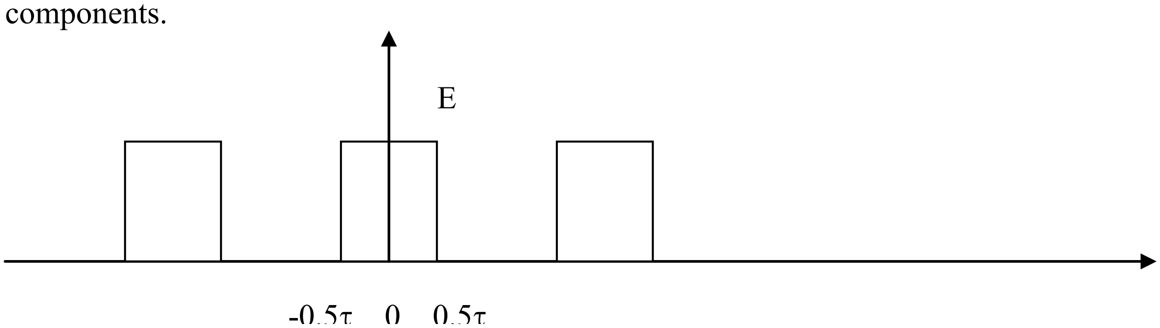
Assignment 4
Due October 14 (Tue), 2008.

=====================Part 1 (no submission is required)==========================

Practice makes perfect. Do and understand all exercises in Chapter 4 of Benoit Boulet's book.

==================Part 2 (Handwritten and submission are required)==================

4.1 A periodic rectangular wave as shown below has a frequency of 5kHz, pulse width τ=20 μs, amplitude E=10 V. Derive the amplitudes of DC, fundamental, second, and third harmonic components.



Answer:

$$a_k = \frac{1}{T}\int_{-\tau/2}^{\tau/2} E e^{-jk\omega_0 t} dt = \frac{E}{-jk\omega_0 T}\int_{-\tau/2}^{\tau/2} e^{-jk\omega_0 t} d(-jk\omega_0 t) = \frac{E}{-jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{E}{-jk\omega_0 T}(e^{-j0.5k\omega_0\tau} - e^{j0.5k\omega_0\tau})$$

$$= \frac{2E}{k\omega_0 T}\sin(0.5k\omega_0\tau)$$

$$= \frac{E}{\pi k}\sin(k\pi\tau/T)$$

$$= \frac{E\tau}{T}\frac{\sin(k\pi\tau/T)}{k\pi\tau/T}$$

$$= \frac{E\tau}{T}\sin c(k\tau/T)$$

$$a_0 = \frac{E\tau}{T} = 10\times 20\times 10^{-6}\times 5000 = 1 \qquad (V)$$

$$a_1 = \frac{E}{\pi}\sin(\pi\tau/T) = \frac{10}{\pi}\sin(\pi\times 20\times 10^{-6}\times 5000) = \frac{10}{\pi}\sin(0.1\pi)$$

$$a_2 = \frac{E}{2\pi}\sin(2\pi\tau/T) = \frac{10}{2\pi}\sin(2\pi\times 20\times 10^{-6}\times 5000) = \frac{5}{\pi}\sin(0.2\pi)$$

$$a_3 = \frac{E}{3\pi}\sin(3\pi\tau / T) = \frac{10}{3\pi}\sin(3\pi \times 20 \times 10^{-6} \times 5000) = \frac{10}{3\pi}\sin(0.3\pi)$$

The amplitude of DC component is a_0=1 V.
As a_1=a_{-1}, the fundamental component is $x_1(t) = a_1 e^{j\omega_0 t} + a_{-1}e^{-j\omega_0 t} = 2a_1\cos(\omega_0 t)$.
The amplitude of fundamental is $2a_1$ =(20/π)sin(0.1π)

As a_2=a_{-2}, the second harmonic component is $x_2(t) = a_2 e^{j2\omega_0 t} + a_{-2}e^{-j2\omega_0 t} = 2a_2\cos(2\omega_0 t)$.
The amplitude of the second harmonic component is $2a_2$ =(10/π)sin(0.2π)=1.871 (V).

