Assignment 2
Due Sep 22 (M) before class.

====================Part 1 (no submission is required)=========================

Practice makes perfect. Do and understand all exercises in Chapter 2 of Benoit Boulet's book.

=================Part 2 (Handwritten and submission are required)==================

2.1 Exercise 2.4 of Boulet's book.

Answer:

$$\sum_{n=-\infty}^{+\infty} \left|(-0.9)^n u[n-4]\right| = \sum_{n=4}^{+\infty} \left|(-0.9)^n\right| = \sum_{n=4}^{+\infty} (0.9)^n = \sum_{m=0}^{+\infty} (0.9)^{m+4}$$

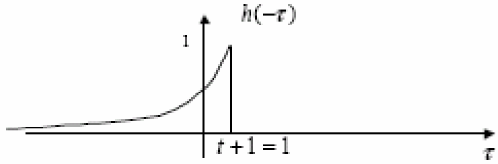
$$= (0.9)^4 \sum_{m=0}^{+\infty} (0.9)^m = (0.9)^4 \frac{1}{1-0.9} = 10(0.9)^4 < +\infty$$

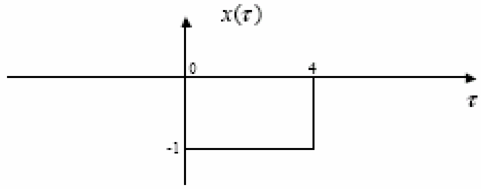
The impulse response is absolutely summable, hence the system is stable. The system is also causal as $h[n] = 0, \; n \leq 0$.

2.2 Exercise 2.6 of Boulet's book

Answer:

SOLUTION 1: Let us time-reverse and shift the impulse response.





The intervals of time of interest are:

$t < -1$: no overlap, so $y(t) = 0$.

$-1 \le t < 3$: overlap over $0 \le \tau < t+1$, so

$$\begin{aligned} y(t) &= \int_0^{t+1} x(\tau)h(t-\tau)d\tau = -\int_0^{t+1} e^{-t+\tau-1}d\tau = -e^{-1-t}\left[e^{\tau}\right]_0^{t+1} \\ &= -e^{-1-t}\left[e^{t+1}-1\right] = e^{-t-1}-1 \end{aligned}.$$

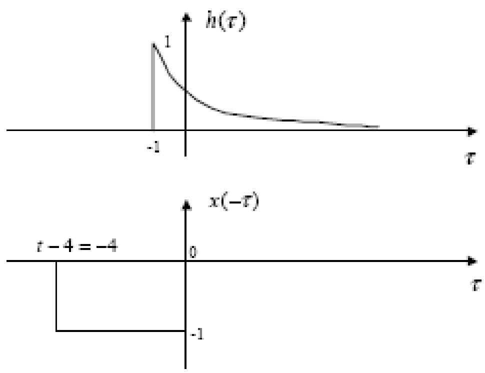
$t \ge 3$: overlap over $0 \le \tau < 4$, so

$$\begin{aligned} y(t) &= \int_0^{4} x(\tau)h(t-\tau)d\tau = -\int_0^{4} e^{-t-1+\tau}d\tau = -e^{-t-1}\int_0^{4} e^{\tau}d\tau = -e^{-t-1}\left[e^{\tau}\right]_0^{4} \\ &= -e^{-t-1}\left[e^{4}-1\right] = -e^{-(t-3)} + e^{-(t+1)} = \left(e^{-1}-e^{3}\right)e^{-t} \end{aligned}.$$

Finally, piecing all three intervals together, we get:

$$y(t) = \begin{cases} 0, & t < -1 \\ e^{-t-1} - 1, & -1 \le t < 3 \\ \left(e^{-1} - e^{3}\right)e^{-t}, & t \ge 3 \end{cases}$$

SOLUTION 2: Time reversing and shifting $x(t)$.



The intervals of interest are:

$t < -1$: no overlap, so $y(t) = 0$.

