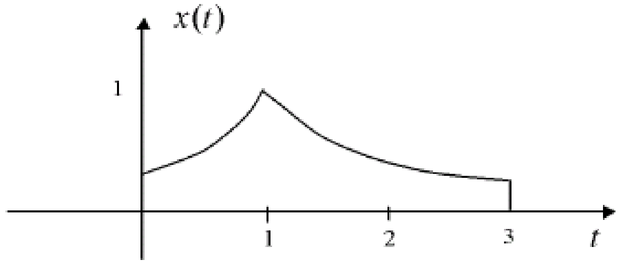
Assignment 5
Due October 20, 2008

===================Part 1 (no submission is required)=======================

Practice makes perfect. Do and understand all exercises in Chapter 5of Benoit Boulet's book.

==================Part 2 (Handwritten and submission are required)================

5.1 Exercise 5.8 of Boulet's book.
Answer:



$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = e^{-1}\int_{0}^{1} e^{(1-j\omega)t}dt + e^{1}\int_{1}^{3} e^{-(1+j\omega)t}dt \\
&= \frac{e^{-1}}{(1-j\omega)}\left[e^{(1-j\omega)t}\right]_0^1 - \frac{e^{1}}{(1+j\omega)}\left[e^{-(1+j\omega)t}\right]_1^3 \\
&= \frac{e^{-1}}{(1-j\omega)}\left[e^{(1-j\omega)} - 1\right] - \frac{e^{1}}{(1+j\omega)}\left[e^{-3(1+j\omega)} - e^{-(1+j\omega)}\right] \\
&= \frac{e^{-j\omega} - e^{-1}}{(1-j\omega)} - \frac{e^{-4-j3\omega} - e^{-j\omega}}{(1+j\omega)} \\
&= \frac{(e^{-j\omega} - e^{-1})(1+j\omega) - (e^{-4-j3\omega} - e^{-j\omega})(1-j\omega)}{1+\omega^2} \\
&= \frac{(e^{-j\omega} + j\omega e^{-j\omega} - e^{-1} - j\omega e^{-1}) - (e^{-4-j3\omega} - j\omega e^{-4-j3\omega} - e^{-j\omega} + j\omega e^{-j\omega})}{1+\omega^2} \\
&= \frac{2e^{-j\omega} - e^{-1} - j\omega e^{-1} - e^{-4-j3\omega} + j\omega e^{-4-j3\omega}}{1+\omega^2}
\end{aligned}
$$

5.2 Exercise 5.10 of Boulet's book.

Answer:

This signal is the convolution of two rectangular pulses $v(t)$ of total width T_1, convolved with an impulse train of period T:

$$x(t) = \underbrace{\sum_{k=-\infty}^{+\infty} \delta(t-kT)}_{p(t)} * \left[v(t) * v(t) \right],$$

where: $$v(t) = \frac{1}{\sqrt{T_1}} \left[u(t+T_1/2) - u(t-T_1/2) \right].$$

By the convolution property,

$$X(j\omega) = \mathcal{F}\left\{ p(t) * v(t) * v(t) \right\} = P(j\omega) V(j\omega)^2 = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\frac{2\pi}{T}) T_1 \mathrm{sinc}^2 (\frac{T_1\omega}{2\pi})$$
$$= \sum_{k=-\infty}^{+\infty} 2\pi \frac{T_1}{T} \mathrm{sinc}^2 (k\frac{T_1}{T}) \delta(\omega - k\frac{2\pi}{T})$$

5.3 Exercise 5.11 of Boulet's book.

Answer:

Let $Y(j\omega)$ be the Fourier transform of a rectangular window of unit magnitude and zero phase (i.e., it is real) from $-W$ to W. Then the Fourier transform of $x(t)$ is

$$X(j\omega) = j\omega Y(j\omega)\,.$$

Using the differentiation property, the signal $x(t)$ is given by:

$$\begin{aligned} x(t) &= \frac{dy(t)}{dt} = \frac{d\frac{W}{\pi}\operatorname{sinc}(\frac{W}{\pi}t)}{dt} \\ &= \frac{W}{\pi}\frac{d}{dt}\frac{\sin(Wt)}{Wt} \\ &= \frac{W}{\pi}\left[\frac{W^2 t\cos(Wt) - W\sin(Wt)}{W^2t^2}\right] \\ &= \frac{W\cos(Wt)}{\pi t} - \frac{\sin(Wt)}{\pi t^2} \end{aligned}$$