As a_3=a_{-3}, the third harmonic component is $x_3(t) = a_3 e^{j2\omega_0 t} + a_{-3}e^{-j2\omega_0 t} = 2a_3\cos(3\omega_0 t)$.
The amplitude of the third harmonic component is $2a_3$ =(20/3π)sin(0.3π).

As a_N=a_{-N}, the third harmonic component is $x_N(t) = a_N e^{jN\omega_0 t} + a_{-N}e^{-jN\omega_0 t} = 2a_N\cos(N\omega_0 t)$.
The amplitude of the third harmonic component is $2a_N$ =2E/(Nπ)sin(Nπτ/T)=(6.366/N)sin(0.3142N).

4.2 Let the Fourier series of a periodic signal be

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

Prove that df(t)/dt is also a periodic signal, and that its FS is

$$\frac{d}{dt}f(t) = \sum_{n=-\infty}^{\infty} j\omega_0 n F_n e^{jn\omega_0 t}$$

Answer:

f(t)=f(t+T)
df(t)/dt=df(t+T)/dt=f ′(t+T)
Thus df(t)/dt is also periodic with period T.

Let the FS of df(t)/dt be

$$\frac{d}{dt}f(t) = \sum_{n=-\infty}^{\infty} E_n e^{jn\omega_0 t}$$

$$E_n = \frac{1}{T}\int_0^T \frac{df(t)}{dt} e^{-jn\omega_0 t} dt = \frac{1}{T}[f(t)e^{-jn\omega_0 t}\Big|_0^T - \int_0^T (-jn\omega_0) f(t) e^{-jn\omega_0 t} dt]$$

$$= jn\omega_0 \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t} dt = jn\omega_0 F_n$$

Thus

$$\frac{d}{dt}f(t) = \sum_{n=-\infty}^{\infty} j\omega_0 n F_n e^{jn\omega_0 t}$$

4.3 Exercise 4.6 of Boulet's book.
Answer:
(a)

$$\begin{aligned} a_k &= \frac{1}{T_1}\int_0^{T_1} v(t)e^{-jk\omega_1 t}dt = \frac{A}{T_1}\int_0^{T_1}\sin(\omega_0 t)e^{-jk\omega_1 t}dt \\ &= \frac{A}{2jT_1}\int_0^{T_1}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)e^{-jk\omega_1 t}dt = \frac{A}{2jT_1}\int_0^{\frac{\pi}{\omega_0}}\left(e^{j\omega_0(1-2k)t} - e^{-j\omega_0(1+2k)t}\right)dt \\ &= \frac{A\omega_0}{2\pi j}\left[\frac{-2}{j\omega_0(1-2k)} - \frac{2}{j\omega_0(1+2k)}\right] \\ &= \frac{2A}{\pi(1-4k^2)} \end{aligned}$$

Thus, $v(t) = \sum_{k=-\infty}^{+\infty}\frac{2A}{\pi(1-4k^2)}e^{-jk\omega_1 t}$ is the Fourier series of the full-wave rectified sinusoid.

(b) Express $v(t)$ as a real Fourier series of the form $v(t) = a_0 + 2\sum_{k=1}^{+\infty}[B_k\cos(k\omega_1 t) - C_k\sin(k\omega_1 t)]$

Answer:

Note that the Fourier series coefficients of $v(t)$ are real. Hence

$$\begin{aligned} v(t) &= a_0 + \sum_{k=1}^{+\infty}(a_k e^{jk\omega_1 t} + a_{-k}e^{-jk\omega_1 t}) \\ &= a_0 + \sum_{k=1}^{+\infty}a_k(e^{jk\omega_1 t} + e^{-jk\omega_1 t}) \\ &= a_0 + 2\sum_{k=1}^{+\infty}a_k\cos(k\omega_1 t) \end{aligned}$$

and we can identify the coefficients $B_k = a_k$, $C_k = 0$. Note that $v(t)$ is an even function, hence it must be an infinite sum of even functions only (cosines and a constant). This is why the C_k coefficients are 0 (the sine function is odd).

4.4 Exercise 4.8 of Boulet's book.

Note: to do problems (b) and (c), you need to study Matlab files linespectrum.m and Fourierseries.m on the CD \Chapter4 and create your m file.

- In the Matlab command window, you type: help eval (and then press the return key), and learn the Matlab built-in function "eval".
- Replace the string '=-j*A/(k*pi)' (in linespectrum.m), which is the FS coefficient value for the signal in Figure 4.3, with the FS coefficient value for the signal in Figure 4.32
- Study lines 93-109 of Fourierseries.m and learn how to sum a number of harmonic components.