$-1 \le t < 3$: overlap over $-1 \le \tau < t$, so

$$\begin{aligned} y(t) &= \int_{-1}^{t} h(\tau)x(t-\tau)d\tau = -\int_{-1}^{t} e^{-\tau-1}d\tau = e^{-1}\left[e^{-\tau}\right]_{-1}^{t} \\ &= e^{-1}\left[e^{-t} - e\right] = e^{-t-1} - 1 \end{aligned}$$

$t \ge 3$: overlap over $t-4 \le \tau < t$, so

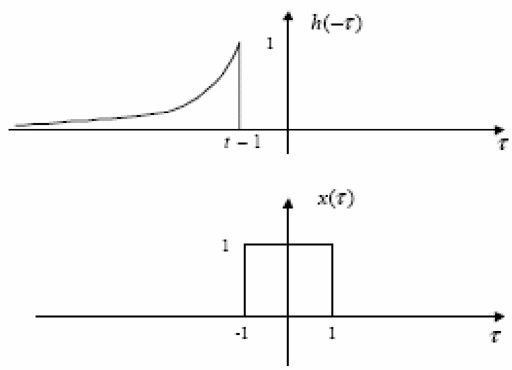
$$\begin{aligned} y(t) &= \int_{t-4}^{t} h(\tau)x(t-\tau)d\tau = -\int_{t-4}^{t} e^{-\tau-1}d\tau = e^{-1}\left[e^{-\tau}\right]_{t-4}^{t} \\ &= e^{-t-1} - e^{-t+3} \end{aligned}$$

Finally, piecing all three intervals together, we get:

$$y(t) = \begin{cases} 0, & t < -1 \\ e^{-t-1} - 1, & -1 \le t < 3 \\ \left(e^{-1} - e^{3}\right)e^{-t}, & t \ge 3 \end{cases}$$

2.3 Exercise 2.8 of Boulet's book.

SOLUTION 1: Let's time-reverse and shift the impulse response.



The intervals of time of interest are:

$t<0$: no overlap, so $y(t)=0$.

$0 \le t < 2$: overlap over $-1 \le \tau < t-1$, so

$$y(t)=\int_{-1}^{t-1} x(\tau)h(t-\tau)d\tau=\int_{-1}^{t-1} e^{1+\tau-t}d\tau=e^{1-t}\left[e^{\tau}\right]_{-1}^{t-1}=e^{1-t}\left[e^{t-1}-e^{-1}\right]=1-e^{-t}.$$

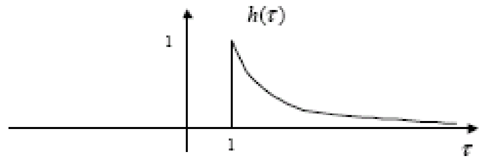
$t \ge 2$: overlap over $-1 \le \tau < 1$, so

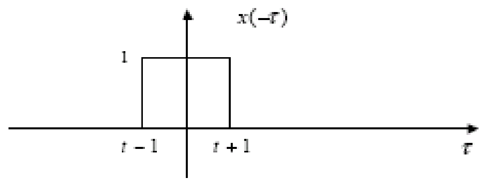
$$\begin{aligned} y(t)&=\int_{-1}^{1} x(\tau)h(t-\tau)d\tau=\int_{-1}^{1} e^{1+\tau-t}d\tau=e^{1-t}\left[e^{\tau}\right]_{-1}^{1}. \\ &=e^{1-t}\left[e^{1}-e^{-1}\right]=e^{-t+2}-e^{-t}=e^{-t}(e^{2}-1) \end{aligned}$$

Finally, piecing all three intervals together, we get:

$$y(t)=\begin{cases} 0, & t<0 \\ 1-e^{-t}, & 0 \le t < 2 \\ e^{-t}(e^{2}-1), & t \ge 2 \end{cases}$$

SOLUTION 2: Time reversing and shifting $x(t)$.





$t < 0$: no overlap, so $y(t) = 0$.

