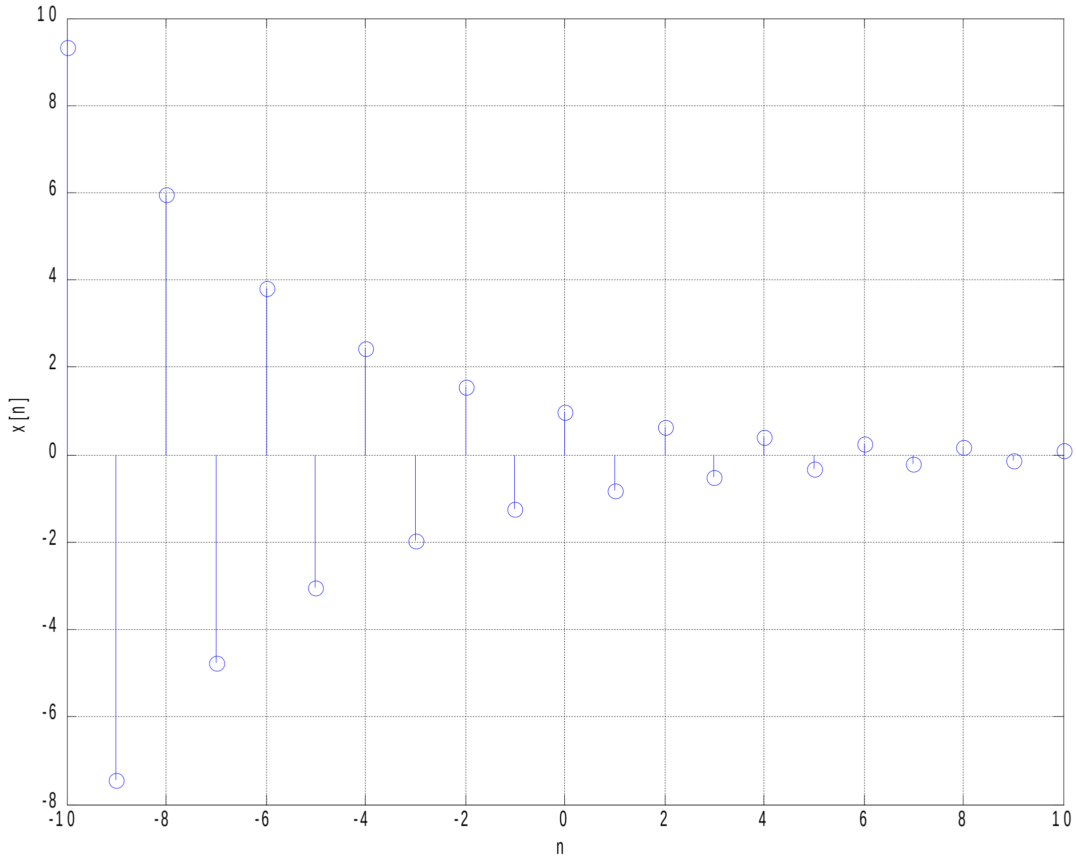
Assignment 1.
Due: September 15, 2008, Monday before class.
Drop finished assignments in the assignment box near the undergraduate Office in the ENGTR building.

==================Part 1 (no submission is required)================================

1. Do and understand all exercises in Chapter 1 of Benoit Boulet's book.
2. Run the Matlab script on page 14-15 of Boulet's book.

