term is the zero-input response (for A=0).
The sum of the first and the last terms is the transient response (determined by system poles)
The sum of the first and the second terms is the zero-state response (for $i(0^-)=0$).