$0 \le t < 2$: overlap over $1 \le \tau < t+1$, so

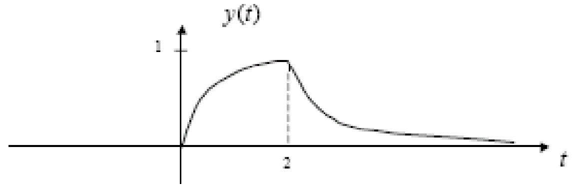
$$y(t) = \int_{1}^{t+1} h(\tau)x(t-\tau)d\tau = \int_{1}^{t+1} e^{1-\tau}d\tau = -\left[e^{1-\tau}\right]_{1}^{t+1}.$$
$$= -\left[e^{-t} - 1\right] = 1 - e^{-t}$$

$t \ge 2$: overlap over $t-1 \le \tau < t+1$, so

$$y(t) = \int_{t-1}^{t+1} h(\tau)x(t-\tau)d\tau = \int_{t-1}^{t+1} e^{1-\tau}d\tau = -\left[e^{1-\tau}\right]_{t-1}^{t+1}.$$
$$= -\left(e^{-t} - e^{2-t}\right) = e^{-t}\left(e^{2} - 1\right)$$

Finally, piecing all three intervals together, we get:

$$y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t}, & 0 \le t < 2 \\ e^{-t}(e^{2} - 1), & t \ge 2 \end{cases}$$



2.4 Exercise 2.10 of Boulet's book.

(a) Compute the impulse response $h(t)$ of the overall system.

Answer:

$$h(t) = h_1(t) * h_2(t) = e^{-t}u(t) * e^{-2t}u(t)$$

$t < 0$: no overlap so $y(t) = 0$

$t \geq 0$:

$$h(t) = \int_0^t h_1(\tau)h_2(t-\tau)d\tau = \int_0^t e^{-\tau}e^{-2(t-\tau)}d\tau = e^{-2t}\int_0^t e^{\tau}d\tau$$

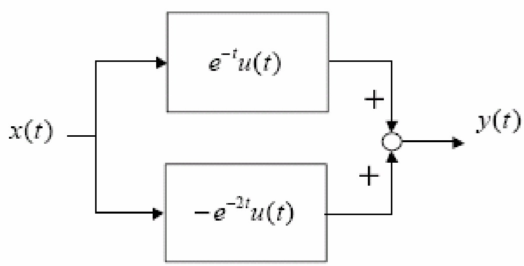
$$= e^{-2t}\left[e^{\tau}\right]_0^t = e^{-2t}\left[e^{t}-1\right] = e^{-t} - e^{-2t}$$

$$h(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

(b) Find an equivalent system (same impulse response) configured as a parallel interconnection of two LTI subsystems.

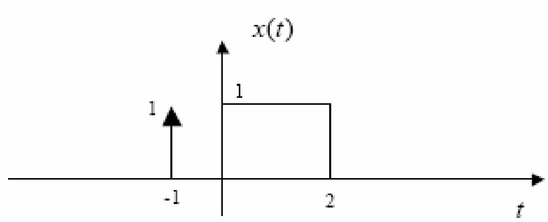
Answer:

From the expression $h(t) = \left(e^{-t} - e^{-2t}\right)u(t)$, it is clear that the system below has the same impulse response:



(c) Sketch the input signal $x(t)$. Compute the output signal $y(t)$.

Answer:



$$y(t) = x(t) * h(t) = x(t) * \left[e^{-t} u(t) - e^{-2t} u(t) \right]$$
$$= \left(e^{-(t+1)} - e^{-2(t+1)} \right) u(t+1) + \underbrace{[u(t) - u(t-2)]}_{p(t)} * \left[e^{-t} u(t) - e^{-2t} u(t) \right]$$
$$= \left(e^{-(t+1)} - e^{-2(t+1)} \right) u(t+1) + \underbrace{p(t) * e^{-t} u(t)}_{y_1(t)} - \underbrace{p(t) * e^{-2t} u(t)}_{y_2(t)}$$

$t < 0: \ y_1(t) = y_2(t) = 0$

$0 \le t \le 2:$

$$y_1(t) = \int_0^t e^{-(t-\tau)} d\tau = e^{-t} \left[e^{\tau} \right]_0^t = 1 - e^{-t}$$

$$y_2(t) = \int_0^t e^{-2(t-\tau)} d\tau = e^{-2t} \left[e^{2\tau} \right]_0^t = 1 - e^{-2t}$$