(a) Compute and sketch (magnitude and phase) the Fourier series coefficients of the sawtooth signal of Figure 4.12.

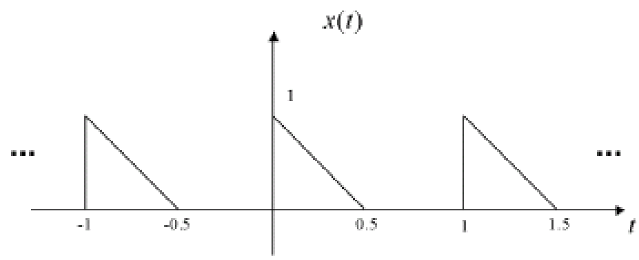


Figure 4.12: Periodic sawtooth signal in Exercise 4.8(a).

Answer:

The Fourier series coefficients are computed:

$$a_0 = \frac{1}{T}\int_0^1 x(t)dt = \int_0^{0.5} (1-2t)dt$$

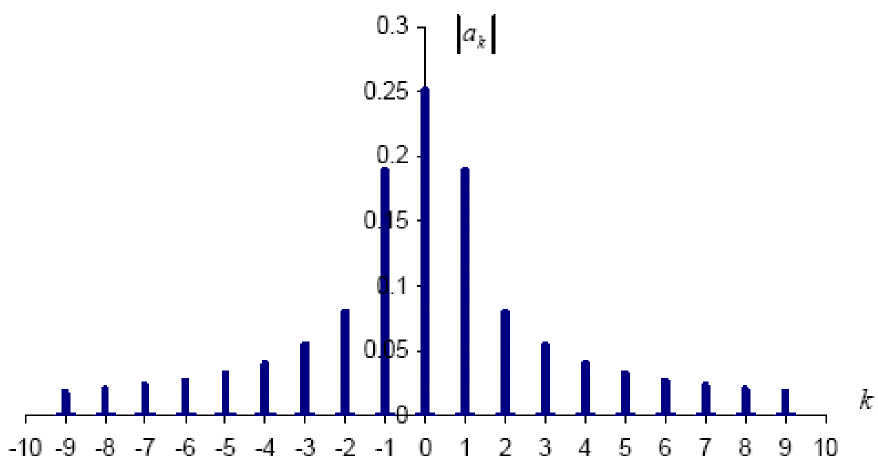
$$= \left[-\frac{1}{4}(1-2t)^2\right]_0^{0.5} = \frac{1}{4}$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_0^1 x(t)e^{-jk2\pi t}dt \\
&= \int_0^{0.5} (1-2t)e^{-jk2\pi t}dt \\
&= \frac{-1}{jk2\pi}\left[(1-2t)e^{-jk2\pi t}\right]_0^{0.5} - \frac{1}{jk\pi}\int_0^{0.5} e^{-jk2\pi t}dt \\
&= \frac{1}{jk2\pi} + \frac{1}{2(jk\pi)^2}\left[e^{-jk2\pi t}\right]_0^{0.5} \\
&= \frac{1}{jk2\pi} - \frac{1}{2(k\pi)^2}\left[e^{-jk\pi} - 1\right] \\
&= \frac{-j}{k2\pi} + \frac{\left[1-(-1)^k\right]}{2(k\pi)^2}, \quad k \neq 0
\end{aligned}
$$

The magnitude is obtained:

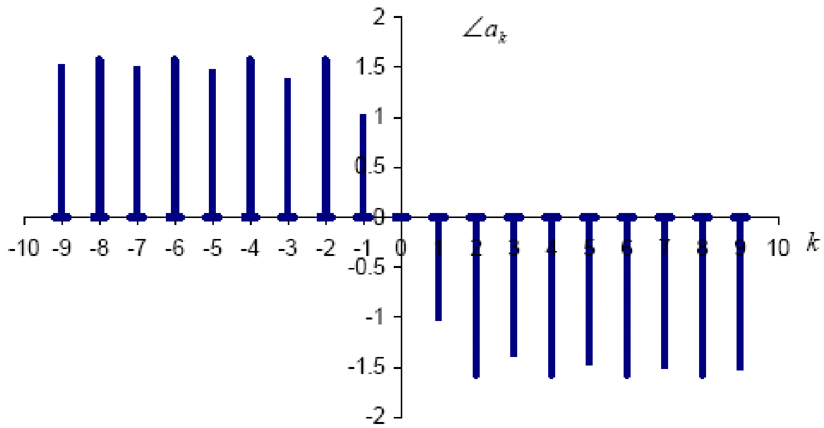
$$
|a_k| = \sqrt{\frac{1}{4(k\pi)^2} + \frac{\left[1-(-1)^k\right]^2}{4(k\pi)^4}} = \frac{1}{2(k\pi)}\sqrt{1 + \frac{\left[1-(-1)^k\right]^2}{(k\pi)^2}},
$$