==================Part 2 (Handwritten and submission are required)==================

1.1 Sketch the signal $x[n]=\alpha^n$, where $\alpha=-0.8$.



1.2 Exercise 1.6 of Boulet's book.

(a) $x(t)=e^{(-2+j3)t}$

Answer:

$$x(t)=e^{(-2+j3)t}=e^{-2t}\left[\cos(3t)+j\sin(3t)\right]=e^{-2t}\cos(3t)+je^{-2t}\sin(3t)$$

(b) $x(t)=e^{-j\pi t}u(t)+e^{(2+j\pi)t}u(-t)$

Answer:

$$\begin{aligned}x(t)&=e^{-j\pi t}u(t)+e^{(2+j\pi)t}u(-t)\\&=\cos(\pi t)u(t)-j\sin(\pi t)u(t)+e^{2t}\cos(\pi t)u(-t)+je^{2t}\sin(\pi t)u(-t)\\&=\cos(\pi t)u(t)+e^{2t}\cos(\pi t)u(-t)+j\left[e^{2t}\sin(\pi t)u(-t)-\sin(\pi t)u(t)\right]\end{aligned}$$

1.3 Exercise 1.8 of Boulet's book.

Compute the convolution: $\delta(t-T)*e^{-2t}u(t)=\int_{-\infty}^{\infty}\delta(\tau-T)e^{-2(t-\tau)}u(t-\tau)d\tau$.

Answer:

$$\begin{aligned}\delta(t-T)*e^{-2t}u(t)&=\int_{-\infty}^{\infty}\delta(\tau-T)e^{-2(t-\tau)}u(t-\tau)d\tau\\&=e^{-2(t-T)}u(t-T)\end{aligned}$$

1.4 Exercise 1.10 of Boulet's book.

Determine whether the following systems are: 1. Memoryless, 2. Time-invariant, 3. Linear, 4. Causal, 5. BIBO Stable. Justify your answers.

(a) $y(t) = \frac{d}{dt}x(t)$, where the time derivative of $x(t)$ is defined as $\frac{d}{dt}x(t) := \lim_{\Delta t \to 0^+} \frac{x(t) - x(t - \Delta t)}{\Delta t}$.

Answer:

1. Memoryless? No.

The output at time t depends on the value of the input at time t^-.

2. Time-invariant? Yes.

$$y_1(t) := Sx(t-T) = \lim_{\Delta t \to 0} \frac{x(t-T) - x(t-T-\Delta t)}{\Delta t} = y(t-T)$$

3. Linear? Yes. Let

$$y_1(t) := Sx_1(t) = \lim_{\Delta t \to 0} \frac{x_1(t) - x_1(t - \Delta t)}{\Delta t}$$
$$y_2(t) := Sx_2(t) = \lim_{\Delta t \to 0} \frac{x_2(t) - x_2(t - \Delta t)}{\Delta t}$$

Then,

$$\begin{aligned} y(t) &= S[ax_1(t) + bx_2(t)] \\ &= \lim_{\Delta t \to 0} \frac{[ax_1(t) + bx_2(t)] - [ax_1(t - \Delta t) + bx_2(t - \Delta t)]}{\Delta t} \\ &= a \lim_{\Delta t \to 0} \frac{x_1(t) - x_1(t - \Delta t)}{\Delta t} + b \lim_{\Delta t \to 0} \frac{x_2(t) - x_2(t - \Delta t)}{\Delta t} \\ &= ay_1(t) + by_2(t) \end{aligned}$$

4. Causal? Yes. The output depends on $x(t)$ and $x(t^-)$.

5. Stable? No. Consider:

$$|x(t)| = |\sin \omega t| \le 1 \Rightarrow \text{ Given } K,\ \exists t_0, \omega \text{ such that } |y(t_0)| = \left|\frac{d}{dt} x(t)\right|_{t_0} = |\omega \cos \omega t_0| = |\omega| > K$$

i.e., the output is unbounded for a sinusoidal input of infinite frequency.

(b) $y(t) = \dfrac{t}{1 + x(t-1)}$

Answer:

1. Memoryless? No, because the output at time t uses the past value of the input $x(t-1)$.

2. Time-invariant? No. The system S is time-varying since:

$$y_1(t) = Sx(t-T) = \frac{t}{1+x(t-1-T)} \neq \frac{t-T}{1+x(t-1-T)} = y(t-T)$$

3. Linear? No. The system S is nonlinear because it does not have the superposition property:

For $x_1(t), x_2(t)$, let $y_1(t) := \frac{t}{1+x_1(t-1)}$, $y_2(t) := \frac{t}{1+x_2(t-1)}$

Define $x(t) = ax_1(t) + bx_2(t)$.