$t > 2:$

$$y_1(t) = \int_0^2 e^{-(t-\tau)} d\tau = e^{-t} \left[e^{\tau} \right]_0^2 = (e^2 - 1) e^{-t}$$

$$y_2(t) = \int_0^2 e^{-2(t-\tau)} d\tau = e^{-2t} \left[e^{2\tau} \right]_0^2 = (e^4 - 1) e^{-2t}$$

Finally, the overall response of the system is:

$$y(t) = \begin{cases} 0 & t < -1 \\ e^{-(t+1)} - e^{-2(t+1)}, & -1 \le t < 0 \\ e^{-(t+1)} - e^{-2(t+1)} - e^{-t} + e^{-2t}, & 0 \le t \le 2 \\ (e^2 + e^{-1} - 1) e^{-t} - (e^4 + e^{-2} - 1) e^{-2t} & t > 2 \end{cases}$$

2.5 It is known that

$x[n] \neq 0$, for $N2 \le n \le N1$

$h[n] \neq 0$, for $N4 \le n \le N3$

$y[n] = x[n] * h[n] \neq 0$, for $N6 \le n \le N5$

Please determine the duration of y[n].

Answer:

The length of x[n] is L1=N1-N2+1. The length of h[n] is L2= N3-N4+1. The length of y[n] is L3=L1+L2-1.

2.6 It is known that the impulse response of a LTI system is $h[n]=2^{-n}$, n>=0. The input signal of the system is x[n]=u[n]-2u[n-5].

Please derive the response of the system to x[n].

Answer:
Let the step response of the system be s[n]. Then, the response of the system to u[n-5] is s[n-5].
Then the response of the system to u[n]-2u[n-5] is s[n]-2s[n-5].

$$s[n]=\sum_{k=0}^{n}h[k]=\sum_{k=0}^{n}2^{-k}=\frac{1-0.5^{n+1}}{1-0.5}, \text{ for } n>=0$$

so, the response to x[n] is

$$y[n]=\frac{1-0.5^{n+1}}{1-0.5}-2\frac{1-0.5^{n-5+1}}{1-0.5}=\frac{-1-0.5^{n}(0.5-2^{5})}{0.5}=-2-0.5^{n}(1-2^{6})=-2+63(1/2)^{n}$$

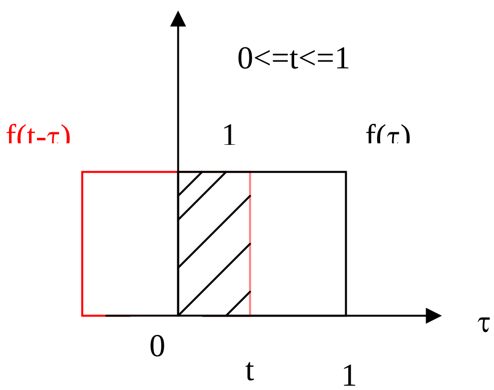
2.7 Calculate the convolution y(t)=f(t)*f(t), where f(t)=u(t)-u(t-1), and sketch y(t).
Answer :
For t<0, f(t-τ) and f(τ) have no overlap, i.e., f(t-τ) f(τ) =0. Thus y(t)=0.
For 0<=t<=1 :

