ECSE306 Midterm 2, November 4, 2008

Name:____________________ **Student number: _________________**

1. (10 points) Fill in "Yes" or "No" according to the properties of the following systems.

Systems	y(t)=dx(t)/dt –2	y[n]-2y[n-1]=x[n]
Linear?	No	Yes
Causal?	Yes	Yes
Stable?	No	No
Time invariant?	Yes	Yes
Memory-less?	Yes	Yes

2. (15 points) Consider the following system described by the difference equation:

$$y[n]-0.9y[n-1]=2x[n]-x[n-10]$$

The system is initially at rest.
(a) (5 points) Determine the homogeneous solution of the system.
(b) (5 points) Compute the impulse response of the system.
(c) (5 points) Compute the response of the system to the input $x[n]=2^{-n}u[n]$.

Answer

(a) The homogeneous equation is $y[n]-0.9y[n-1]=0$
The solution that satisfies the homogeneous equation is $y_h[n]=A0.9^n$.

(b) First obtain $h_a[n]$, the impulse response of the system $y[n]-0.9y[n-1]=x[n]$.
$h_a[n]$ satisfies

$$ha[n]-0.9ha[n-1]=\delta[n]$$

Let $h_a[n]= C0.9^n u[n]$.
Then,

$$C0.9^n u[n]-0.9\ C\ 0.9^{n-1}u[n-1]= \delta[n]$$

For n=0, the above is

$$C\ 0.9^0 u[0]-0.9\ C0.9^{0-1}u[0-1]= \delta[0]=1$$

Thus, C=1, and $h_a[n]= 0.9^n u[n]$.

Next, obtain $h[n]$, the impulse response of $y[n]-0.9y[n-1]=2x[n]-x[n-10]$ by applying linear superposition and time invariance properties of LTI systems:

$h[n]=2h_a[n]-h_a[n-10]$
$=2(0.9^n u[n])- 0.9^{n-10}u[n-10].$

(c) Now that we have obtained the impulse response, the response of the system to the input $x[n]= 2^{-n}u[n]$ can be obtained using the convolution:

$$y[n]=h[n]*x[n]=2\sum_{k=-\infty}^{\infty}0.9^k u[k]2^{-(n-k)}u[n-k]-\sum_{k=-\infty}^{\infty}0.9^{(k-10)}u[k-10]2^{-(n-k)}u[n-k]$$

$$=2\sum_{k=0}^{\infty}0.9^k 2^{-(n-k)}u[n-k]-\sum_{k=10}^{\infty}0.9^{(k-10)}2^{-(n-k)}u[n-k]$$

$$=2\times 2^{-n}(\sum_{k=0}^{n}0.9^k 2^k)u[n]-2^{-n}0.9^{-10}(\sum_{k=10}^{n}0.9^k 2^k)u[n-10]$$

$$=2\times 2^{-n}\frac{1-1.8^{n+1}}{1-1.8}u[n]-2^{-n}0.9^{-10}2^{10}0.9^{10}\frac{1-1.8^{n-9}}{1-1.8}u[n-10]$$

$$=2\times 2^{-n}\frac{1-1.8^{n+1}}{1-1.8}u[n]-2^{-(n-10)}\frac{1-1.8^{n-9}}{1-1.8}u[n-10]$$

Important Notes:

(a) The second convolution can be directly obtained by delaying the first term by 10 and dividing it by –2 (according to the property of LTI systems).

(b) Make sure append u[n] and u[n-10] to reflect *the ranges of* ***n*** *in the two convolutions.*

(c) The final solution consists of homogeneous solutions and particular solutions:

$$y[n]=2\times 2^{-n}\frac{1-1.8^{n+1}}{1-1.8}u[n]-2^{-(n-10)}\frac{1-1.8^{n-9}}{1-1.8}u[n-10]$$

$$=\frac{2}{-0.8}2^{-n}u[n]+\frac{3.6}{0.8}0.9^n u[n]+\frac{1}{0.8}2^{-(n-10)}u[n-10]-\frac{1.8}{0.8}0.9^{n-10}u[n-10]$$

$$=-\frac{5}{2}2^{-n}u[n]+\frac{9}{2}0.9^n u[n]+\frac{5}{4}2^{-(n-10)}u[n-10]-\frac{9}{4}0.9^{n-10}u[n-10]$$

(d) You can obtain the solution from homogeneous solutions and particular solutions, as follows.