5.4 Prove that the integral of a signal *f(t)* and its Fourier transform are the following FT pair:

$$\int_{-\infty}^{t} f(\tau)d\tau \overset{FT}{\longleftrightarrow} \frac{1}{j\omega}F(\omega) + \pi F(\omega)\delta(\omega)$$

where *F(ω)* is the Fourier transform of *f(t).*
Hint: View the integral as the convolution of f(t) and u(t). The FT of u(t) is given in Lecture 15 or Appendix D of Boulet's book. Apply the convolution property of FT.

Answer:

$$\int_{-\infty}^{t} f(\tau)d\tau = f(t) * u(t) \overset{FT}{\leftrightarrow} F(\omega)[\frac{1}{j\omega} + \pi\delta(\omega)] = \frac{F(\omega)}{j\omega} + \pi F(\omega)\delta(\omega)$$

5.5 Prove the following properties of Fourier transform, where $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of *x(t)* and y(t), respectively.
a. Time shift property of FT

$$x(t-t_0) \overset{FT}{\leftrightarrow} X(\omega)e^{-j\omega t_0}$$

Answer:

$$\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)}d\tau = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = e^{-j\omega t_0}X(\omega)$$

b. Frequency shift property of FT:

$$x(t)e^{j\omega_0 t} \overset{FT}{\leftrightarrow} X(\omega-\omega_0)$$

Answer:

$$\int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(\omega-\omega_0)$$

c. Symmetry properties of FT

$$x(-t) \overset{FT}{\leftrightarrow} X(-\omega)$$

Answer:

$$\int_{-\infty}^{\infty} x(-t)\,e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)\,e^{j\omega\tau}dt = X(-\omega)$$

d. Conjugate symmetry property of FT

$$\overset{*}{x}(t) \overset{FT}{\leftrightarrow} \overset{*}{X}(-\omega)$$

Answer:

$$\int_{-\infty}^{\infty} \overset{*}{x}(t)e^{-j\omega t}dt = \{\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\}^* = \{X(-\omega)\}^*$$

e. Convolution property of FT

$$x(t)*y(t) \overset{FT}{\leftrightarrow} X(\omega)Y(\omega)$$

Answer:

$$x(t)*y(t) = \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau\}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\{\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega t}dt\}d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\{\int_{-\infty}^{\infty} y(t-\tau)e^{-j\omega(t-\tau)}d(t-\tau)\}d\tau$$

$$= \{\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau\}Y(\omega) = X(\omega)Y(\omega)$$

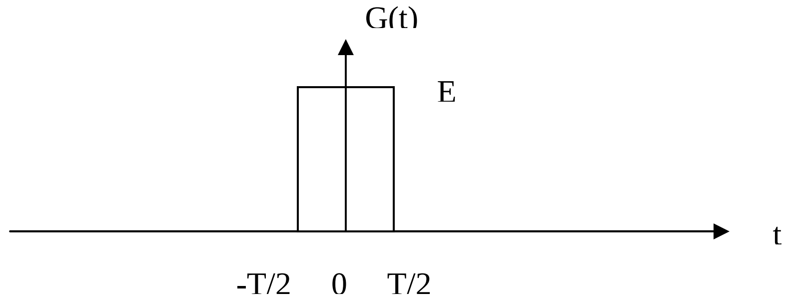
f. Multiplication property of FT

$$x(t)y(t) \overset{FT}{\leftrightarrow} \frac{1}{2\pi}X(\omega)*Y(\omega)$$

Answer: the inverse FT of the right hand side is:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\frac{1}{2\pi}X(\omega)*Y(\omega)\}e^{-j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}\frac{1}{2\pi}X(\Omega)Y(\omega-\Omega)d\Omega\}e^{-j\omega t}d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\Omega)\{\int_{-\infty}^{\infty}\frac{1}{2\pi}Y(\omega-\Omega)\}e^{-j\omega t}d\omega\}d\Omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\Omega)e^{-j\Omega t}\{\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(\omega-\Omega)\}e^{-j(\omega-\Omega)t}d(\omega-\Omega)\}d\Omega$$
$$=\frac{1}{2\pi}\{\int_{-\infty}^{\infty}X(\Omega)e^{-j\Omega t}d\Omega\}y(t)=x(t)y(t)$$

5.6 Apply properties of FT to obtain the FT of f(t)=G(t)cos(ω_0t), where G(t) is a rectangular wave with amplitude E and width T, as shown below.