and it is plotted below.



The phase is given by:

$$\angle a_k = \arctan\left(\frac{-1/(2k\pi)}{-\left[1-(-1)^k\right]/2(k\pi)^2}\right).$$



(b) Express $x(t)$ as its real Fourier series of the form:

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty}[B_k \cos(k\omega_1 t) - C_k \sin(k\omega_1 t)]$$

Answer:

Let $a_k = \alpha_k + j\beta_k$. The fundamental frequency is $\omega_1 = 2\pi$. Note that the Fourier series coefficients can be written as:

$$x(t) = a_0 + \sum_{k=1}^{+\infty}(a_k e^{jk\omega_1 t} + a_{-k}e^{-jk\omega_1 t}) = a_0 + \sum_{k=1}^{+\infty}(a_k e^{jk\omega_1 t} + a_k^* e^{-jk\omega_1 t})$$

$$= a_0 + \sum_{k=1}^{+\infty}(\alpha_k + j\beta_k)(\cos(k\omega_1 t) + j\sin(k\omega_1 t)) + (\alpha_k - j\beta_k)(\cos(k\omega_1 t) - j\sin(k\omega_1 t))$$

$$= a_0 + \sum_{k=1}^{+\infty} 2\alpha_k \cos(k\omega_1 t) - 2\beta_k \sin(k\omega_1 t)$$

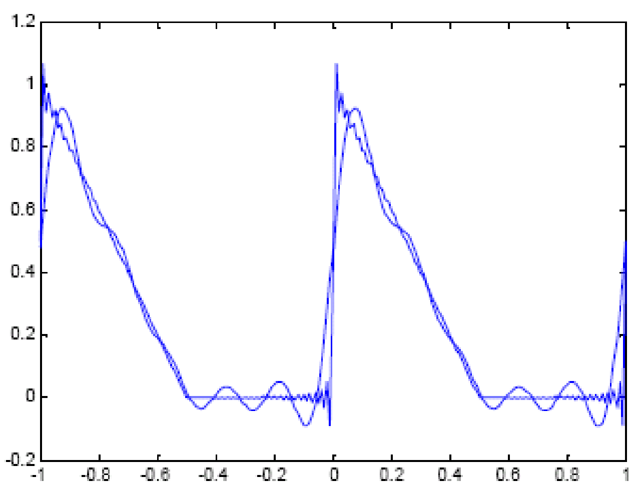
and we can identify the coefficients $B_k = \alpha_k = \dfrac{\left[1-(-1)^k\right]}{2(k\pi)^2}$, $C_k = \beta_k = \dfrac{-1}{k2\pi}$. Thus,

$$x(t) = \frac{1}{4} + \sum_{k=1}^{+\infty}\frac{\left[1-(-1)^k\right]}{(k\pi)^2}\cos(k2\pi t) + \frac{1}{k\pi}\sin(k2\pi t).$$

(c) Use MATLAB to plot, superimposed on the same figure, approximations to the signal over two periods by summing the first 5, and the first 50 harmonic components of $x(t)$, i.e., by plotting $\tilde{x}(t) = \sum_{-N}^{N} a_k e^{jk\frac{2\pi}{T}t}$. Discuss your results.

Answer:

5 and 50 harmonic components:



The fifty-harmonic approximation is obviously a better representation of the sawtooth waveform, but significant ripple are seen in the plots of both truncated Fourier series, especially around the discontinuities.