Step 1: obtain the homogeneous solution $y_h[n]$ and the particular solution $y_p[n]$ for the deference equation $y[n]-0.9y[n-1]=x[n]$.

$$y_h[n]=A0.9^n u[n]$$

$$y_p[n]=B2^{-n}u[n]$$

$$y_1[n]=y_h[n]+y_p[n]=A0.9^n u[n]+B2^{-n}u[n]$$

A and B can be determined from:

$$y[0]-0.9y[-1]=x[0]\Rightarrow A+B=1$$

$$y[1]-0.9y[0]=x[1]\Rightarrow A0.9+B0.5-0.9(A+B)=0.5$$

Hence $A=9/4,\ B=-5/4$

And, the solution to the equation y[n]-0.9y[n-1]=x[n] is

$$y_1[n] = y_h[n] + y_p[n] = \frac{9}{4}0.9^n u[n] - \frac{5}{4}2^{-n}u[n]$$

Step 2: The solution to y[n]-0.9y[n-1]=2x[n] can be obtained by applying linearity property of LTI systems: $y_2[n] = 2y_1[n] = \frac{9}{2}0.9^n u[n] - \frac{5}{2}2^{-n}u[n]$.

Finally, the solution to y[n]-0.9y[n-1]=2x[n]-x[n-10] can be obtained by applying time invariance property of LTI systems:

$$y\,[n] = 2y_1[n] - y_1[n-10] = \frac{9}{2}0.9^n u[n] - \frac{5}{2}2^{-n}u[n] - \frac{9}{4}0.9^{n-10}u[n-10] + \frac{5}{4}2^{-(n-10)}u[n-10]$$

This result is identical to that obtained using the convolution above. Two methods reached the same solution.

(d) You can also use the inverse z-transform to obtain the impulse response from H(z):

$$Y(z)(1-0.9z^{-1}) = X(z)(2-z^{-10})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2-z^{-10}}{1-0.9z^{-1}}$$

$$h[n] = 2(0.9)^n u[n] - (0.9)^{n-10}u[n-10]$$

3. (15 points) The input signal to a system is *x(t)=A sin(2π t /T).* The output of the system is a periodic signal shown in Fig.1, where *T* is the period of the signals *x(t)* and *y(t)*, and *A* is the amplitudes of *x(t)* and of *y(t).*
The THD is defined as:

$$THD = \sqrt{\frac{\sum_{k=2}^{\infty}|a_k|^2}{|a_1|^2}}100\%$$

(a) (5 points) Calculate the total average power of *y(t)* over one period *T*;
(b) (5 points) Calculate the amplitude of the fundamental component *B sin(2π t /T)* contained the output *y(t).*
(c) (5 points) Calculate the THD caused by the system (hint: apply Parseval theorem).

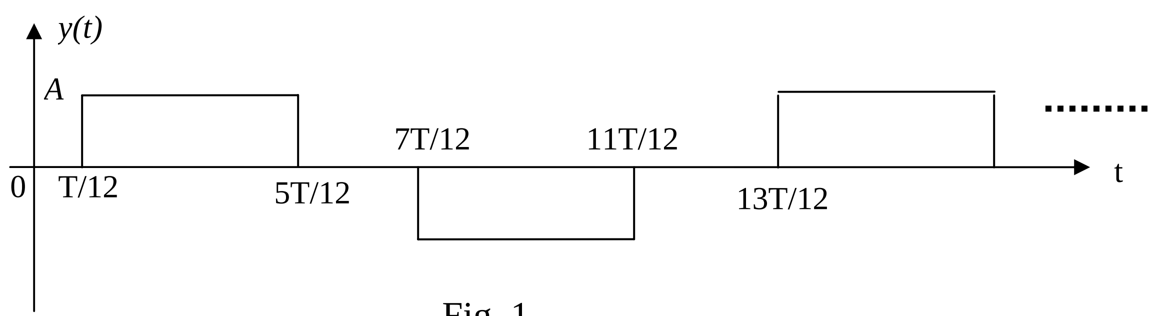


Fig. 1

Answer:

(a)

$$P = \frac{1}{T}\int_T |y(t)|^2\ dt = \frac{1}{T}(A^2\frac{4T}{12} + A^2\frac{4T}{12}) = \frac{2A^2}{3}$$