Answer:

$$f(t)=G(t)\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

From Appendix D of Boulet’s book, the FT of G(t) is $ET\sin(\omega T/2)/(\omega T/2)$

Applying the frequency shift property of the FT as shown in problem 5.5b, we have the following FT pairs:

$$f(t)=G(t)\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\overset{FT}{\leftrightarrow}\frac{E\sin((\omega+\omega_0)T/2)}{\omega+\omega_0}+\frac{E\sin((\omega-\omega_0)T/2)}{\omega-\omega_0}$$

Thus, the FT of f(t) is $\frac{E\sin((\omega+\omega_0)T/2)}{\omega+\omega_0}+\frac{E\sin((\omega-\omega_0)T/2)}{\omega-\omega_0}$

5.7 The duality property of FT means that for any FT pair, there exists a dual pair with the time and frequency variables interchanged (see Lecture 15). Apply the duality property and other properties of FT and obtain the time-domain signals of the following Fourier transforms:

a. F(ω)=$\delta(\omega-\omega_0)$
b. F(ω)=u($\omega+\omega_0$)- u($\omega-\omega_0$)

Answer :

a. It is known that the following Fourier transform pair exists:

$$\delta(t) \overset{FT}{\leftrightarrow} 1$$

Then, according to duality property of the FT (see Lecture 15), the following FT pair exists:

$$1 \overset{FT}{\leftrightarrow} 2\pi\delta(-\omega)$$

Since δ(x) is an even function, then $1 \overset{FT}{\leftrightarrow} 2\pi\delta(\omega)$. According to the frequency shift property as shown in 5.5b, then the following FT pair exists:

$$e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

Thus, the time domain signal of the FT $F(\omega)=\delta(\omega-\omega_0)$ is $\frac{1}{2\pi}e^{j\omega_0 t}$.

b. From Appendix D of Boulet’s book or Lecture 15, it is known that the unit step signal and its FT constitute the following FT pair:

$$u(t) \overset{FT}{\leftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$$

According to the duality property of the FT, there exists the following FT pair:

$$\frac{1}{jt} + \pi\delta(t) \overset{FT}{\leftrightarrow} 2\pi u(-\omega)$$

According to the time/frequency scaling property of the FT and the above FT pair, the following FT pair exists:

$$\frac{1}{-jt} + \pi\delta(-t) \overset{FT}{\leftrightarrow} 2\pi u(\omega)$$

According to the frequency shift property of FT and the above FT, the following FT pair exists:

$$\frac{e^{j\omega_0 t}}{-jt} + \pi e^{j\omega_0 t}\delta(-t) \overset{FT}{\leftrightarrow} 2\pi u(\omega - \omega_0)$$

According to the property of δ(t), the above FT pair can be written as

$$e^{j\omega_0 t}\frac{1}{-jt} + \pi\delta(t) \overset{FT}{\leftrightarrow} 2\pi u(\omega - \omega_0)$$

Substituting ω_0 in the above Eq. with $-\omega_0$, we have the following FT pair:

$$e^{-j\omega_0 t}\frac{1}{-jt} + \pi\delta(t) \overset{FT}{\leftrightarrow} 2\pi u(\omega + \omega_0)$$

Thus,

$$(e^{j\omega_0 t} - e^{-j\omega_0 t})(\frac{1}{jt}) \overset{FT}{\leftrightarrow} 2\pi[u(\omega+\omega_0) - u(\omega-\omega_0)]$$

Hence, $(e^{j\omega_0 t} - e^{-j\omega_0 t})(\frac{1}{2j\pi t}) = \frac{\sin(\omega_0 t)}{\pi t} \overset{FT}{\leftrightarrow} u(\omega+\omega_0) - u(\omega-\omega_0)$

Therefore, the time-domain signal given by the FT $F(\omega)=u(\omega+\omega_0)- u(\omega-\omega_0)$ is $\frac{\sin(\omega_0 t)}{\pi t}$.

Please check the above results for (a) and (b) obtained using the FT properties are the same as those using the inverse FT equation:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega.$$