Then $y(t) = \frac{t}{1+ax_1(t-1)+bx_2(t-1)} \neq \frac{at}{1+x_1(t-1)} + \frac{bt}{1+x_2(t-1)} = ay_1(t) + by_2(t)$

4. Causal? Yes. The system is causal as the output is a function of past and current values of the input $x(t-1)$ and $x(t)$ only.

5. Stable? No. The system is unstable since for the bounded input $x(t) = -1 \Rightarrow \left|y(t)\right| = +\infty$, i.e., the output is unbounded.

(c) $y(t) = 2tx(2t)$

Answer:

1. Memoryless? No. The output at time $t > 0$ depends on future value of the input $x(2t)$.

2. Time-invariant? No.

$$y_1(t) = Sx(t-T) = 2tx(2t-T) \neq 2(t-T)x(2(t-T)) = y(t-T)$$

3. Linear? Yes. Let

$$y_1(t) := Sx_1(t) = 2tx_1(2t)$$
$$y_2(t) := Sx_2(t) = 2tx_2(2t)$$

Then

$$
\begin{aligned}
y(t) &= S[ax_1(t) + bx_2(t)] \\
&= 2t[ax_1(2t) + bx_2(2t)] \\
&= a2tx_1(2t) + b2tx_2(2t) \\
&= ay_1(t) + by_2(t)
\end{aligned}
$$

4. Causal? No. The output at time $t > 0$ depends on future value of the input $x(2t)$.

5. Stable? No. Consider the constant input $x(t) = B \Rightarrow$ for any K, $\exists T$ such that $|y(T)| = |2TB| > K$,

namely, for times $T > \frac{K}{|2B|}$. Hence, the output is unbounded.

(d) $y[n] = \sum_{k=-\infty}^{n} x[k-n]$

Answer:

1. Memoryless? No. The output at time n is computed using past values of the input, e.g., for $n = 1$, we need all past values of the input.

2. Time-invariant? No. Let $y_1[n] := Sx[n-N] = \sum_{k=-\infty}^{n} x[k-n-N]$:

$$
\begin{aligned}
y[n-N] &= \sum_{k=-\infty}^{n-N} x[k-(n-N)] = \sum_{k=-\infty}^{n-N} x[k-n+N], \text{ let } m = k+N \\
&= \sum_{m=-\infty}^{n} x[m-n] \neq y_1[n]
\end{aligned}
$$

3. Linear? Yes. Let $y_1[n] := Sx_1[n] = \sum_{k=-\infty}^{n} x_1[k-n]$, $y_2[n] := Sx_2[n] = \sum_{k=-\infty}^{n} x_2[k-n]$. Then, the output of the system with $x[n] := ax_1[n] + bx_2[n]$ is given by:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{n} x[k-n] = \sum_{k=-\infty}^{n} ax_1[k-n] + bx_2[k-n] \\
&= a\sum_{k=-\infty}^{n} x_1[k-n] + b\sum_{k=-\infty}^{n} x_2[k-n] = y_1[n] + y_2[n]
\end{aligned}
$$

4. Causal? No. For negative times, the output $y[n]$ depends on the future value $x[n]$.

5. Stable? No. For the bounded input signal: $x[n] = B \Rightarrow |y[n]| = \left|\sum_{k=-\infty}^{n} x[k-n]\right| = \left|\sum_{k=-\infty}^{n} B\right| = +\infty$.