(b)

$$a_1 = \frac{1}{T}\int_0^T y(t)e^{-j\omega_0 t}dt = \frac{1}{T}[\int_{T/12}^{5T/12} Ae^{-j\omega_0 t}dt - \int_{7T/12}^{11T/12} Ae^{-j\omega_0 t}dt]$$

$$= \frac{A}{-j\omega_0 T}[e^{-j\frac{2\pi}{T}\frac{5T}{12}} - e^{-j\frac{2\pi}{T}\frac{T}{12}}] + \frac{A}{j\omega_0 T}[e^{-j\frac{2\pi}{T}\frac{11T}{12}} - e^{-j\frac{2\pi}{T}\frac{7T}{12}}]$$

$$= \frac{A}{-j\omega_0 T}[e^{-j\frac{5\pi}{6}} - e^{-j\frac{\pi}{6}}] + \frac{A}{j\omega_0 T}[e^{-j\frac{11\pi}{6T}} - e^{-j\frac{7\pi}{6T}}]$$

$$= \frac{A}{-j\omega_0 T}[e^{-j\frac{5\pi}{6}} - e^{-j\frac{\pi}{6}}] + \frac{A}{j\omega_0 T}[-e^{-j\frac{5\pi}{6T}} + e^{-j\frac{\pi}{6T}}]$$

$$= \frac{2A}{-j\omega_0 T}[e^{-j\frac{5\pi}{6}} - e^{-j\frac{\pi}{6}}] = \frac{2A\sqrt{3}}{j2\pi} = \frac{A\sqrt{3}}{j\pi}$$

Because y(t) is real, then $a_{-k} = \overset{*}{a}_k$ and $a_{-1} = \overset{*}{a}_1 = -\frac{A\sqrt{3}}{j\pi}$.

The fundamental component contained in y(t) is

$$a_1 e^{j\omega_0 t} + a_{-1}e^{-j\omega_0 t} = \frac{2A\sqrt{3}}{\pi}(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}) = \frac{2A\sqrt{3}}{\pi}\sin(\omega_0 t)$$

Thus, the amplitude of the fundamental component is

$$B = \frac{2A\sqrt{3}}{\pi}$$

(c) According to Parseval theorem,

$$\frac{1}{T}\int_T |x(t)|^2\ dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Thus

$$\frac{1}{T}\int_T |x(t)|^2\ dt = |a_0|^2 + 2|a_1|^2 + 2\sum_{k=2}^{\infty} |a_k|^2$$

As $a_0=0$, hence

$$\sum_{k=2}^{\infty}|a_k|^2=\{\frac{1}{T}\int_T|x(t)|^2\,dt-|a_0|^2-2|a_1|^2\}/2=\{\frac{2A^2}{3}-2|a_1|^2\}/2$$

$$=\{\frac{2A^2}{3}-2|\frac{\sqrt{3}A}{\pi}|^2\}/2=A^2(\frac{1}{3}-\frac{3}{\pi^2})$$

$$THD=\sqrt{\frac{\sum_{k=2}^{\infty}|a_k|^2}{|a_1|^2}}=\frac{A\sqrt{1/3-3/\pi^2}}{A\sqrt{3}/\pi}=\frac{\sqrt{\pi^2-9}}{3}100\%=31.08\%$$

4. (10 points) Determine the Fourier transform of $f(t)=G(t)\sin(\omega_0 t)$, where G(t) is a rectangular wave with amplitude E and width T, as shown in Fig. 2.

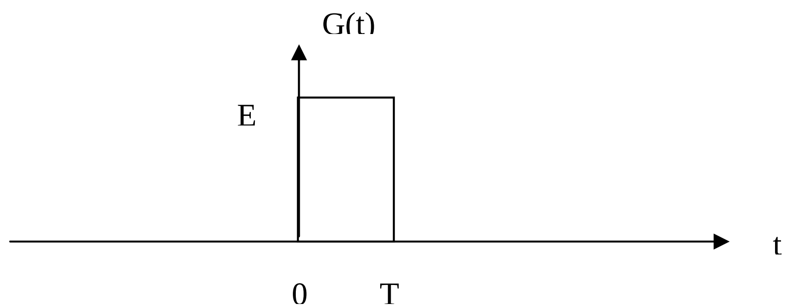


Fig. 2

Answer:

Delay $t_0=T/2$, looking up the FT pair Table and applying the delay property of FT, we have

$$G(t)\leftrightarrow G(j\omega)=e^{-j\omega T/2}TE\,\mathrm{sinc}(\frac{T}{2\pi}\omega)$$

$$\sin(\omega_0 t)=\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\leftrightarrow\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$$