(d) The sawtooth signal $x(t)$ is the input to an LTI system with impulse response $h(t) = e^{-t}\sin(2\pi t)u(t)$. Let $y(t)$ denote the resulting periodic output. Find the frequency response $H(j\omega)$ of the LTI system. Give expressions for its magnitude $|H(j\omega)|$ and phase $\angle H(j\omega)$ as functions of ω. Find the Fourier series coefficients b_k of the output $y(t)$. Use your computer program of (c) to plot an approximation to the output signal over two periods by summing the first 50 harmonic components of $y(t)$. Discuss your results.

Answer:

Answer:

The frequency response of the system is given by

$$
\begin{aligned}
H(j\omega) &= \int_0^{+\infty} e^{-t}\sin(2\pi t)e^{-j\omega t}\,dt \\
&= \frac{1}{2j}\int_0^{+\infty} (e^{j2\pi t} - e^{-j2\pi t})e^{-(1+j\omega)t}\,dt \\
&= \frac{1}{2j}\int_0^{+\infty} (e^{-(1+j(\omega-2\pi))t} - e^{-(1+j(\omega+2\pi))t})\,dt \\
&= \frac{1}{2j(1+j(\omega-2\pi))} - \frac{1}{2j(1+j(\omega+2\pi))} \\
&= \frac{(1+j(\omega+2\pi))-(1+j(\omega-2\pi))}{2j(1+j(\omega-2\pi))(1+j(\omega+2\pi))} \\
&= \frac{2\pi}{1-\omega^2+4\pi^2+j2\omega}
\end{aligned}
$$

Magnitude:

$$|H(j\omega)| = \frac{2\pi}{\left[(1-\omega^2+4\pi^2)^2+4\omega^2\right]^{1/2}}$$

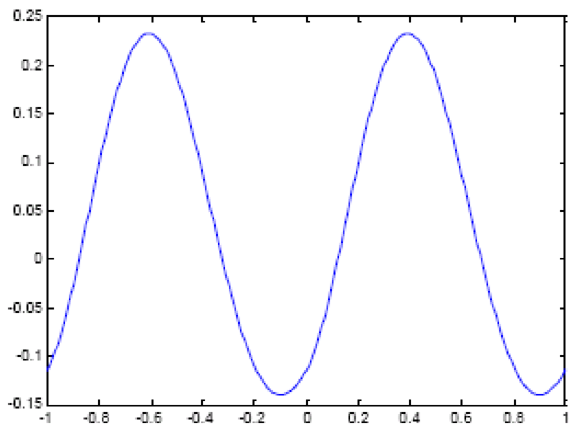
Phase:

$$\angle H(j\omega) = \arctan\left(\frac{-2\omega}{1-\omega^2+4\pi^2}\right)$$

Spectral coefficients of the output signal:

$$b_k = H(jk\omega_1)a_k = \frac{2\pi}{1-(k2\pi)^2+4\pi^2+j4k\pi}\left(\frac{-j}{k2\pi}+\frac{\left[1-(-1)^k\right]}{2(k\pi)^2}\right)$$
$$= \frac{1}{1-(k2\pi)^2+4\pi^2+j4k\pi}\left(\frac{\left[1-(-1)^k\right]-jk\pi}{k^2\pi}\right)$$

Truncated Fourier series with $N=50$ harmonic components:



The filter essentially lets only the DC and fundamental components pass through it.

4.5 Exercise 4.10 of Boulet's book.

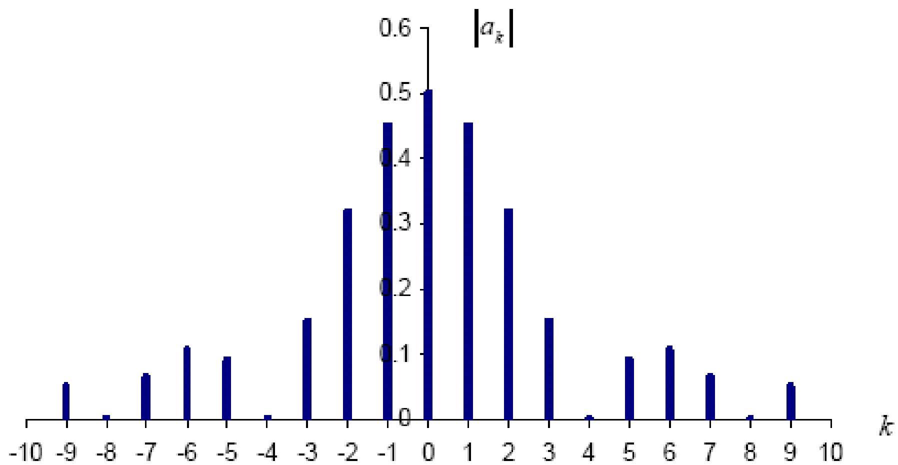
Answer (a)