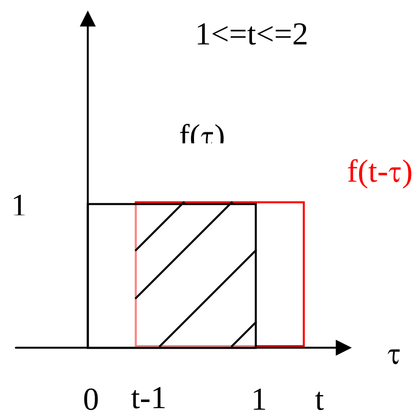
$$y(t)=\int_{0}^{t}d\tau=t$$

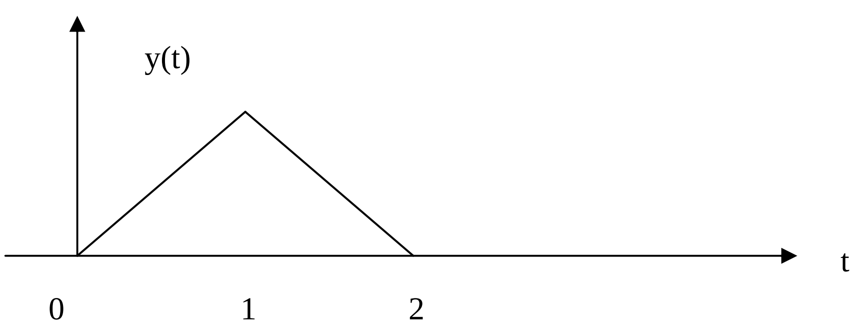
For 1<=t<=2 :

$$y(t)=\int_{t-1}^{1}d\tau=2-t$$

For t>=2 : y(t)=0

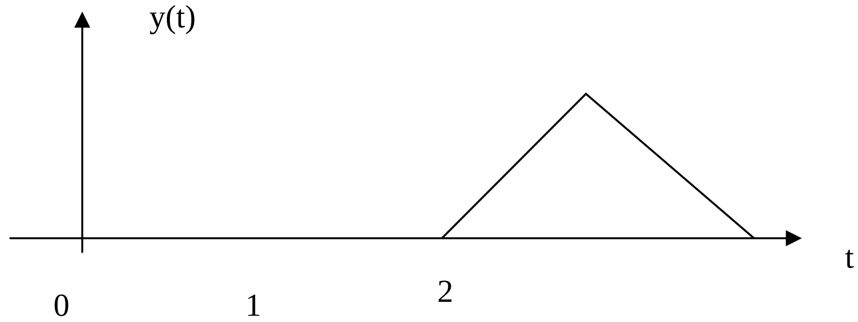






Note : Combining graphical method and analysis method makes convolutions clear and simple.

2.8 Calculate the convolution y(t)=f(t)*f(t), where f(t)=u(t-1)-u(t-2), and sketch y(t).
Answer : y(t) is the delayed version of the answer to 2.7: shift y(t) obtained in 2.7 to the right by 2.



2.9 $f_1(t)=\delta(t)$, $f_2(t)=\cos(\omega t+45^o)$, calculate $f_1(t)* f_2(t)$

Answer : $\delta(t)* f_2(t)= f_2(t)=\cos(\omega t+45^o)$,

2.10 Calculate $f_1(t)* f_2(t)$, where $f_1(t)=\sin(t)u(t)$, $f_2(t)=u(t-1)$. (hint: step response of $f_1(t)$)

Answer : $f_1(t)$ can be viewed as an impulse response of a system. The step response of the system defined by $f_1(t)$ is the running integral:

$$s(t) = \int_0^t \sin(\tau)u(\tau)u(t-\tau)d\tau = \int_0^t \sin(\tau)d\tau = -\cos t + 1 \text{ for } t>0$$

Then, the response of the system defined by $f_1(t)$ to u(t-1) is s(t-1)=1-cos(t-1) for t>1, i.e., $f_1(t)* f_2(t)=[1-\cos(t-1)]u(t-1)$.

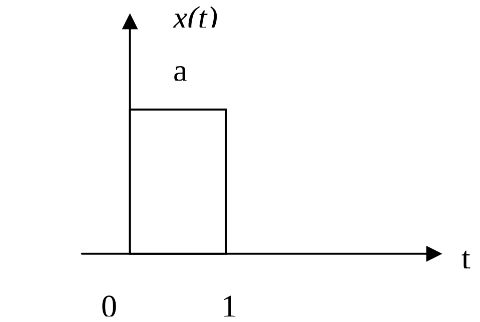
2.11 Calculate $f_1(t)* f_2(t)$, where $f_1(t)=1$, $f_2(t)=e^{-(t+1)}u(t+1)$
Answer : 1

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau = \int_{-\infty}^{t+1} e^{-(t-\tau+1)}u(t-\tau+1)d\tau = \int_{-\infty}^{t+1} e^{-(t-\tau+1)}d\tau$$

