$$H_B \; j\omega \; = \frac{\omega_c}{j\omega - \omega_c \; \frac{-}{\sqrt{\;}} + j\frac{}{\sqrt{\;}} \quad j\omega - \omega_c \; \frac{-}{\sqrt{\;}} - j\frac{}{\sqrt{\;}}}$$

$$H_B \; j\omega \; = \frac{A}{j\omega - \omega_c \; \frac{-}{\sqrt{\;}} + j\frac{}{\sqrt{\;}}} + \frac{A}{j\omega - \omega_c \; \frac{-}{\sqrt{\;}} - j\frac{}{\sqrt{\;}}}$$

$$h_B \; t \; = \sqrt{\;}\, \omega_c e^{\frac{-\omega_c}{\sqrt{\;}} t} \qquad \frac{\omega_c}{\sqrt{\;}} t \; u \; t$$

$$\omega \qquad s \; = \omega_c \; \frac{-}{\sqrt{\;}} + j\frac{}{\sqrt{\;}} \qquad s \; = \omega_c \; \frac{-}{\sqrt{\;}} - j\frac{}{\sqrt{\;}}$$

$$H_B \; s \; = \frac{\omega_c}{s - \omega_c \; \frac{-}{\sqrt{\;}} + j\frac{}{\sqrt{\;}} \quad s - \omega_c \; \frac{-}{\sqrt{\;}} - j\frac{}{\sqrt{\;}}} = \frac{\omega_c}{s - s \quad s - s} = \frac{A}{s - s} + \frac{A}{s - s}$$

$$A \; = H_B \; s \;\; s - s \; \Big|_{s = s} \; = \frac{\omega_c}{s \; - s} = \frac{\omega_c}{\sqrt{\;}\; j}$$

$$A \; = H_B \; s \;\; s - s \; \Big|_{s = s} \; = \frac{\omega_c}{s \; - s} = -\frac{\omega_c}{\sqrt{\;}\; j}$$

$$\frac{A}{j\omega - s}$$

$$A\, e^{s\,t} u \; t \; \overset{FT}{\leftrightarrow} \frac{A}{j\omega - s} \qquad for \qquad s \;\; <$$

$$h_B \; t \; = \; A e^{s \; t} + A \; e^{s \; t} \; u \; t \; = \frac{\omega_c}{\sqrt{\;} \; j} \; e^{\omega_c \; \frac{-}{\sqrt{\;}} + j \frac{}{\sqrt{\;}} \; t} - e^{\omega_c \; \frac{-}{\sqrt{\;}} - j \frac{}{\sqrt{\;}} \; t} \; u \; t$$

$$= \frac{\omega_c}{\sqrt{\;} \; j} e^{\frac{-\omega_c t}{\sqrt{\;}}} \; e^{j \frac{\omega_c}{\sqrt{\;}} t} - e^{-j \frac{\omega_c}{\sqrt{\;}} t} \; u \; t \; = \sqrt{\;} \omega_c e^{\frac{-\omega_c t}{\sqrt{\;}}} \qquad \frac{\omega_c t}{\sqrt{\;}} \; u \; t$$

6. 2 Exercise 6.4 of Boulet’s book.

Compute the step response of the LTI system $H(s) = \dfrac{6(s+1)}{s(s+3)}$, $\operatorname{Re}\{s\} > 0$.

Answer:

$$S(s) = \frac{1}{s} H(s) = \frac{6(s+1)}{s^2(s+3)}, \;\; \operatorname{Re}\{s\} > 0$$
$$= \underbrace{\frac{-4/3}{s+3}}_{\operatorname{Re}\{s\}>-3} + \underbrace{\frac{4/3}{s}}_{\operatorname{Re}\{s\}>0} + \underbrace{\frac{2}{s^2}}_{\operatorname{Re}\{s\}>0}$$

Using Table D.4 of basic Laplace transform pairs, we obtain the step response:

$$s(t) = -\frac{4}{3} e^{-3t} u(t) + \frac{4}{3} u(t) + 2tu(t).$$

6.3 Exercise 6.6 of Boulet’s book.

Suppose that the LTI system described by $H(s)=\frac{2}{(s+3)(s-1)}$ is known to be stable. Is this system causal? Compute its impulse response $h(t)$.

Answer:

There are three possible ROC's that could be associated with this transfer function. Only one ROC contains the imaginary axis and leads to a stable system: $-3<\mathrm{Re}\{s\}<1$. With this ROC, the system is noncausal as the ROC is not a right half-plane. Impulse response:

$$\begin{aligned} H(s) &= \frac{2}{(s+3)(s-1)}, \quad -3<\mathrm{Re}\{s\}<1 \\ &= \underbrace{\frac{-1/2}{s+3}}_{\mathrm{Re}\{s\}>-3} + \underbrace{\frac{1/2}{s-1}}_{\mathrm{Re}\{s\}<1} \end{aligned}$$

which yields: $h(t)=-\frac{1}{2}e^{-3t}u(t)-\frac{1}{2}e^{t}u(-t)$.

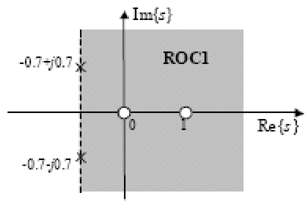
6.4 Exercise 6.8 of Boulet's book.