Multiplication in the time domain corresponds to convolution in the frequency domain, and we have

$$G(t)\sin(\omega_0 t) \leftrightarrow \frac{1}{2\pi}G(\omega) * \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$$

$$= \frac{1}{2j}\{G(\omega-\omega_0)-G(\omega+\omega_0)\}$$

$$= \frac{1}{2j}TE\{e^{-j(\omega-\omega_0)T/2}\sin c(\frac{T(\omega-\omega_0)}{2\pi}) - e^{-j(\omega+\omega_0)T/2}\sin c(\frac{T(\omega+\omega_0)}{2\pi})\}$$

5. (15 points) Consider the causal LTI system initially at rest described by

$$2\frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 2x(t).$$

(a) (7 points) Determine the transfer function of the system *H(s)* and its *ROC*.

(b) (8 points) Determine the magnitude and phase frequency responses of the system $|H(j\omega)|$ and $\angle H(j\omega)$.

Answer:

(a)

$$Y(s)[2s+1] = X(s)[s^2 - s - 2]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 - s - 2}{2s+1}$$

ROC : Re{s}>-1/2

(b)

$$H(j\omega) = \frac{-\omega^2 - j\omega - 2}{2j\omega + 1}$$

$$|H(j\omega)| = \frac{\sqrt{\omega^2 + (2+\omega^2)^2}}{\sqrt{1+4\omega^2}}$$

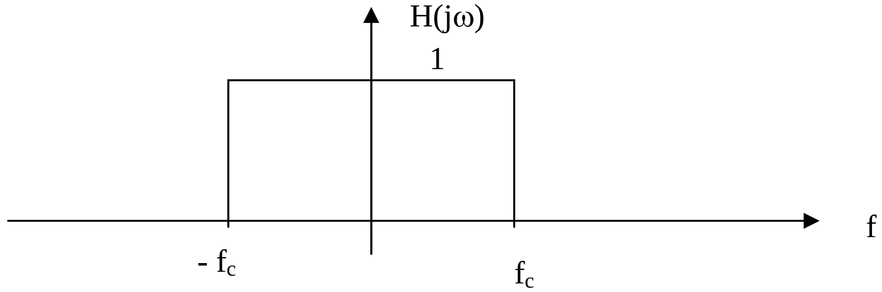
$$\angle H(j\omega) = \pi + \arctan\frac{\omega}{2+\omega^2} - \arctan\frac{2\omega}{1}$$

6. (10 points) Consider an ideal low-pass filter with cutoff frequency f_c=200 Hz, as shown in Fig. 3.

(a) (5 points) Determine the impulse response h(t) of the low-pass filter.

(b) (5 points) If the input signal is $x(t)=e^{-t}u(t)$, determine |Y(jω)|, the magnitude spectrum of the output signal y(t).

Fig. 3



Answer:

(a) $\omega_c = 400\pi$.

$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{1}{2\pi j t}[e^{j\omega_c t} - e^{-j\omega_c t}] = \frac{1}{\pi t}\sin(\omega_c t)$$

(b) The Fourier transform of x(t) is

$$X(j\omega) = \frac{1}{j\omega + 1}$$

Its spectrum is

$$|X(j\omega)| = \frac{1}{\sqrt{\omega^2 + 1}}$$

The spectrum of the output is

$$|Y(j\omega)| = \begin{cases} \frac{1}{\sqrt{\omega^2+1}}, & |\omega| < \omega_c \\ 0, & otherwise \end{cases}$$

7. (10 points) Consider an LTI system with transfer function

$$H(s) = \frac{s^2 + 2}{s^2 + s + 4}$$

(a) (2 points) Computer the zeros and poles of the system.
(b) (2 points) Indicate the ROC for the system to be stable.
(c) (6 points) Determine the impulse response of the system.

Answer:
(a) the zeros are $j\sqrt{2}, and -j\sqrt{2}$. The poles are $-1/2 \pm j\sqrt{15}/2$
(b) For the system to be stable, the ROC must include the jω axis. Hence, ROC∈Re{s}>-1/2.
(c)

$$H(s)=\frac{s^2+2}{s^2+s+4}=\frac{(s\ \ +1/2)^2+15/4-s-7/4}{(s\ \ +1/2)^2+15/4}=1-\frac{s+7/4}{(s\ \ +1/2)^2+15/4}$$