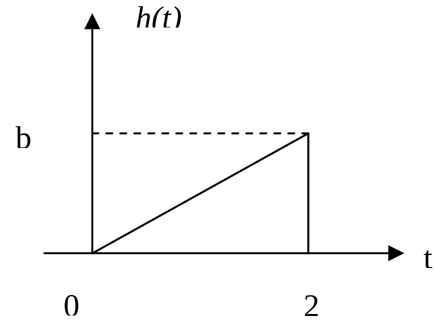
$$= e^{-t+\tau-1}\Big|_{\tau=-\infty}^{\tau=t+1} = 1-0=1$$

2.12 Calculate the convolution y(t)=x(t)*h(t), where x(t) and h(t) are shown below.
Answer : The output values for different ranges of time shift *t* can be calculated from the corresponding overlapping areas (triangle or trapezoid are formed when shifting x(-τ) from the left to the right).

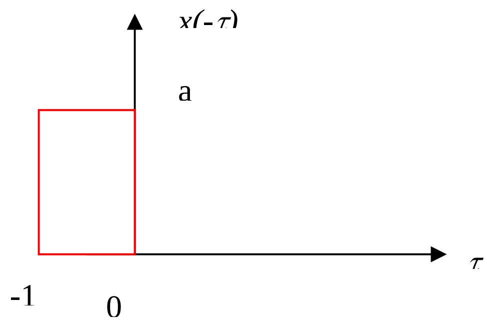
$$y(t) = \begin{cases} \frac{ab}{4}t^2, & 0 \le t \le 1 \\ \frac{ab}{4}(2t-1), & 1 \le t \le 2 \\ \frac{ab}{4}(3+2t-t^2), & 2 \le t \le 3 \\ 0, & otherwise \end{cases}$$



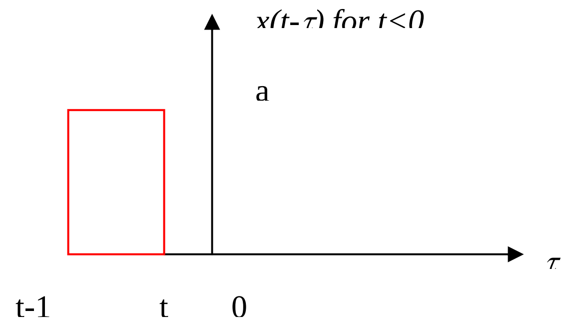
x(t)
a
t
0
1



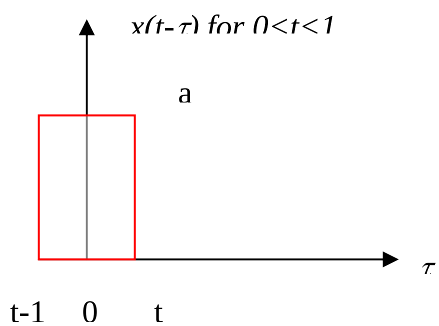
h(t)
b
t
0
2



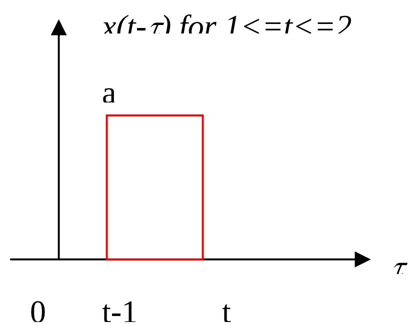
x(-τ)
a
τ
-1
0



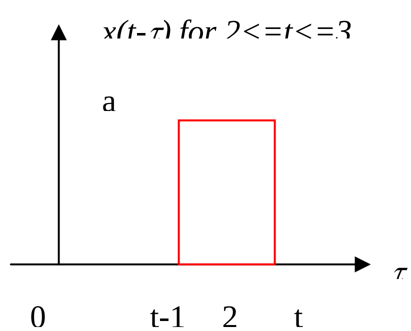
x(t-τ) for t<0
a
τ
t-1
t
0



x(t-τ) for 0<t<1
a
τ
t-1
0
t



x(t-τ) for 1<=t<=2
a
τ
0
t-1
t



x(t-τ) for 2<=t<=3
a
τ
0
t-1
2
t