Assignment 3

Due September 29 (M), 2008.

=====================Part 1 (no submission is required)=========================

Practice makes perfect. Do and understand all exercises in Chapter 3 of Benoit Boulet's book.

==================Part 2 (Handwritten and submission are required)==================

3.1 Show that if the response of an LTI system to x(t) is the output y(t), then the response of the system to

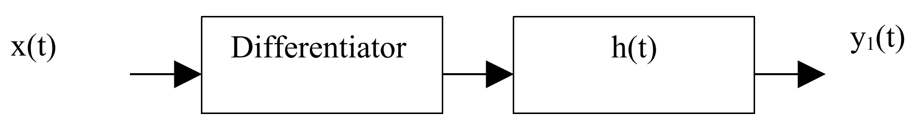
$x'(t) = \frac{dx(t)}{dt}$ is $y'(t) = \frac{dy(t)}{dt}$

Do this problem in three different ways:

1) Directly from the properties of linearity and time invariance and the fact that

$$x'(t) = \lim_{h \to 0} \frac{x(t) - x(t-h)}{h}$$

2) By differentiating the convolution integral.
3) By representing the derivative operation using an LTI system called "differentiator", whose response to the input x(t) is x′(t), and applying the commutative property of LTI systems.



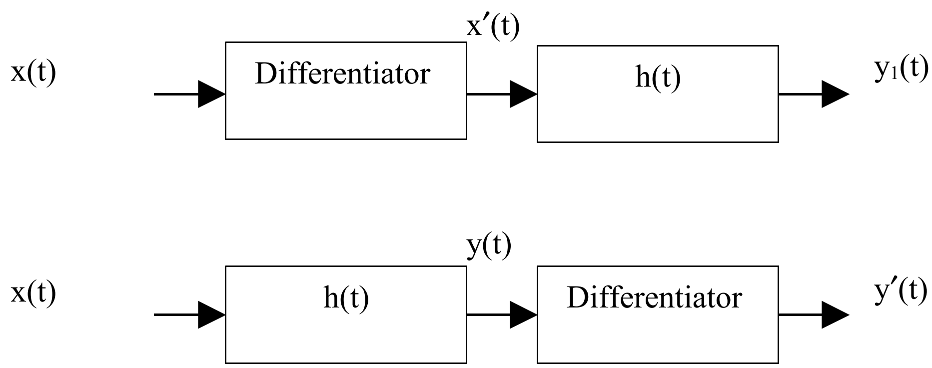
Answer:

1) From the property of time invariance of LTI systems, we know that if y(t) is the response to x(t), then the input x(t-h) lead to the output y(t-h).

From the property of linearity of LTI systems, we have additivity and homogeneity, we know the input [x(t)-x(t-h)]/h lead to the output [y(t)-y(t-h)]/h. taking limit as h→0, and the input [x(t)-x(t-h)]/h becomes x'(t) and the output becomes y'(t).

2) $y'(t) = \frac{d}{dt}\int_{-\infty}^{\infty} x(t-l)h(l)dl = \int_{-\infty}^{\infty} \frac{d}{dt}[x(t-l)]h(l)dl = \int_{-\infty}^{\infty} x'(t-l)h(l)dl = x'(t) * h(t)$

3)



The above 2 systems are equivalent according to the commutative property of LTI systems. Thus $y'(t)=y_1(t)=x'(t)*h(t)$.

3.2 Derive the step response of the system described by the differential equation:

$$\frac{dy(t)}{dt} + ay(t) = \frac{dx(t)}{dt} - 2x(t) \qquad (3.2)$$

For the system to be BIBO stable, please specify the range of the parameter a.

Answer:

Step 1. Solve the setep response of the system with input being x(t)=u(t):

$$\frac{dy(t)}{dt} + ay(t) = u(t) \qquad (3.2a)$$

The solution of Eq. (3.2a) is

$$y_1(t)=y_h(t)+y_p(t)$$

where $y_h(t)$ satisfies the homogeneous equation:

$$\frac{dy_h(t)}{dt} + ay_h(t) = 0$$

$y_h(t)$ should have the form $y_h(t)=Ae^{st}$. Substituting $y_h(t)$ in the above equation with Ae^{st}, the following hold:

$$Ae^{st}(s+a)=0.$$

Thus, for $y_h(t)$ to be non-zero, the following holds:

$$s=-a, \text{ and } y_h(t)=Ae^{-at}.$$

$y_p(t)$ should have the same form as the input u(t) for t>0. Let $y_p(t)=K$. Then

$$y_1(t)= y_h(t)+y_p(t)= (Ae^{-at}+K)u(t).$$

Substitute y(t) in Eq. (3.2a) with above $y_1(t)$:

$$\frac{d}{dt}[(Ae^{-at}+K)u(t)]+a(Ae^{-at}+K)u(t)=u(t)$$

Then we have

$$(-aAe^{-at})u(t)+(Ae^{-at}+K)\delta(t)+a(Ae^{-at}+K)u(t)=u(t)$$

i.e., $$aKu(t)+(Ae^{-at}+K)\delta(t)=u(t)$$

for the two sides of the above equation to be equal, the singularity functions on the left-hand and right-hand sides must match:

$$aK=1 \qquad (3.2b)$$

and $$A+K=0 \qquad (3.2c)$$

thus $K=1/a,\ A=-K=-1/a$

Therefore, the solution of Eq. (3.2a) is

$$y_1(t)=[-\frac{1}{a}e^{-at}+\frac{1}{a}]u(t)$$

Step 2: According to the principle of superposition of LTI systems, the step response of Eq. (3.2) is

$$\begin{aligned}s(t)&=\frac{dy_1(t)}{dt}-2y_1(t)=\frac{d}{dt}[(-\frac{1}{a}e^{-at}+\frac{1}{a})u(t)]-2(-\frac{1}{a}e^{-at}+\frac{1}{a})u(t)\\&=(-\frac{1}{a}e^{-at}+\frac{1}{a})\delta(t)+e^{-at}u(t)-2(-\frac{1}{a}e^{-at}+\frac{1}{a})u(t)\\&=[(1+\frac{2}{a})e^{-at}-\frac{2}{a}]u(t)\end{aligned}$$

For the system to be BIBO stable, the real part of the zero of the characteristic polunomial must be negative. As the zero of the characteristic polynomial is s=-a, thus Re{-a}<0 is required for the system to be stable.

Note:

1. The above solution s(t) is the integral of the impulse response $h_1(t)$ given in Exercise 3.1 of Boulet's book.
2. The method in this anwer for obtaining step response does not use the initial conditions at $t=0^+$ to determin the coefficients *A* and *K*. They were obtained by matching the singularity functions on the left-hand and right-hand sides. This is because the differential equation holds for $-\infty < t < \infty$, including the instant *t*=0 when the input u[t] jumps from 0 to 1.

3.3 Exercise 3.6 of Boulet’ book.

Answer:

The characteristic polynomial of this system is:

$$p(s) := 2s^2 - 2s - 24 = 2(s+3)(s-4).$$

Since one of the zeros of this polynomial has a positive real part, i.e., $s_1 = 4$, the system is unstable.

3.4 Exercise 3.8 of Boulet’ book.

Answer:

Let $y[n]$ be the total amount in the account at the beginning of year n, and $x[n]$ be the amount deposited at the beginning of year n (included in $y[n]$.) The following first-order, causal LTI difference system initially at rest describes the evolution of the bank account:

$$y[n] = (1+r)y[n-1] + x[n].$$

Note that this is an unstable system as the zero of the first-order characteristic polynomial, which is $1+r=1.06$, is larger than one. The amount at the end of the 50th year can be computed by recursion in order to find the response of the system to the step input: $x[n]=1000u[n-1]$:

$$
\begin{aligned}
n=1: \quad & y[1]=1.06y[0]+1000u[0] \\
& =1000 \\
n=2: \quad & y[2]=1.06y[1]+1000u[1] \\
& =1.06(1000)+1000 \\
n=3: \quad & y[3]=1.06y[2]+1000u[2] \\
& =(1.06)^2(1000)+1.06(1000) \\
& \vdots \\
& y[n]=1.06y[n-1]+1000u[n-1] \\
& =1000\sum_{k=1}^{n}(1.06)^{k-1} \\
& \vdots
\end{aligned}
$$

At the end of the 50th year:

$$
\begin{aligned}
(1+r)y[50] &= (1.06)1000\sum_{k=1}^{50}(1.06)^{k-1} = 1060\sum_{m=0}^{49}(1.06)^m \\
&= 1060\frac{1-(1.06)^{50}}{1-(1.06)} = 1060\frac{1-(1.06)^{50}}{-0.06} = \$307,756
\end{aligned}
$$

3.5 Exercise 3.10 of Boulet' book.

Answer:

Step 1: Set up the problem to calculate the impulse response of the left-hand side:

$$2\frac{dh_a(t)}{dt}+4h_a(t)=\delta(t)$$

Step 2: Find the initial condition of the corresponding homogeneous equation at $t=0^+$ by integrating the above differential equation from $t=0^-$ to $t=0^+$. Note that the impulse will be in the term $\frac{dh_a(t)}{dt}$, so $h_a(t)$ will have a finite jump at most. Thus we have:

$$\int_{0^-}^{0^+} \frac{dh_a(\tau)}{d\tau} d\tau = h_a(0^+) = 0.5,$$

hence $h_a(0^+) = 0.5$ is our initial condition for the homogeneous equation for $t > 0$:

$$2\frac{dh_a(t)}{dt} + 4h_a(t) = 0.$$

Step 3: The characteristic polynomial is $p(s) = 2s + 4$ and it has one zero at $s = -2$, which means that the homogeneous response has the form $h_a(t) = Ae^{-2t}$ for $t > 0$. The initial condition allows us to determine the constant A:

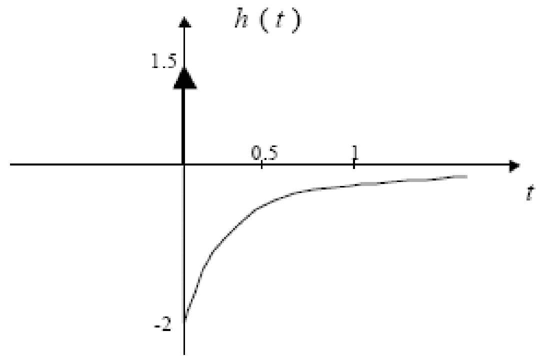
$$h_a(0^+) = A = 0.5,$$

so that
$$h_a(t) = 0.5e^{-2t}.$$

Step 4: Apply the right-hand side of the differential equation to $h_a(t)$:

$$\begin{aligned} h(t) &= 3\frac{dh_a(t)}{dt} + 2h_a(t) \\ &= 3\frac{d}{dt}\left(0.5e^{-2t}u(t)\right) + e^{-2t}u(t) \\ &= -3e^{-2t}u(t) + 1.5\delta(t) + e^{-2t}u(t) \\ &= -2e^{-2t}u(t) + 1.5\delta(t) \end{aligned}$$

Sketch of impulse response:



3.6 Exercise 3.12 of Boulet's book.

Answer:

Step 1: Set up the problem to calculate the impulse response of the left-hand side of the equation

$$\frac{d^2 h_a(t)}{dt^2} + 2\frac{dh_a(t)}{dt} + 2h_a(t) = \delta(t)$$

Step 2: Find the initial conditions of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{d^2 h_a(t)}{dt^2}$, so $\frac{dh_a(t)}{dt}$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{d^2 h_a(\tau)}{d\tau^2} d\tau = \frac{dh_a(0^+)}{dt} = 1 ,$$

hence $\frac{dh_a(0^+)}{dt} = 1$ is one of our two initial conditions for the homogeneous equation for $t > 0$

$$\frac{d^2 h_a(t)}{dt^2} + 2\frac{dh_a(t)}{dt} + 2h_a(t) = 0 .$$

Since $\frac{dh_a(t)}{dt}$ has a finite jump from $t = 0^-$ to $t = 0^+$, the other initial condition is $h_a(0^+) = 0$.

Step 3: The characteristic polynomial is $p(s) = s^2 + 2s + 2$ and it has zeros at $s_1 = -1 + j, s_2 = -1 - j$, which means that the homogeneous response has the form $h_a(t) = Ae^{(-1+j)t} + Be^{(-1-j)t}$ for $t > 0$. The initial conditions allow us to determine the constants A and B:

$$h_a(0^+) = 0 = A + B ,$$

$$\frac{dh_a(0^+)}{dt} = 1 = (-1 + j)A + (-1 - j)B$$

so that $A = -\frac{j}{2}, B = \frac{j}{2}$ and

$$\begin{aligned} h_a(t) &= -\frac{j}{2}e^{(-1+j)t}u(t) + \frac{j}{2}e^{(-1-j)t}u(t) \\ &= \frac{e^{(-1+j)t} - e^{(-1-j)t}}{2j}u(t) \\ &= e^{-t}\sin(t)u(t) \end{aligned} .$$

Step 4: Apply the right-hand side of the differential equation to $h_a(t)$ to obtain $h(t)$.

$$\begin{aligned}
h(t) &= -3\frac{dh_a(t)}{dt} + h_a(t) \\
&= -3\frac{d}{dt}\left[e^{-t}\sin t u(t)\right] + e^{-t}\sin t u(t) \\
&= -3\left[-e^{-t}\sin t u(t) + e^{-t}\cos t u(t) + e^{-t}\sin t\delta(t)\right] + e^{-t}\sin t u(t) \\
&= e^{-t}\cos t u(t) + 4e^{-t}\sin t u(t)
\end{aligned}$$