This is just our regular rectangular wave with $T_1 = 0.5$ and of amplitude 2 but that has been "lowered" by 1 and time-delayed by 0.5. Fourier series coefficients are:

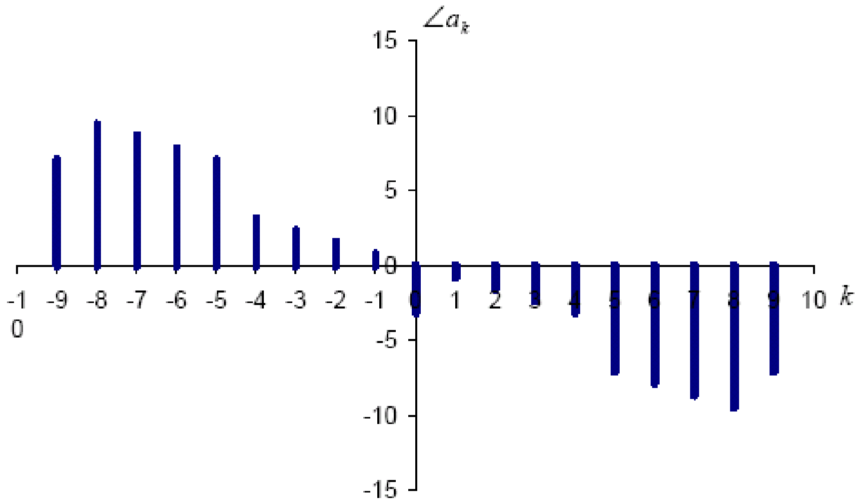
$$a_0 = \frac{1}{T}\int_0^1 x(t)dt = -\frac{1}{2},$$

$$a_k = 2\frac{2T_1}{T}\text{sinc}(k\frac{2T_1}{T})e^{-jk\frac{2\pi}{T}T_1}$$
$$= \frac{1}{2}\text{sinc}(k\frac{1}{4})e^{-jk\frac{\pi}{4}}, \quad k \neq 0$$

Magnitude: $$|a_k| = \frac{1}{2}\left|\text{sinc}(k\frac{1}{4})\right|.$$



Phase: $\angle a_k = \frac{1}{2}\text{sinc}(k\frac{1}{4})e^{-jk\frac{\pi}{4}} = -k\frac{\pi}{4} - \frac{\pi}{2}\left\{1 - \text{sgn}\left(\text{sinc}(k\frac{1}{4})\right)\right\}$



(b) $x(t) = \sin(10\pi t) + \cos(20\pi t)$

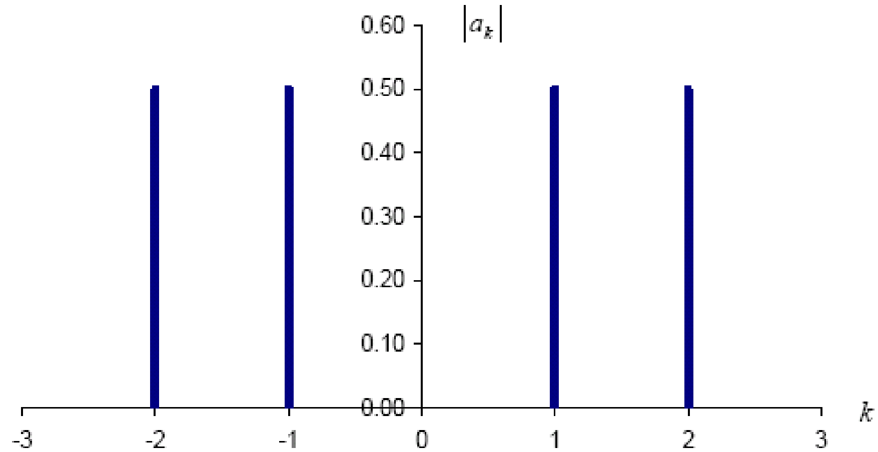
Answer:

$$x(t) = \frac{1}{2j}\left(e^{j10\pi t} - e^{-j10\pi t}\right) + \frac{1}{2}\left(e^{j20\pi t} + e^{-j20\pi t}\right)$$
$$= \frac{1}{2}e^{-j20\pi t} + \frac{j}{2}e^{-j10\pi t} - \frac{j}{2}e^{j10\pi t} + \frac{1}{2}e^{j20\pi t}$$