(e) $y[n] = x[n] + nx[n+1]$

Answer:

1. Memoryless? No. The current value of the output depends on the next value of the input $x[n+1]$.

2. Time-invariant? No.

$$
y_1[n] := x[n-N] + nx[n-N+1] \neq x[n-N] + (n-N)x[n-N+1] = y[n-N]
$$

3. Linear? Yes. Let $y_1[n] := x_1[n] + nx_1[n+1]$, $y_2[n] := x_2[n] + nx_2[n+1]$. Then,

$$
\begin{aligned}
y[n] &= S(ax_1[n] + bx_2[n]) = ax_1[n] + bx_2[n] + nax_1[n+1] + nbx_2[n+1] \\
&= ax_1[n] + nax_1[n+1] + bx_2[n] + nbx_2[n+1] = ay_1[n] + by_2[n]
\end{aligned}
$$

4. Causal? No. The current output $y[n]$ depends on the next value of the input $x[n+1]$.

5. Stable? No.

For $x[n] = B$, given any $K > 0$,

$\exists N$ such that $|y[N]| = |x[N] + Nx[N+1]| = |(1+N)B| > K$, namely when $N > \dfrac{K}{B} - 1$.

(f) $y[n] = x[n] + x[n-2]$

Answer:

1. Memoryless? No. The output $y[n]$ depends on a previous value of the input $x[n-2]$

2. Time-invariant? Yes. $y_1[n] := Sx[n-N] = x[n-N] + x[n-2-N] = y[n-N]$

3. Linear? Yes. Let $y_1[n] := Sx_1[n] = x_1[n] + x_1[n-2]$, $y_2[n] := Sx_2[n] = x_2[n] + x_2[n-2]$.

Then,

$$\begin{aligned} y[n] &= S(ax_1[n] + bx_2[n]) = (ax_1[n] + bx_2[n]) + (ax_1[n-2] + bx_2[n-2]) \\ &= ax_1[n] + ax_1[n-2] + bx_2[n] + bx_2[n-2] = ay_1[n] + by_2[n] \end{aligned}$$

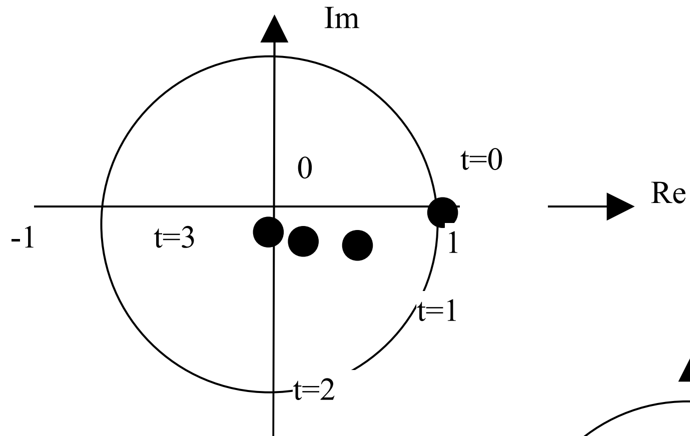
4. Causal? Yes. The current value of the output $y[n]$ depends only on the current and previous values of the input $x[n], x[n-2]$.

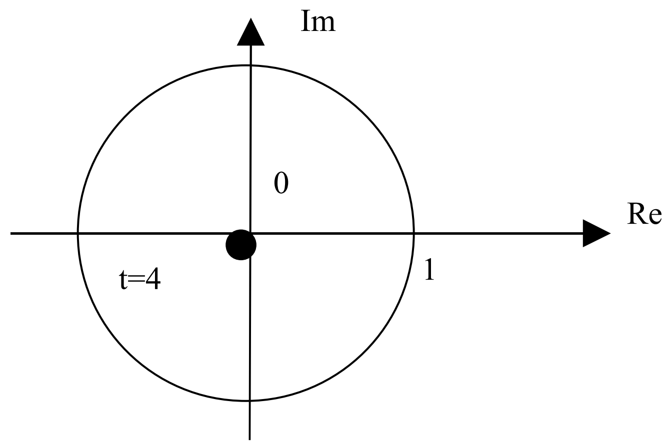
5. Stable? Yes. Given any $B > 0$, for $|x[n]| < B$, $|y[n]| \le |x[n]| + |x[n-2]| = 2B, \forall n$.