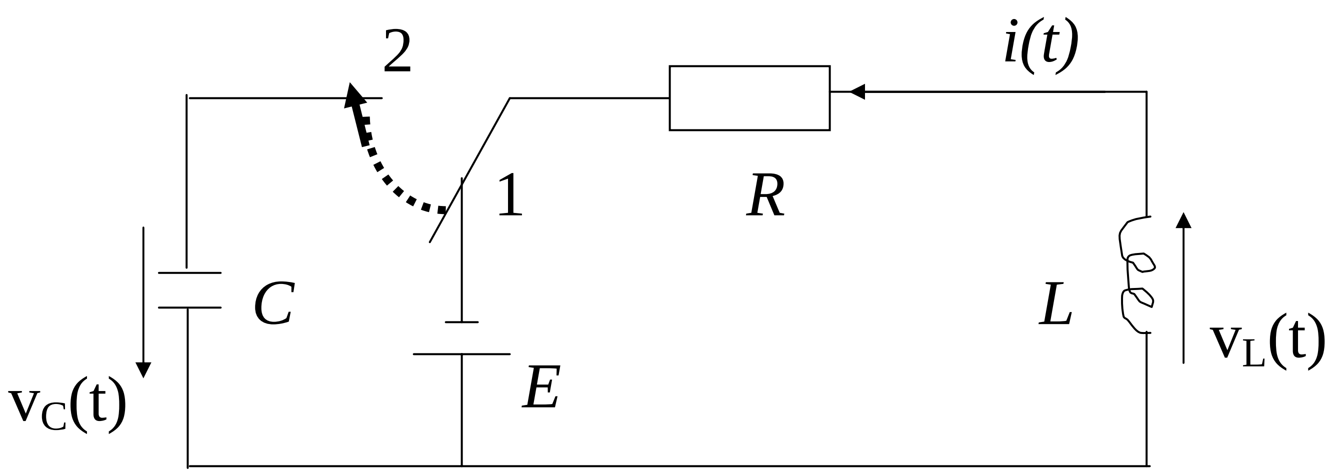
$$=1-\frac{s+1/2}{(s\ \ +1/2)^2+15/4}-\frac{5/4}{(s\ \ +1/2)^2+15/4}$$

$$\leftrightarrow h(t)=\delta(t)-e^{-t/2}\cos(\sqrt{15}t/2)u(t)-\frac{5}{4}\frac{2}{\sqrt{15}}e^{-t/2}\sin(\sqrt{15}t/2)u(t)$$

8. (15 points) Consider the circuit below. C is a capacitor, L is an inductor and R is a resister. Before time t=0, the system is steady. At time t=0, the switch is turned on 2 from 1.

(a) (12 points) Determine the *i(t)* for t≥0.

(b) (2 points) Indicate the condition for the system to be stable for t≥0.



Answer:

(a) For t>=0, the current *i(t)* is determined by:

$$\frac{1}{C}\int_0^t i(\tau)d\tau+Ri(t)+L\frac{di(t)}{dt}=0\,. \qquad \text{(Eq.8(a))}$$

Taking unilateral Laplace transform of the above Eq., we have

$$\frac{1}{Cs}I(s)+RI(s)+L[sI(s)-i(0^-)]=0\,.$$

$$I(s)+RCsI(s)+LCs^2I(s)-LCsi(0^-)=0$$

$$I(s)=\frac{LCsi(0^-)}{LCs^2+RCs+1}=\frac{si(0^-)}{s^2+Rs/L+1/LC}=\frac{A}{s-p_1}+\frac{B}{s-p_2}$$

where $i(0^-)=E/R$. Taking the inverse unilateral Laplace transform of I(s), we have

$$i(t) = [Ae^{p_1 t} + Be^{p_2 t}]u(t)$$

where

$$p_1 = \frac{-R/L + \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$p_2 = \frac{-R/L - \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

and

$$A = \frac{p_1 i(0^-)}{p_1 - p_2} = \frac{p_1 E/R}{p_1 - p_2}$$

$$B = \frac{p_2 i(0^-)}{p_2 - p_1} = \frac{p_2 E/R}{p_2 - p_1}$$

(b) For the system to be stable, the poles should be in the left half s-plane, i.e., $Re\{p_{1,2}\}<0$, which is satisfied always.

Note:
Some students first took derivative of Eq. 8(a) and then did Unilateral Laplace transform and included $i(0^-)=E/R$ to all derivatives of i(t). This is incorrect because after taking additional derivative, the information about the voltage of the capacitor can not be recovered. This mistake is charged one point only.