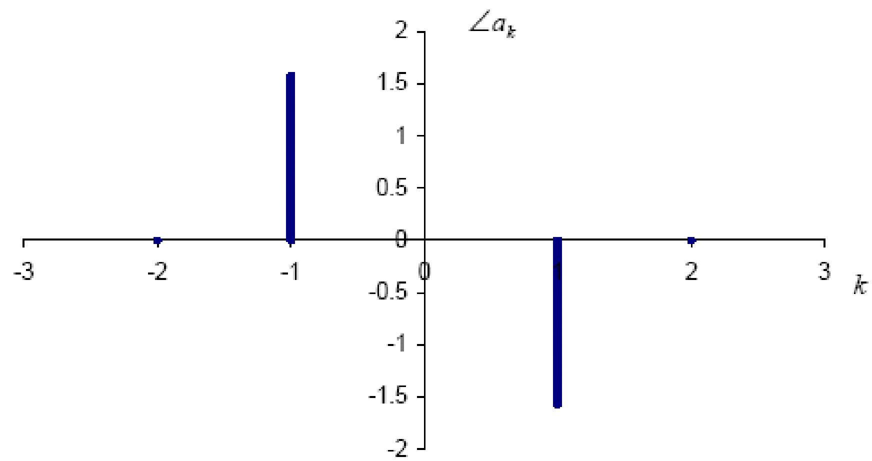
Thus,

$$\omega_0 = 10\pi,\ \ a_0 = 0$$
$$a_{-2} = \frac{1}{2}, a_{-1} = \frac{j}{2}, a_1 = -\frac{j}{2}, a_2 = \frac{1}{2}, \quad a_k = 0, \quad \text{otherwise}$$

Magnitude: $|a_0| = 0,\ |a_{-2}| = \frac{1}{2},\ |a_{-1}| = \frac{1}{2},\ |a_1| = \frac{1}{2},\ |a_2| = \frac{1}{2},\quad |a_k| = 0,\quad$ otherwise .



Phase: $\angle a_0 = 0,\ \angle a_{-2} = 0,\ \angle a_{-1} = \frac{\pi}{2},\ \angle a_1 = -\frac{\pi}{2},\ \angle a_2 = 0,\quad \angle a_k = 0,\quad$ otherwise .



4.6

a. Answer

The Fourier series of x(t)y(t) is

$$c_k = \frac{1}{T}\int_T \Big(\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \sum_{l=-\infty}^{\infty} b_l e^{jl\omega_0 t}\Big) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T \Big(\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_n b_l e^{j(n+l)\omega_0 t} e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_n b_l \int_T (e^{j(n+l)\omega_0 t} e^{-jk\omega_0 t}) dt = \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_n b_l \delta(k-(n+l)) = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

(b) Answer:

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$\overset{*}{x}(t) = \sum_{k=-\infty}^{\infty} \overset{*}{a}_k \, e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \overset{*}{a}_{-k} \, e^{jk\omega_0 t}$$

$$As \qquad y(t) = \overset{*}{x}(t)$$

$$Thus \qquad b_k = \overset{*}{a}_{-k}$$

From part (a),

$$c_0 = \sum_{n=-\infty}^{\infty} a_n b_{-n} = \sum_{n=-\infty}^{\infty} a_n \overset{*}{a}_{-n} = \sum_{n=-\infty}^{\infty} a_n \overset{*}{a}_{-n} = \sum_{n=-\infty}^{\infty} | a_n |^2$$

From the Fourier series analysis equation, we have

$$c_k = \frac{1}{T} \int | x(t) |^2 \, e^{-jk\omega_0 t} dt$$

with k=0, we have

$$c_0 = \frac{1}{T} \int_T | x(t) |^2 \, dt = \sum_{n=-\infty}^{\infty} | a_n |^2$$