1.5 x(t)= $2^{-t} e^{j100t}$, the signal can be represented using a spiral trajectory in the complex plan. Please indicate the points corresponding to t=0, 1, 2, 3, 4.

Answer:

t	0	1	2	3	4
|x(t)|	1	0.5	0.25	0.125	0.0625
Angle of x(t) in radians	0	15×2π+5.7522	31*2π+5.2213	47*2π+4.6903	63*2π+4.1593
Angle of x(t) in degree	0	329.57	299.1585	268.7344	238.3103





1.6 Determine the fundamental period for each of the following signals, if it is periodic.

a. Sin(2t)+cos(3t)

Answer : 2π

b. $e^{j0.5\pi n} + e^{-j0.7\pi n}$

Answer : The minimum N to make 0.5πN=m2π and 0.7πN=k2π, where k and m are prime, is N=20.

1.7. Determine the even and odd components of the signals

a. x(t)=sin(t)+cos(3t)

Answer:

$$x_{even}(t) = \frac{\sin(t) + \cos(3t) + \sin(-t) + \cos(-3t)}{2} = \cos(3t)$$

$$x_{odd}(t) = \frac{\sin(t) + \cos(3t) - \sin(-t) - \cos(-3t)}{2} = \sin(t)$$

b. x[n]=u[n]+nu[-n-1]

Answer:

$$x_{even}[n] = \frac{u[n] + nu[-n-1] + u[-n] - nu[n-1]}{2}$$

$$x_{odd}[n] = \frac{u[n] + nu[-n-1] - u[-n] + nu[n-1]}{2}$$

1.8 Unit ramp function is determined as *R(t)= tu(t)*. Derive the first and second derivatives of the ramp function.

$$R'(t) = \frac{dR(t)}{dt} = u(t) + t\delta(t)$$

$$R''(t) = \frac{dR'(t)}{dt} = \delta(t) + \delta(t) + t\delta'(t) = 2\delta(t) + t\delta'(t)$$

1.9 The derivative of the impulse signal *)δ(t)* is the unit doublet, denoted as *δ′(t)*. Assuming *f(t)* is continuous at t_0, prove the following relationships:

a. $\int_{-\infty}^{\infty} f(t)\delta'(t-t_0)dt = -f'(t_0)$

Answer:

$$\int_{-\infty}^{\infty} f(t)\delta'(t-t_0)dt$$

$$= \int_{-\infty}^{\infty} f(t)d\delta(t-t_0)$$

$$= f(t)\delta(t-t_0)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t-t_0)f'(t)dt$$

$$= -f'(t_0)$$

b. $f(t)\delta'(t)=f(0)\delta'(t)-f'(0)\delta(t)$

Answer :

$\because$ *{f(t) δ(t)}′ = f′(t) δ(t) + f(t)δ′(t)=f′(0) δ(t)+ f(t)δ′(t)*

$\because$ *{f(t) δ(t)}′=f(0) δ′(t)*

$\therefore$ *f(t)δ′(t)=f(0) δ′(t)-f′(0) δ(t)*

c. *δ′(-t)= - δ′(t)*

Answer :

$$\delta'(t) = \frac{d\delta(t)}{dt}$$

$$\delta'(-t) = \frac{d\delta(-t)}{dt} = -\delta'(t)$$

1.10 Show that x(t)=sin(t) and y(t)=sin(2t) are orthogonal over the interval [t_0, t_0+T], and indicate the value of T.

$$\int_{t_0}^{t_0+T} \sin t \sin 2t dt = \int_{t_0}^{t_0+T} \left(\frac{e^{jt} - e^{-jt}}{2j}\right)\left(\frac{e^{j2t} - e^{-j2t}}{2j}\right)dt$$

$$= \int_{t_0}^{t_0+T} \left(\frac{e^{j3t} + e^{-j3t} - e^{jt} - e^{-jt}}{-4}\right)dt$$

$$= -\frac{1}{2}\int_{t_0}^{t_0+T} (\cos 3t - \cos t)dt$$

$$= -\frac{1}{6}\sin 3t\Big|_{t_0}^{t_0+T} + \frac{1}{2}\sin t\Big|_{t_0}^{t_0+T}$$

$$= 0$$

where T is 2π, the fundamental period of cos(t).