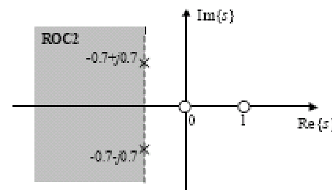
Consider an LTI system with transfer function $H(s)=\frac{s(s-1)}{s^2+\sqrt{2}s+1}$. Sketch all possible ROCs of $H(s)$ on a pole-zero plot and compute the associated impulse responses $h(t)$. Indicate for each impulse response whether it corresponds to a system that is causal/stable.

Answer:

$$H(s)=\frac{s(s-1)}{(s+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}})(s+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}})}$$

There are two possible ROCs that could be associated with this transfer function. Only one ROC leads to a stable system: ROC1.





Impulse response with ROC1: stable and causal.

$$H(s) = \underset{\forall s}{1} + \underbrace{\frac{-1.21 - j0.5}{s + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}}_{\mathrm{Re}\{s\} > -\frac{1}{\sqrt{2}}} + \underbrace{\frac{-1.21 + j0.5}{s + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}}}_{\mathrm{Re}\{s\} > -\frac{1}{\sqrt{2}}}$$

$$h(t) = \delta(t) + e^{-\frac{1}{\sqrt{2}}t}\left[-2.42\cos(\frac{1}{\sqrt{2}}t) + \sin(\frac{1}{\sqrt{2}}t)\right]u(t)$$

Impulse response with ROC2: unstable, anticausal.

$$H(s) = \underset{\forall s}{1} + \underbrace{\frac{-1.21 - j0.5}{s + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}}_{\mathrm{Re}\{s\} < -\frac{1}{\sqrt{2}}} + \underbrace{\frac{-1.21 + j0.5}{s + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}}}_{\mathrm{Re}\{s\} < -\frac{1}{\sqrt{2}}}$$

$$h(t) = \delta(t) - e^{-\frac{1}{\sqrt{2}}t}\left[-2.42\cos(\frac{1}{\sqrt{2}}t) + \sin(\frac{1}{\sqrt{2}}t)\right]u(-t)$$

6.5 Exercise 6.10 of Boulet's book.

(a) Find the impulse response of the system $H(s) = \frac{3s^2 - 3s - 6}{s^3 + 12s^2 + 120s + 200}$, $\mathrm{Re}\{s\} > -2$. Hint: this system has a pole at -2.

Answer:

$$H(s) = \frac{3s^2 - 3s - 6}{s^3 + 12s^2 + 120s + 200}, \;\; \mathrm{Re}\{s\} > -2$$
$$= \frac{3s^2 - 3s - 6}{(s+2)(s^2 + 10s + 100)}, \;\; \mathrm{Re}\{s\} > -2$$
$$= \underbrace{\frac{0.14286}{s+2}}_{\mathrm{Re}\{s\}>-2} + \underbrace{\frac{1.4286 + j1.4104}{s + 5 - j5\sqrt{3}}}_{\mathrm{Re}\{s\}>-5} + \underbrace{\frac{1.4286 - j1.4104}{s + 5 + j5\sqrt{3}}}_{\mathrm{Re}\{s\}>-5}$$

Thus,

$$h(t) = 0.14286e^{-2t}u(t) + \left(1.4286 + j1.4104\right)e^{(-5+j5\sqrt{3})t}u(t) + \left(1.4286 - j1.4104\right)e^{(-5-j5\sqrt{3})t}u(t)$$
$$= 0.14286e^{-2t}u(t) + 2e^{-5t}\,\mathrm{Re}\left\{\left(1.4286 + j1.4104\right)e^{j5\sqrt{3}t}\right\}u(t)$$
$$= 0.14286e^{-2t}u(t) + 2e^{-5t}\left(1.4286\cos(5\sqrt{3}t) - 1.4104\sin(5\sqrt{3}t)\right)u(t)$$
$$= 0.14286e^{-2t}u(t) + e^{-5t}\left(2.8572\cos(5\sqrt{3}t) - 2.8208\sin(5\sqrt{3}t)\right)u(t)$$

(b) Find the settling value of the step response of $H(s)$ given in (a).

Answer:

Using the final value theorem: $\lim_{t \to +\infty} s(t) = H(0) = \frac{-6}{200} = -0.03$.

$$x \;\; t \;\; = e^{- \;\; t} u \;\; t \;\; \leftrightarrow \frac{\quad}{s +} \qquad ROC \in \qquad s \;\; > -$$

$$x \;\; t \;\; = e^{- \;\; t} u \;\; t \;\; \leftrightarrow \frac{\quad}{s +} \qquad ROC \in \qquad s \;\; > -$$

$$x \;\; t - \qquad \leftrightarrow \frac{e^{- \;\; s}}{s +} \qquad ROC \in \qquad s \;\; > -$$

$$x \;\; -t + \qquad \leftrightarrow \frac{e^{- \;\; s}}{- s +} \qquad ROC \in \qquad s \;\; <$$

$$x\ \ t- \quad x\ \ -t+ \quad \leftrightarrow \left[\frac{e^{-\ s}}{s+}\right]\left[\frac{e^{-\ s}}{-s}\right] \quad ROC \supseteq \quad s\ >- \quad \cap \quad s\ <$$

β

β β

$$X\ s\ = \frac{}{s+} + \frac{}{s+\beta} \qquad s\ > \qquad - \qquad \beta$$

β

β