1.11 derive the values of the following functions

a. $\int_{-\infty}^{\infty} f(5-2t)dt$, where *f(t)=2δ(t-3)*.

Answer :

$$\int_{-\infty}^{\infty} 2\delta(5-2t-3)dt = \int_{-\infty}^{\infty} 2\delta(-2t+2)dt = -\int_{\infty}^{-\infty} 2\delta(2t+2)dt = 1$$

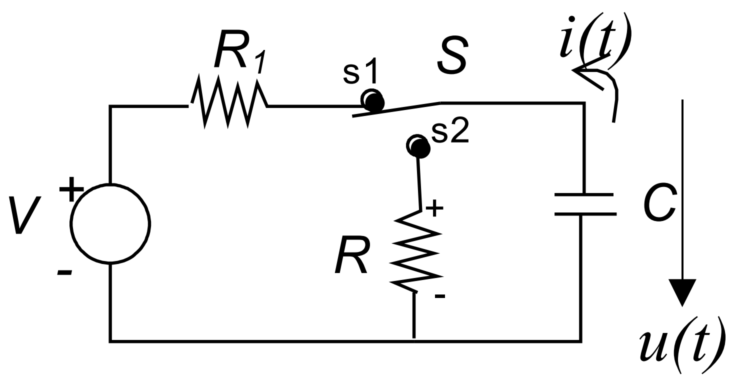
b. $\int_{-2\pi}^{2\pi} (1+t)\delta(\cos t)dt$

Answer:

$$\int_{-2\pi}^{2\pi}(1+t)\delta(\cos t)dt = \int_{-2\pi}^{2\pi}(1+t)\{\delta(t+3\pi/2)+\delta(t+\pi/2)+\delta(t-3\pi/2)+\delta(t-\pi/2)dt = 4$$

1.12 For the following figure, assume the capacitor C has zero charge initially. The reference directions of the voltage *u(t)* and the current *i(t)* of the capacitor C are shown in the figure. At time t=0, s1 is turned on.

a) Please derive the mathematical expression of *i(t)* for non-zero R1.
b) If R1 is zero, please give the mathematical model of the signal *i(t)*, and plot *i(t)*.



Answer:

a.

$$-\frac{1}{C}\int_{-\infty}^{t} i(t)dt - R_1 i(t) = V$$

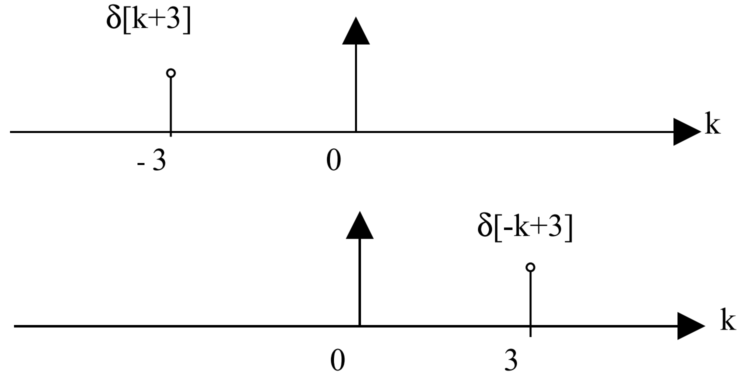
$$\frac{di(t)}{dt} + \frac{1}{R_1 C}i(t) = 0$$

$$i(t) = \frac{-V}{R_1}e^{\frac{-t}{R_1 C}}u(t)$$

b. $i(t) = -CV\delta(t)$

1.13 plot

a. δ[-k+3]



b. u[-k-1]